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Mathematical Flow Determination in Open Channel by Method of Characteristics

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Abstract

In this paper grid method of continues forward characteristics and rectangular grid broken backward characteristics is presented, for the solution of saint Venant equations for free surface flow. The method yield depth and velocity hydrographs at predetermined distances from which depths and velocity are obtained. The distance steps used in this method need not be equal, and may be chosen in such a manner as to deal with each channel geometry and spatial distribution of lateral inflows. Studies so far show that there are minimal differences in accuracy between the direct and iterative formulation of method.

1.0 Introduction

There are basically two characteristic methods for the solution of the Saint Venant Equations for free surface flows [10]. These are the characteristic grid method and the rectangular grid broken characteristic method. Each of these methods is capable of a direct and an iterative formulation. Amein [2] used a direct formulation of the characteristic grid method for stream flow routing. Liggett and Woolhiser [7] and Fletcher and Hamilton [4] used iterative formulations of the characteristic grid method for routing overland flow and flood routing in an irregular channel, respectively. Baltzer and Lai [3] used a direct formulation of the rectangular grid broken characteristic method for simulation of unsteady flows in waterways. The characteristic grid method essentially uses a grid of continuous forward and backward characteristics. The rectangular grid broken characteristic method uses a rectangular grid and broken forward and backward characteristics. This paper presents a method that uses specified distances and a grid made of continuous forward characteristics and broken backward characteristics. Direct and iterative formulations of the method are detailed and their accuracies demonstrated.

Saint Venant Equations

The Saint Venant Equations exist in various forms. Their derivations and basic assumptions made are available in the literature [8, 9]. The general form of the Saint Venant Equations may be stated as follows [6]:

$$A\frac{dv}{dx} + vB\frac{dy}{dx} + B\frac{dy}{dt} + v\left(\frac{dA}{dx}\right), = \mathrm{q} \tag{1}$$

$$\frac{dy}{dx} + \frac{v}{g}\frac{dv}{dx} + \frac{1}{g}\frac{dv}{dt} = S_o - S - \frac{q^v}{g^A} \tag{2}$$

in which, at a horizontal distance x from the origin at time t, y = depth of water; A = area of the water section; B = width of channel at the water surface; v = the velocity; q = lateral inflow per unit length of channel; S = the friction slope; $\left(\partial A/\partial x\right)$, = rate of variation of A with respect to x when y is held constant, S_o = the bed slope at distance x from origin; and g = acceleration due to gravity.

Characteristic Form

Equations 1 and 2 may be expressed in characteristic form by the methods given in Refs [1, 4 6] as follows:

$$\frac{1}{c}\frac{dy}{dt} + \frac{1}{g}\frac{dv}{dt} = S_o - S - \frac{q}{gA}(v-c) - \frac{vc}{gA}\left(\frac{\partial A}{\partial x}\right) \tag{3}$$

$$\frac{dx}{dt} = v + c \tag{4}$$

$$-\frac{1}{c}\frac{dy}{dt} + \frac{1}{g}\frac{dv}{dt} = S_o - S - \frac{q}{gA}(v-c) - \frac{vc}{gA}\left(\frac{\partial A}{\partial x}\right) \tag{5}$$

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$$\frac{dx}{dt} = v - c \tag{6}$$

in which $c = \sqrt{gA/B}$. The compatibility equation (3) holds along positive or forward characteristics given by equation (4) and the compatibility equation (5) holds along negative or backward characteristics given by equation (6). The right-hand sides of equation (3) and (5) are hereafter referred to as G_f and G_b, respectively, for convenience.

Initial and Boundary Conditions

It is assumed that there is steady flow in the channel initially. At the upstream boundary, in general, the conditions may be subcritical, critical, or supercritical. When conditions are subcritical or critical, the upstreams boundary condition may be given as a discharge hydrograph or a stage hydograph. When conditions are supercritical, both hydrogaphs are necessary. In general at the downstream boundary the conditions may be subcritical, critical, or supercritical also. If conditions are subcritical or critical, a stage discharge relationship would exist.

Scheme or Calculation

Fig. 1 shows the scheme of characteristics used in the calculation. The specified distances at which t, y and v are to be calculated are shown by the broken lines. A_1, A_o is a continuous positive characteristic, A_3, B_1, A_3, G_2, A_4, G_3 etc are broken negative characteristics initiated from points A_2, A_3, A_4, etc. If the values of t, y and v are known at points A_1, A_2, A_3 etc, the solution is first advanced to points B_1, G_2, G_3 etc, by the methods detailed in the sections that follow. The values of t, y and v at B_2, B_3, B_4, etc (specified distances) are then determined by interpolation from the values at points B_1, G_2, G_3, etc. The calculations are commended from a forward characteristic below the unsteady flow region and advanced successively from one forward characteristic to the next until the desired time.

