*Journal of the Nigerian Association of Mathematical Physics*
***Volume* 19 (November, 2011), pp 285 – 290**
***© J. of NAMP***

# Analysis of Turbulent Boundary Layers and Friction in Smooth Tubes at High Prandtl and Schmidt Numbers

***Chuks Okoli***
**Department of Civil Engineering,**
**University of Benin, Benin City. Nigeria**

## *Abstract*

***This paper presents an analytical solution for velocity distribution over the entire flow depth for open channel or pipe radius. The concept upon which this solution is based, was first introduced by [5] in 1877. He expressed viscous shear terms of the Reynold equation as the product of normal velocity gradient and his mixing coefficient. It suggests that the parabolic distribution of turbulent diffusivity should be replaced by one for which the rate of change of turbulent diffusivity with distance away from the wall is zero at the wall. Since the solutions presented herein include both Laminar and turbulent shear, they may be more generally applicable than previous approximate solutions. Velocity defect law and velocity distribution law satisfy the boundary conditions for $\frac{du}{dy}$ and U at both the flow boundary and at the top of the boundary layer. This represents a distinct improvement over the logarithmic equation $\frac{\bar{\varepsilon}_m}{\upsilon} = \infty$ which can satisfy only one of these four boundary conditions, and over the power function which violates the boundary conditions for $\frac{du}{dy}$. The concept and solutions presented herein will be useful in modeling and interpreting velocity distribution in open channel and pipe flow.***

## 1.0 Introduction

Previous solutions for velocity distributions of uniform flow in channels and pipes have been obtained for either the fully turbulent or the fully laminar regions of the flow. Experimental data define the transition between thee regions, but because of the different flow scales used to characterize the two flow regions, a universal distribution function has not been developed The purpose of this paper is to present a new analytical solution for the velocity distribution over the entire flow depth or pipe radius.

## Analysis

The concept upon which this solution is based, was first introduced by [5] in 1877. He expressed viscous shear terms of the Reynolds equations as the product of the normal velocity gradient and his mixing coefficient, $\rho\varepsilon_m$ in which $\rho$ is the fluid density; and $\varepsilon_m$ is the eddy viscosity or turbulent diffusivity. Consideration of the total shear between fluid elements as the sum of turbulent and viscous parts leads to the expression of the total shear stress, $\tau$ as

$$\frac{\tau}{\rho} = \left(v + \varepsilon_m\right)\frac{dU}{dy} \tag{1}$$

in which v = the kinematic fluid viscosity. For uniform flow with hydrostatic pressure the governing equations require a linear distribution of shear stress over the flow depth or pipe radius.

$$\tau = \tau_o\left(1 - \frac{y}{y_r}\right) \tag{2}$$

in which $\tau_o$ = the shear stress, on the fixed boundary, and $y_r$ = the flow depth or pipe radius.
Combining equations (1) and (2) then gives the well-known force balance equation for uniform flow:

Corresponding authors: E-mail: okolics2002@yahoo.com, *Tel*.: +2348034277819.

$$U_*^2\left(1-\frac{y}{y_r}\right)=\left(\varepsilon_m+v\right)\frac{dU}{d\upsilon} \tag{3}$$

in which U* = the shear velocity defined by

$$U_* = \sqrt{\frac{\tau_o}{\rho}} \tag{4}$$

A parabolic distributions of turbulent diffusivity has been suggested by data from several investigations [6]. This distribution is expressed in terms of the mean diffusivity

$\bar{\varepsilon}_m$ by

$$\frac{\varepsilon,}{\bar{\varepsilon}_m} = 6\left(\frac{y}{y_r}\right)\left(1-\frac{y}{y_r}\right) \tag{5}$$

Equation (5) is simply a parabola giving average value of $\frac{\varepsilon_m}{\bar{\varepsilon}_m}$ over the flow depth or pipe radius. This equation s identical to the classical equation if $\bar{\varepsilon}_m$ is defined by

$$\bar{\varepsilon}_m = \frac{kU_* y_r}{6} \tag{6}$$

Solutions of equation (3) have been obtained for laminar and turbulent cases separately, but a previous solution for combined laminar and turbulent shear is not known to the writer. If $\varepsilon_m$ is negligibly small, equation (3) yields the parabolic velocity distribution of purely laminar flow:

$$\frac{vU}{U_*^2 y_r} = n - \frac{1}{2}n^2 \tag{7}$$

in which n = y/$y_r$. If µ is neglected, the solution of equation (3) for $\varepsilon_m$, given by equation (5), is the classical logarithmic velocity distribution similar to that proposed by Prandtl in 1933 [4]:

$$\frac{\bar{\varepsilon}_m\left(U-U_{\max}\right)}{U_*^2 y_r} = \frac{1}{6} In\,(n) \tag{8}$$