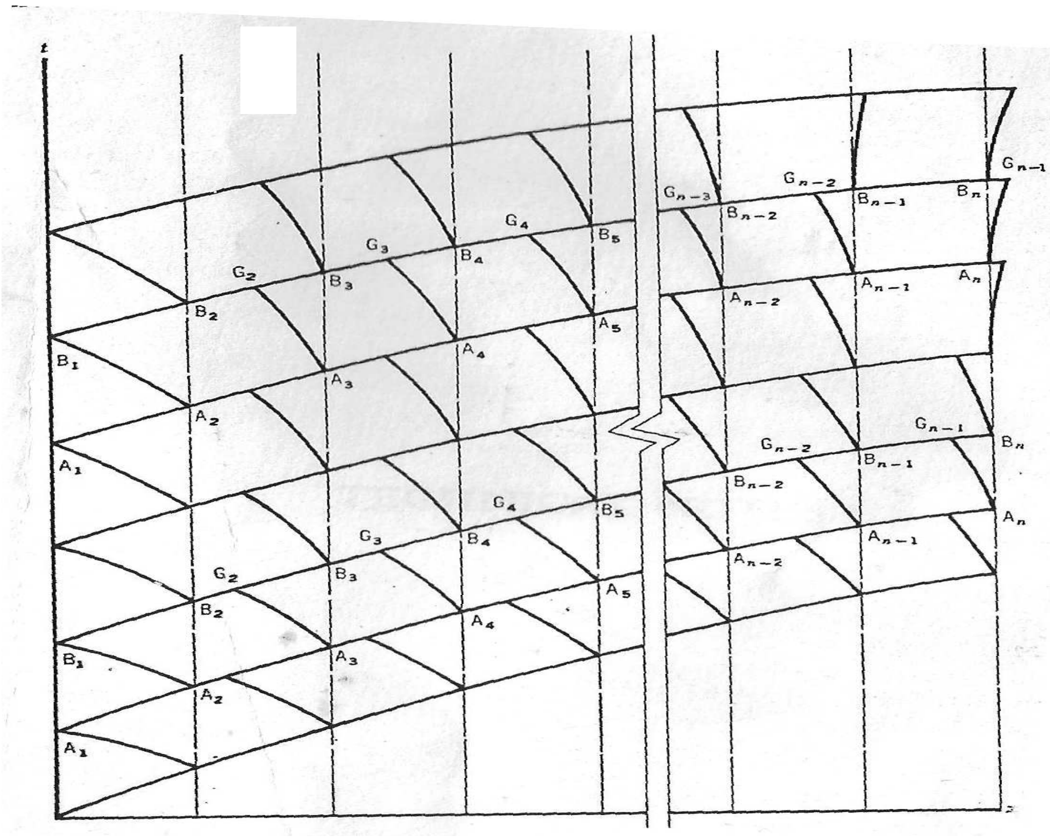

Fig. 1 Point Letting for scheme of Calculation

In advancing the solution from points A_1, A_2, A_3 etc to points B_1, G_2, G_3, etc one of three types of calculations may be involved at the upstream boundary. These are (1) the conditions there are subcritical and the slope of the negative characteristic, A_2B_1 is not too large or too small (upstreams and case 1); (2) the conditions there are subcritical and slope of the negative characteristic A_2 B_1 is either too large or too small (upstream end case 2); and (3) conditions there are critical or supercritical (upstream end case 3). At the interior points such as G_2, G_3, G_4 etc the same type of calculations are involved whatever the conditions. At the downstream end either of two types of calculations may arise: (1) The conditions there are subcritical or critical (downstream end case 1); and (2) the conditions there are supercritical (downstream end case 2).

Iterative Formulations

In the iterative formulations herein, the slope of the characteristics (i.e. $v_f + c_f$ and $v_b - c_b$), the coefficients of the terms $\frac{dy}{dt}$ (i.e $\frac{1}{c_f}$ and $\frac{1}{c_b}$) and the terms G_f andG_b are taken as evaluated using mean values of x, t, y, and v between the points concerned. The iteration method used is the Newton-Raphson iteration method for the solution of simultaneous non-linear equations (Appendix 1). Parameter such s c, A,B,R,N, $\left(\frac{\partial A}{\partial x}\right)$, q, and S which occur in the terms G_f and G_b or individually in the finite difference equations herein are functions of some or all of he variables, x, y, and v, and are therefore known at each stage of the iteration process for the particular values of x, t, y and v being used at that stage.

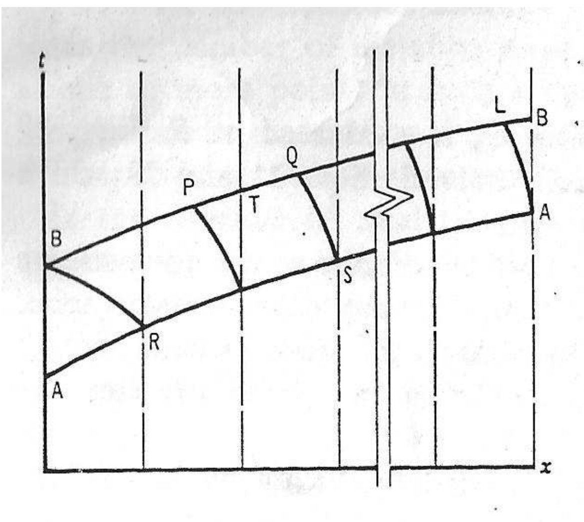

Fig. 2: Point Letting for Upstream End Case 1, 1, Interior Point and Down-stream End Case 1

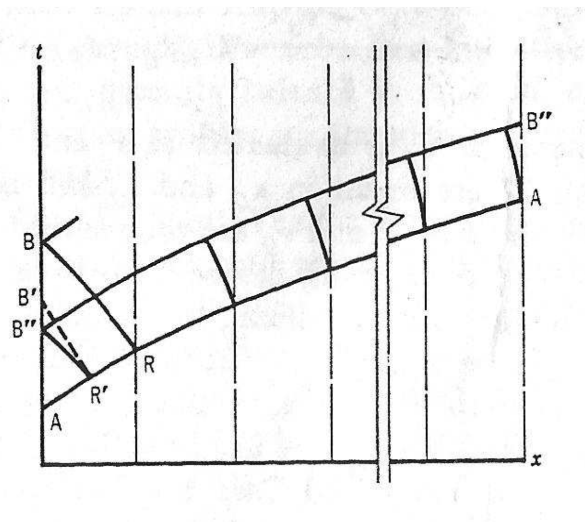

Fig. 3: Point Letting for Upstream End Case 2

Upstream End Case 1: The equation to the negative characteristics, BR and the corresponding compatibility equation are expressed (Fig. 2) in finite differences and the following equations result:

$$x_B - x_B - (v_b - c_b)(t_B - t_B) = 0 \tag{7}$$

$$\frac{y_B - y_B}{c_b} + \frac{v_B - v_B}{g} - G_b\left(t_B - t_B\right) = 0 \tag{8}$$

If the upstream boundary condition is given as a discharge hydrograph Q(0,t), the following equation also holds.

$$v_B = \frac{Q(0,t_B)}{A(y_B,0)} = 0 \tag{9}$$

Equations (7,) (8) and (9) have unknown y_B, v_B, and t_B and are nonlinear. They are solved by the Newton-Raphson iteration method (Appendix 1), assuming starting values obtained by putting $v_R = v_B$, $= y_R$. If he upstream boundary condition is given as a depth hydrogaph, then in effect equations (7) and (8) would have two unknowns v_B and t_B and can be solved by the New-Raphson iteration method.

Upstream End Case 2: In order to ensure adequate accuracy of the computation (Fig. 3), the time interval AB between successive forward characteristics originating from the upstream boundary should not be large. Likewise in order to exercise some control over the computation time, it may sometimes be desirable that AB should not be too small. These are dealt with as follows: (1) Choosing of $R'B'$ as $(v_A - c_A)$; (2) values of v, y, t at R' are nest determined by interpolation from values at the grid points along the positive characteristic ARA; and (3) the calculations for Upstream End Case 1 are then used with the values at R' in place of those at R to determine the values of v, y and t at B'' .

Upstream End Case 3 - When conditions at the upstream boundary are supercritical (fig. 4), the positive and negative characteristics from the upstream boundary are as shown on the left in the figure. In this case the depth y_B and velocity v_B (or discharge Q_B) have to be supplied at time t_B. If conditions are critical the negative characteristic AL will coincide with AB and v_B will be a function of y_B. In this case the time t_B is chosen arbitrarily. Equation (9) with c_B in place of v_B gives a nonlinear equation in y_B which can be solved by the Newton-Raphson method.

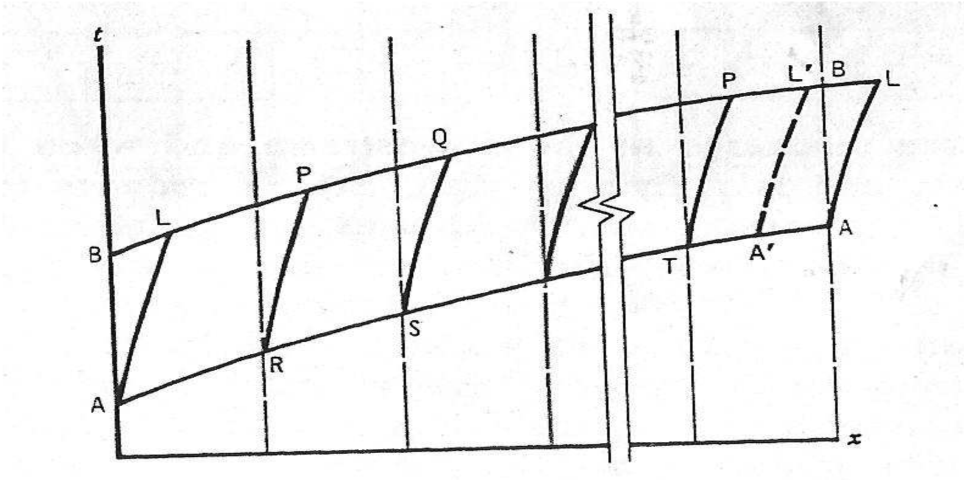

Fig. 4: Point Letting for Upstream End Case 3, Interior Point, and Down-stream End Case 1