The complete solution is given in the following for flows with both laminar and turbulent characteristics. Equation (3) is expressed as

$$\frac{\left(\bar{\varepsilon}+v\right)dU}{U_*^2 y_r} = \frac{\left(1+\frac{v}{\bar{\varepsilon}_m}\right)(1-n)dn}{\frac{\varepsilon_m}{\bar{\varepsilon}_m}+\frac{v}{\bar{\varepsilon}_m}} \tag{9}$$

Substituting equation (5) into equation (9) gives

$$\frac{\left(\bar{\varepsilon}+v\right)dU}{U_*^2 y_r} = \frac{\left(1+\frac{v}{\bar{\varepsilon}_m}\right)(1-n)dn}{\frac{v}{\bar{\varepsilon}_m}+6n-6n^2} \tag{10}$$

The right side of equation (10) can be expanded as

$$\frac{\left(\bar{\varepsilon}+v\right)dU}{U_*^2 y_r} = \frac{\left(1+\frac{v}{\bar{\in}_m}\right)}{12}\left[\frac{\left(6-12n\right)dn}{\frac{v}{\bar{\varepsilon}_m}+6n-6n^2}+\frac{6\,\mathrm{dn}}{\frac{v}{\bar{\varepsilon}_m}+6n-6n^2}\right] \tag{11}$$

The fraction of the last term of equation (11) can be expressed as

$$F = \frac{6}{\frac{v}{\bar{\varepsilon}_2} + 6n - 6n^2} = \frac{4}{D^2 - (2n-1)^2} \qquad (12)$$

in which

$$D = \left(1 + \frac{2}{3}\frac{v}{\bar{\varepsilon}_m}\right)^{1/2} \qquad (13)$$

The fraction denoted by F can then be expressed as the sum of two partial fractions based on its factors:

$$F = \frac{2}{D}\left(\frac{1}{D + 2n - 1} + \frac{1}{D - 2n + 1}\right) \qquad (14)$$

Substituting equation (14) into equation (11) gives

$$\frac{(\bar{\varepsilon}_m + v)dU}{U_*^2 y_r} = \frac{1 + \frac{v}{\bar{\varepsilon}_m}}{12}\left[\frac{(4 - 8n)dn}{D^2 - 1 + 4n - 4n^2} + \frac{2\,\mathrm{dn}}{\mathrm{D(D + 2n - 1)}} - \frac{-2\,\mathrm{dn}}{\mathrm{D(D - 2n + 1)}}\right] \qquad (15)$$

Each of the terms on the right side of equation (15) are of the form dz/z so that integration gives

$$\frac{(\bar{\varepsilon}_m + v)U}{U_*^2 y_r} = \frac{1 + \frac{v}{\bar{\varepsilon}_m}}{12}\left[\mathrm{In}(\mathrm{D}^2 - (2n-1)^2) + \frac{1}{D}\mathrm{In(D + 2n - 1)} - \frac{1}{\mathrm{D}}\mathrm{In(D - 2n + 1)}\right]_o^n \qquad (16)$$

Completing the evaluation of equation (16) at the indicated limits of integration gives

$$\frac{(\bar{\varepsilon}_m + v)U}{U_*^2 y_r} = \frac{1 + \frac{v}{\bar{\varepsilon}_m}}{12}\left[\mathrm{In}\left(\frac{\mathrm{D}^2 - (2n-1)^2}{D^2 - 1}\right) + \frac{1}{D}\mathrm{In}\left(\frac{\mathrm{D + 2n - 1}}{\mathrm{D - 2n + 1}}\right)\left(\frac{\mathrm{D + 1}}{\mathrm{D - 1}}\right)\right] \qquad (17)$$

$$\frac{(\bar{\varepsilon}_m + v)U_{\max}}{U_*^2 y_r} = \frac{1\left(+\frac{v}{\bar{\varepsilon}_m}\right)}{6D} In\left(\frac{D+1}{D-1}\right) \qquad (18)$$

Subtracting equation (17) from equation (18) gives the velocity defect form of the velocity distribution:

$$\frac{(\bar{\varepsilon}_m + v)(U_{\max} - U)}{U_*^2 y_r} = \frac{1\left(+\frac{v}{\bar{\varepsilon}_m}\right)}{12}\left[\frac{1}{\mathrm{D}}\mathrm{In}\left[\left(\frac{\mathrm{D + 1}}{D^2 - 1}\right)\left(\frac{\mathrm{D - 2n + 1}}{\mathrm{+ -2n - 1}}\right)\right] - \mathrm{In}\left(\frac{\left(D^2 - 2n - 1\right)^2}{D^2 - 1}\right)\right] \qquad (19)$$