Interior Points- The values x,t , y, and v are known at P and S (Figs. 2 and 4). The values at Q are to be calculated first. The equations to the characteristics PQ and SQ and their corresponding compatibility equations are expressed as follows in finite differences:

$$x_Q - x_p - (v_f + c_f)(t_Q - t_p) = 0 \tag{10}$$

$$\frac{y_Q - y_p}{c_f} + \frac{v_Q - v_p}{g} - G_f(t_Q - t_p) = 0 \tag{11}$$

$$x_Q - x_s - (v_b - c_b)(t_Q - t_s) = 0 \tag{12}$$

$$-\frac{y_Q - y_s}{c_b} + \frac{v_Q - v_s}{g} - G_b(t_Q - t_s) = 0 \tag{13}$$

The preceding equations have among them four unknowns (x_Q, t_Q, y_Q, and v_Q) and are nonlinear. They are solved by the Newton-Raphson iteration method (Appendix 1) assuming starting values for X_Q, t_Q, y_Q and v_Q that may be obtained by putting $v_f = v_p$, $c_f = c_p$, , $v_b = v_s$, $c_b = c_s$, and evaluating G_f at P and G_b at S. The values of t, y, and v at the specified distances such as at T are then obtained by linear interpolation from the values at P and Q.

Downstream End Case 1: The values of x,t, y, and v are known at L; at B the unknowns are t_B, y_B, and v_B (Fig. 2). The equation to the forward characteristic LB and the corresponding compatibility equation are expressed as follows in finite differences

$$x_B - x_L - (v_f + c_f)(t_B - t_L) = 0 \tag{14}$$

$$\frac{y_B - y_L}{c_f} + \frac{v_B - v_L}{g} - G_f(t_B - t_L) = 0 \tag{15}$$

The following equation is obtained from the downstream boundary condition.

$$v_B - \frac{Q_B}{A(y_B, x_B)} \tag{16}$$

in which Q_B = the discharge at the downstream end when the depth is y_B. If the downstream end is a free overfall, equation (16) reduces to $v_B - c_B = 0$. Equations (14), (15) and (16) have among them unknowns y_B, v_B, and t_B and are nonlinear. They are solved by the Newton-Raphson iteration method (Appendix 1) using starting values $t_B = {(x_B - x_L)}/{(v_L + c_L)} + t_L$, $y_B = y_L$ and $v_B = v_L$.

Downstream End Case 2: The values at L are calculated as for the interior point on the assumption that beyond the downstream boundary, the channel slope, channel section, and Manning's N are the same as at A (Fig. 4). The values of A' is then chosen such that AA'=BL. The values of y, v, and t at A' are determined by linear interpolation from the values at T and A. The values of x, t, y and v at L' are then determined using the calculations for an interior point. If L' falls between B and L, the procedure is repeated until it falls to the left of B. The values of y, v, and t at B are then determined by extrapolation of the vales at P and L'. If the distance BL is very small, values of t, y, and v at B may be obtained directly by interpolation from the values at P and L.

Direct Formulations

Upstream End Case 1: The following equations take the place of equations (7), (8) and (9) (see Fig. 2).

$$x_B - x_R - (v_R - c_R)(t_B - t_R) = 0 \tag{17}$$

$$-\frac{y_B - y_R}{c_R} + \frac{v_B - v_R}{g} - G_b(t_B - t_R) = 0 \tag{18}$$

$$VB - \frac{Q(0,t_s)}{A(y_B,0)} = 0 \tag{19}$$

The term G_b is evaluated at R. Equation (17) yields t_B. Substitution for v_B (from equation 19) in equation (18) yields a nonlinear equation in y_B which can be solved by the Newton-Raphson method. Equation (19) then gives v_B, If the upstream boundary condition (Fig. 2) is given as a depth hydrogaph, y_B is known at time t_B. Equation (18) then gives v_B.

Upstream End Case 2: All the statements made under the iterative formulation apply here also.

Upstream End case 3: All the statements made under the iterative formulation apply here also.

Interior Points – The following equations take the place of equations (10), (11), (12) and (13).

$$x_Q - x_p - (v_p + c_r)(t_Q - t_r) = 0 \tag{20}$$

$$\frac{y_Q - y_p}{c_p} + \frac{v_Q - v_p}{g} - G_f(t_Q - t_r) = 0 \tag{21}$$

$$x_Q - x_s - (v_s - c_s(t_Q - t_s) = 0 \tag{22}$$

$$-\frac{y_Q - y_s}{c_2} + \frac{v_Q - v_s}{g} - G_b(t_Q - t_s) = 0 \tag{23}$$

The term G_f is evaluated at P and the term G_b is evaluated at S. Equations (20) and (22) are linear in x_Q and t_Q and are easily solved. Equations (21) and (23) which are linear in y_Q and v_Q are the solved for y_Q and v_Q.

Interior Point just before Downstream End: If conditions at the downstream boundary are subcritical, the calculation is to be as for any other interior point. However, if conditions at the downstream boundary are critical or change form critical to supercritical and vice versa, the calculations under the iterative formulation are to be sued for this point.

Downstream End Case 1: The following equations take the place of equations (14), (15) and (16).

$$x_B - x_L - (v_L + c_L)(t_B - t_L) = 0 \tag{24}$$

$$\frac{y_B - y_L}{c_L} + \frac{v_B - v_L}{g} - G_f(t_B - t_L) = 0 \tag{25}$$

$$v_B - \frac{Q_B}{a(y_B, x_B)} = 0 \tag{26}$$

The term G_f is evaluated at L. Equation (24) gives t_B. Substitution for v_B (from equation (26)) in equation (25) gives a nonlinear equation in y_B that can be solved by the Newton-Raphson method. Equation (26) then gives v_B.

Downstream End Case 2: All the statements made under the iterative formulation apply here also.

Interpolation between Grid Points

If P and Q are two grids on a forward characteristic (see Figs. 2 and 4) the values t_r, y_r, and v_r corresponding to a specified distance x_r on the characteristic are obtained as follows: (1) The value of t_r is first obtained by linear interpolation from the values t_Q and t_p at x_Q and x_p, respectively; (2) it is assumed that y and v vary linearly with the characteristic curve length between the two grid points considered; and (3) since the characteristic between P and Q is assumed as a straight line, y and v may be taken to vary linearly with x or t between these two points.

Values at Predetermined Times

The calculations detailed earlier yield values of t, y, and v corresponding to predetermined values of x. At each of these value of x, values of y ad v can be determined for any predetermined time by linear interpolation. Thus values y and v can be obtained on any predetermined rectangular grid.

Stability and Accuracy

Numerical experiments confirm that the new method is stable. It maintains steady-state solutions successfully (to within a small fraction of a percentage when the number of distance steps is large enough). Perturbations introduced at one or more points in such a manner that quantity balance is satisfied, do not produce any instability. Perturbed steady-state profiles are found to approach the unperturbed steady-state profiles with time.

In the absence of analytical solutions to the Saint Venant equations, the accuracy of any solution cannot in general be determined precisely. However, it may be said that a solution that is to be termed accurate should satisfy the following criteria, quantity balance must be satisfied at all times. Large increases in the number of distance steps should produce negligible changes in the computed solutions at all grid points. Solutions satisfying these criteria can be obtained by the new method. Computations using the new method have also been compared with computations using the iterative formulation of the characteristics grid method. Depths have been obtained at grid points on a predetermined rectangular grid using the two methods. These have been found to agree very closely in all cases that have been studied.

For the purpose of observing quantity balance a parameter called herein as the "percentage quantity deficit" at a section distant L from the upstream end at time t is defined as 100D/E in which D and E are

$$D = \int_O^t Q(0,t)dt + \int_0^t \int_0^L q(x,t)dxdt + \int_O^L A[x, y(x,0)]dx$$

$$\int_O^t v(L,t)A[L, y(L,t)]dt - \int_O^L A(x,y)(\mathrm{x},\mathrm{t})dx \tag{27}$$

$$E = \int_O^t Q(0,t)dt + \int_O^t \int_O^L q(x,t)dxdt \tag{28}$$

The equation of continuity requires that the percentage quantity deficit should be negligible for all values of t. However, there are some errors inherent in its evaluation. These errors are dependent on the time steps and distance steps used in the evaluation of the integrals in the expression D and E. Thus the percentage quantity deficit may not turn out t be negligible even for accurate solutions. Nevertheless it should be sufficiently small at all times if adequately small time and distance steps are used in its evaluation. In the examples that follow time steps of 1 sec have been used for this purpose. The distance and standard deviation of the percentage quantity deficit are found to approach zero with decrease in the size of the distance steps. In cases where conditions at the upstream end are a stated under upstream end case 2, the choice of the interval of time AB' (Fig. 3) is found to have an effect on the percentage quantity deficit, the smaller values for AB' producing smaller values for percentage quantity deficit. The percentage quantity deficit appears to give some indication of the relative accuracy of solutions for the same method and the same problem. Continuous monitoring of it throughout a computation could help to judge the adequacy of the distance steps used and the time interval AB' chosen for upstream end case 2.

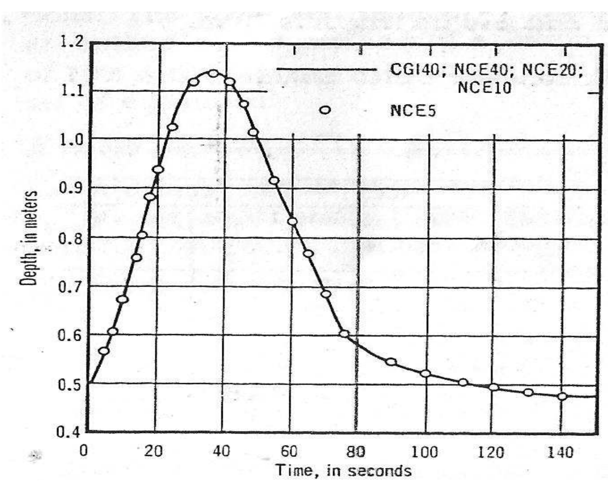

Fig. 5: Depth Hydrographs at Up-stream End in Example 1 (1m=3.28ft)