Diving equation (17) by equation (18) gives the ratio form of the velocity distribution:

$$\frac{U}{U_{\max}} = \frac{D}{2In\left(\frac{D+1}{D-1}\right)}\left\{In\left(\frac{D^2 - (2n-1)^2}{D^2 - 1}\right) + \frac{1}{D} In\left[\left(\frac{D + 2n - 1}{D - 2n + 1}\right)\left(\frac{D+1}{D-1}\right)\right]\right. \qquad (20)$$

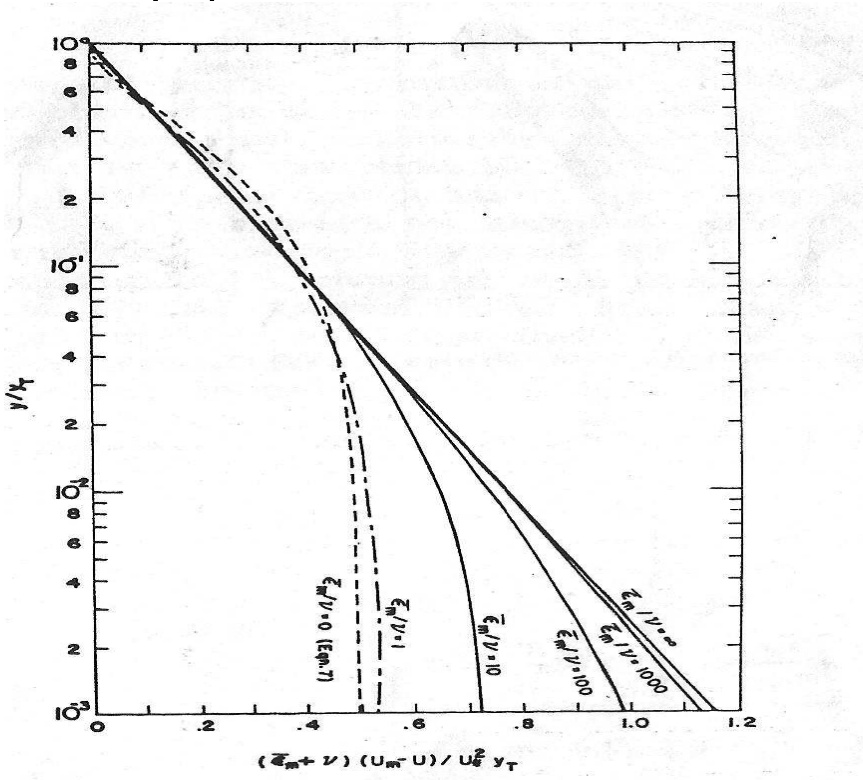


Fig. 1: Defect forms of velocity distributions

The predicted velocity distributions for selected values of $\frac{\nu}{\bar{\varepsilon}_m}$ are depicted in Figs. 1 and 2 for the defect and ratio forms of the velocity distributions, respectively. Equation (19) is seen to approach the laminar and turbulent solutions as limiting distributions in Fig. 1. As the flow boundary is approached, different curves for different ratios of $\frac{\bar{\varepsilon}_m}{\nu}$ are defined.

The curves defined by equation (20) become increasingly steep with increasing values of $\frac{\bar{\varepsilon}_m}{\nu}$ This illustrates the reason a power law function with constant exponent of the form (3).

$$\frac{U}{U_{max}} = \left(\frac{y}{y_r}\right)^m \qquad (21)$$

cannot serve as a general model. For an m value of $\frac{1}{8}$ and a value of $\frac{\bar{\varepsilon}_m}{\nu}$ of 1,000 equations 20 and 21 agree satisfactorily over the upper 90% of the flow depth. A change in the value $\frac{\bar{\varepsilon}_m}{\nu}$ would require a change in m to approximate the velocity distribution with equation (21).

For velocity distributions near a flow boundary, depiction of $U^* = U/U_*$ as a function $y^+ = U_* \frac{y}{\nu}$ is customary [1, 2,5]. However, the equations derived herein do not give unique relationships in this format. Equation (17) can be expressed as

$$U^{+} = \frac{\frac{Rv}{\bar{\varepsilon}_m}}{12}\left[ In\left( D^{2} - \frac{\left(\frac{2y^{+}}{R} - 1\right)}{D^{2} - 1}\right) + \frac{1}{D} In\left( \frac{D + \frac{2y^{+}}{R} - 1}{D - \frac{2^{+}}{R} + 1}\,\frac{D+1}{D-1}\right)\right] \qquad (22)$$

in which $R = U_* y_r/v$ = the shear velocity Reynolds number. Likewise, for purely laminar flow equation (7) can be expressed as:

$$U^{+} = y^{+}\left(1 - \frac{1}{2}\frac{y^{+}}{R}\right) \qquad (23)$$

Equations (22) and (23) suggest that $U^+$ depends on not only $y^+$ but also R and $\frac{v}{\bar{\varepsilon}_m}$.