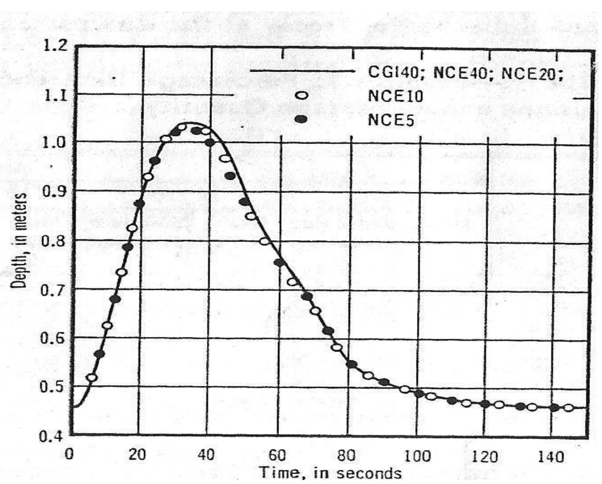

Fig. 6: Depth Hydrographs at x =12m in Example 1 (1m=3.28ft)

Applications

For convenience the following abbreviations are used in this section. The new characteristic method is referred to in its direct formulation by NCE and in its iterative formulation by NCI. The iterative formulation of the characteristic grid method is referred to by CGI. A number attached to any of these abbreviations is to be taken to denote the number of steps used in the calculation, e.g. CG140 denotes the iterative formulation of the characteristic grid method using 40 distance steps.

Two examples of flood routing are dealt with following: (1) Conditions at the downstream boundary are critical throughout; and (2) conditions as the downstream boundary change from critical to supercritical, remain supercritical for some time, and then revert to critical. In the first example, computations by NCE5, NCE10, NCE20, and NCE40 are compared with computations by CG140. The differences between some of the hydrographs are so small that it is not possible to show them visibly with the scale used for the graphs. The same line or set of points is therefore used to represent more than one computed hydrograph. However, the differences are shown quantitatively in the tables that follow, through their mean values and standard deviation. All computations were done on a IBM360/65 machine.

Example 1: The channel in which the flood routing is done has a section that is circular (radius of 1m) up to a depth of 1.25m and has vertical sides above this. It has a constant bed slope of 0.005, Manning's N = 0.01. The channel discharges freely at the downstream end and its length is 20m.

Table 1 – Example 1: Percentage Differences between CG140 Solutions and NCE Solutions and Percentage Quantity Deficits

	METHOD									
	NCE40		NCE20		NCE10		NCE5		CG140	
Location 1	Main value 2	Standard deviation 3	Main value 4	Standard deviation 5	Main value 6	Standard deviation 7	Main value 8	Standard deviation 9	Main value 10	Standard deviation 11
Upstream end r=12m	0.2310 0.3668	0.1231 0.1795	0.1684 0.2625	0.4246 0.7309	0.3205 0.5038	0.7399 1.340	0.7399 1.3490	0.6133 0.6176		
Downstream end	0.1576	0.1246	0.2044	0.1802	0.2393	0.5105	0.5796	0.6505		
Percentage quantity deficits	-0.3942	0.0664	-0.6616	0.0860	-1.0617	0.2091	-2.2018	0.6711	-0.037	0.064

A discharge hydrograph is superimposed at the upstream end on a steady flow of $0.8m^3/s$ to give

$$
\begin{aligned}
&Q(0,t) = 0.8 + \sin\frac{\pi t}{50}; 0 \le t < 25\\
&Q(0,t) = 1.8 : 25 \le t < 50\\
&Q(0,t) = 0.8 + \sin\frac{\pi(75-t)}{50}; 50 \le t < 75\\
&Q(0,t) = 0.8; 75 \le \mathrm{t}
\end{aligned}
\tag{29}
$$

The following lateral inflow hydrograph is also superimposed over the whole length of the channel

$$
\begin{aligned}
&q = 0.06\sin\frac{\pi t}{50}; 0 \le t < 50\\
&q = 0; 50 \le \mathrm{t}
\end{aligned}
\tag{30}
$$

The computations are continued up to t = 150 sec; figs 5, 6 and 7 show the depth hydrographs computed by CG140, NCE40, NCE10 and NCE5 at the upstream end (x = 0), at x = 12 m and at the downstream end (x = 20m), respectively. Table 1 shows the percentage differences between the depth hydrographs by CG140 and the depth hydrogaphs by NCE40, NCE20, NCE10, and NCE5 through their mean values and standard deviations. Table 1 also shows the variations of the percentage quantity deficits through their mean values and standard deviations. In this example, conditions are critical at the downstream boundary throughout.

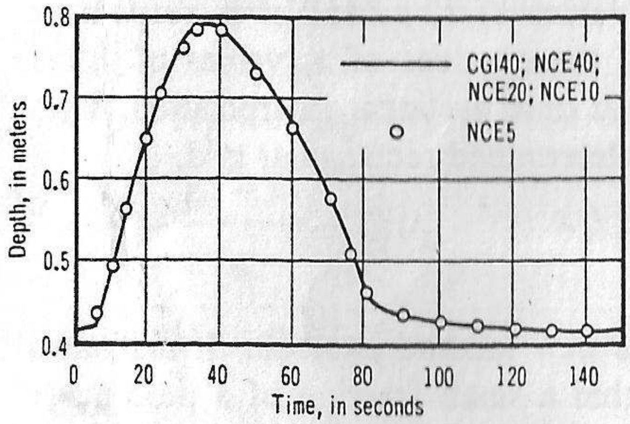

Fig. 7: Depth Hydrographs at Down-stream End in Example 1 (1m=3.28ft)

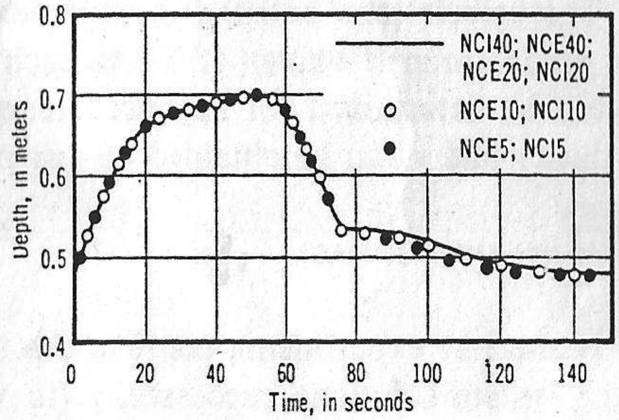

Fig. 8: Depth Hydrographs at Up-stream in Example 2 (1m=3.28ft)

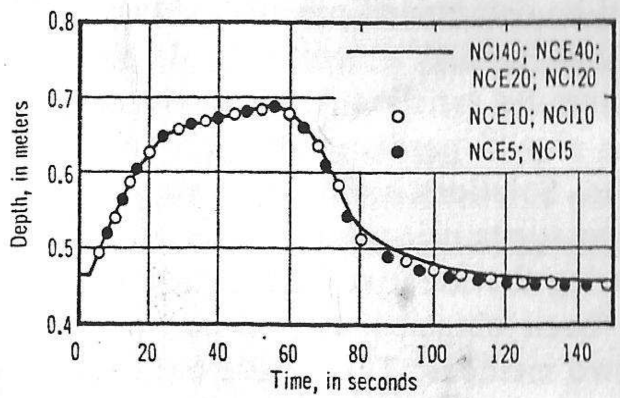

Fig. 9: Depth Hydrographs at x = 12m in Example 2 (1m=3.28ft)

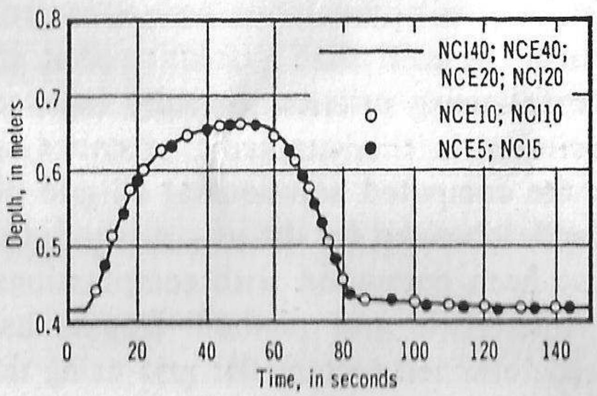

Fig. 10: Depth Hydrographs at Downstream in Example 2 (1m=3.28ft)

Example 2: The particulars of the channel are the same as for example 1. A discharge hydrograph is superimposed at the upstream end on the same steady flow to give

$$
\begin{aligned}
Q(0,t) &= 0.8 + 1.2\sin\frac{\pi t}{50};\ 0 \le t < 25 \\
Q(0,t) &= 2.0;\ 25 \le t < 50 \\
Q(0,t) &= 0.8 + 1.2\sin\frac{\pi(75-t)}{50};\ 50 \le t < 75 \\
Q(0,t) &= 0.8;\ 75 \le t
\end{aligned}
\tag{31}
$$

The computations are continued up to t = 150 sec. Figures 8,9 and 10 show depth hydrogaphs computed by NC140, NCE40, NC120, NCE20, NC110, NCE10, NC15, NCE5 at the upstream end (x = 0), at x = 12 m and at the downstream end (x = 20m), respectively. Table 2 shows the percentage differences between the NC140 depth hydrographs and the depth hydrographs by NC120, NC10, NC15 and the percentage quantity deficits through their mean values and standard deviations. Table 3 shows the percentage differences between the NC140 depth hydrographs and the depth hydrographs by NCE40, NCE20, NCE10, NCE5 and he percentage quantity deficits through their mean values and standard deviations. In this example conditions at the downstream boundary are super critical from t = 7.27 sec to t=40.27 sec and critical at other times.

Table 2 – Example 1: Percentage Differences between CG140 Solutions and NCE Solutions and Percentage Quantity Deficits

	METHOD									
	NCE40		NCE20		NCE10		NCE5		CG140	
Location 1	Main value 2	Standard deviation 3	Main value 4	Standard deviation 5	Main value 6	Standard deviation 7	Main value 8	Standard deviation 9	Main value 10	Standard deviation 11
Upstream end r=12m	0.0289 0.0082	0.1337 0.1547	0.0935 0.1107	0.1417 0.1417	0.3220 0.4520	0.3026 0.2690	0.9425 1.1930	0.9069 0.8568		
Downstream end	0.0822	0.1467	0.1332	0.2626	0.2405	0.3602	0.5017	0.4674		
Percentage quantity deficits	- 0.3310	0.1592	- 0.3432	0.1292	- 0.4726	0.1212	0.1542	-0.1542	- 0.1935	0.0754

Table 3: Example 2: Percentage Differences between NC140 Solution and Solutions by NC120, NC110 and NC15 and Percentage Quantity Deficits

	METHOD							
	NCE20		NCE10		NCE5		CG140	
Location 1	Main value 2	Standard deviation 3	Main value 4	Standard deviation 5	Main value 6	Standard deviation 7	Main value 8	Standard deviation 9
Upstream end r = 12m	0.07911 0.1402	0.1317 0.1498	0.3711 0.5352	0.4843 0.5111	0.9935 1.2365	1.0983 1.0931		
Downstream end	0.0318	0.1513	0.1588	0.2436	0.4150	0.3759		
Percentage quantity deficits	-0.2022	0.1230	-0.2939	0.1374	-0.4962	0.2203	-0.1935	0.0754

Conclusions

In the preceding pages, a characteristic method that uses specified distances and a grid made of continuous forward characteristics and broken backward characteristics has been presented. The method yields depth and velocity hydrographs at predetermined distances from which depths and velocities on a predetermined rectangular grid can be easily obtained. Computations using this method agree very closely with computations using the iterative formulation of the characteristic grid method. The distance steps used in this method need not be equal and may be chosen in such a manner as to deal most effectively with the particular channel geometry and the spatial distribution of lateral inflows. Studies so far show the differences in accuracies between the direct and iterative formulation of method to be minimal. The direct formulation of the method seems to be more promising than its iterative formulation. It is easier to program and takes less computing time. The name "characteristic method of specified distances" is suggested for this method.

Appendix 1 – Newton-Raphson Iteration [5]

Solutions to sets of nonlinear equations are called for in the iterative formulations. The maximum number of unknowns involved is four. The following is an outline f the method of solution. Let E, F, G and H be four functions of the variables x, t, v and y and let solutions be required to the following set of equations.

$$E(x,t,v,y)=0;\quad F(x,t,v,y)=0;\quad G(x,t,v,y)=0;\quad H(x,t,v,y,)=0 \qquad \text{(A1)}$$

Let k denote the iteration number. Taylor expansions on the assumption that x_{k+1}, t_{k+1}, v_{k+1}, and v_{k+1}, and y_{k+1} are exact solutions lead to the following equations with the convention that the function E, F,G and H and their partial derivatives E_x, E_t, E_v, E_y, F_x, F_t, etc are evaluated at x_k, t_k, v_k and y_k

$$\begin{aligned}
&(x_{k+1}-x_k)E_x+(t_{k+1}-t_k)E_t+(v_{k+1}-v_k)E_v+(y_{k+1}-y_k)E_y=-E\\
&(x_{k+1}-x_k)F_x+(t_{k+1}-t_k)F_t+(v_{k+1}-v_k)F_v+(y_{k+1}-y_k)F_y=-F \qquad \text{(A2)}\\
&(x_{k+1}-x_k)G_x+(t_{k+1}-t_k)G_t+(v_{k+1}-v_k)G_v+(y_{k+1}-y_k)G_y=-G\\
&(x_{k+1}-x_k)H_x+(t_{k+1}-t_k)H_t+(v_{k+1}-v_k)H_v+(y_{k+1}-y_k)H_y=-H
\end{aligned}$$

The preceding are four linear equations in unknowns $(x_{k+1}-x_k)$, $(t_{k+1} - t_k)$, $(v_{k+1} - v_k)$, and $(y_{k+1}-y_k)$ and can be solved using Cramer's rule. The iterations are continued until the values of $(_{k+1} - x_k)$, $(t_{k+1} - t_k)$, $(v_{k+1} - v_k)$, and $(y_{k-1} - y_k)$ are less than an accuracy criterion. In the examples presented in this paper, an accuracy criterion of 0.00000000001 has been used.

Notation

The following symbols are used in this paper:

A = area of water section at distance x from origin at time t;

B = breadth of channel at water surface at distance x from origin at time t;

c = $\sqrt{gA/B}$ = celerity;

$G_f = S_o - S - \left[q(v-c)/(gA)\right] - \left[vc/(gA)\left(\partial A/\partial x\right)\right]$;

$G_b = S_o - S - \left[q(v+c)/(gA)\right] + \left[vc/(gA)\left(\partial A/\partial x\right)\right]$;

g = acceleration due to gravity;

N = Manning's constant;

Q = discharge at distance x from origin at time t;

q = lateral inflow per unit length at distance x from origin at time t;

R = hydraulic mean depth at distance x from origin at time t;

S_o= bed slope at distance x from origin;

S = friction slope obtained from the Manning formular = $v^2N^2/R^{4/3}K^2$ in which K= 1 when SI units are used and 1.49 when FPS units are used;

t = time;

v = velocity at distance x from origin at time t;

x = horizontal distance from origin (upstream boundary); and

y = depth at distance x from origin at time t.

Subscripts

f = positive or forward characteristic; and

b = negative or backward characteristic

Special Symbols

CGI = iterative formulation of characteristic grid method;

NCI = iterative formulation of new characteristic method;

NCE = direct formulation of new characteristic method and ;

A number attached to the symbols, CHI, NCI, or NCE denotes the number of distance steps in which the calculations have been done.

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