Actual data for the inner region of the flow indicate that the flow remains more nearly laminar for $y^+ < 23$ than equation (22) suggests and experiences the full turbulent effects for larger $y^+$. These comparisons are depicted in Figure 3 in relation to the data used by Diessler [2].

For cases where a more accurate description of the velocity distribution in the inner region is necessary, a semi-empirical form of equation (23) can be applied for $y^+ \leq 23$.

$$U^{+} = y^{+}\left(1 - \frac{\frac{1}{2}y^{+}}{26}\right) \qquad (24)$$

This equation is the viscous equation for R = 26 but one that seems to account for combined turbulent and viscous effects for larger R. This expression is closely approximated by equation (22) for $R\frac{v}{\bar{\varepsilon}_m} = 100$ for $y^+$ less than about 15. Although equation (22) appears to depend on R and $\frac{v}{\bar{\varepsilon}_m}$ separately, plots for combination of these parameters giving the same product are indistinguishable. For $y^+ \geq 23$, equation (16) may still be applied with lower limits of 23/R for $y^+$ and 12.8 for $U^+$ replacing the zero limits. The resulting expression is

$$U^{+} = 12.8 + \frac{\frac{Rv}{\bar{\varepsilon}_m}}{12}\left[ In\left( \frac{D^{2} - \left(\frac{2y^{+}}{R} - 1\right)^{2}}{D^{2} - \left(\frac{46}{R} - 1\right)^{2}}\right) + \frac{1}{D} In\left( \frac{D + \frac{2y^{+}}{R} - 1}{D - \frac{2y^{+}}{R} + 1}\,\frac{D - \frac{46}{R} + 1}{D + \frac{46}{R} - 1}\right)\right] \qquad (25)$$

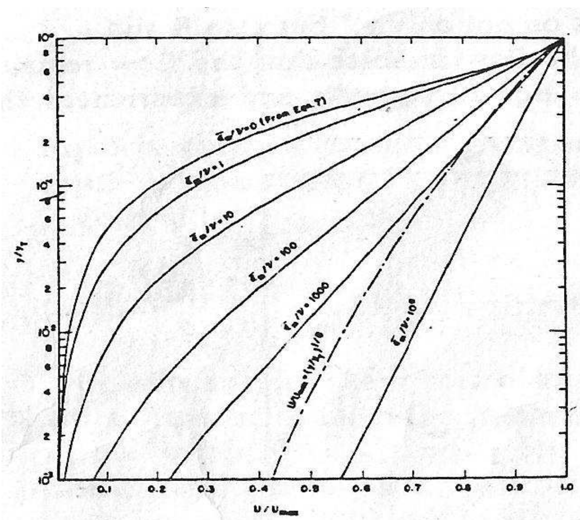

Fig.2: Ratio Form of velocity Distributions

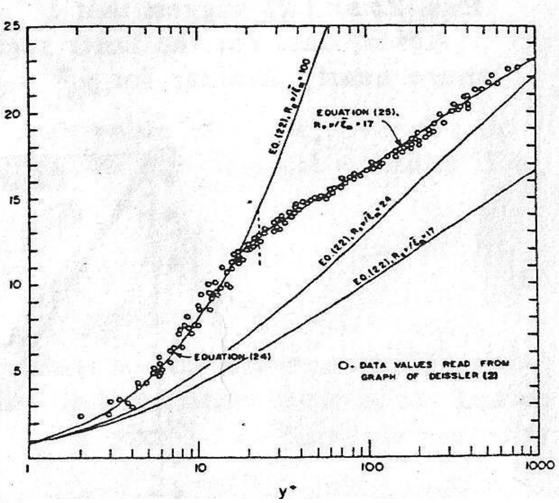


Fig.3: Velocity Distribution in inner Region

Equation (25) for $R\frac{v}{\bar{\varepsilon}_m}$=17 is shown superimposed on the data in Fig. 3. The distinction between $R\frac{v}{\bar{\varepsilon}_m}$ values of 100 and 17 for the inner and outer flow regions may be interpreted as a rather change in the turbulent intensity experienced by the two regions. It suggests that the parabolic distribution of turbulent diffusivity should be replaced by one for which the rate of change of turbulent diffusivity with distance away from the wall is zero at the wall. A gamma distribution has this property but its functional form does not give an analytical solution.

Since the solutions presented herein include both laminar and turbulent shear; they may be more generally applicable than previous approximate solutions. Equations (19) and (20) satisfy the boundary conditions for $\frac{dU}{dy}$ and U at both the flow boundary and at the top of the boundary layer. This represents a distinct improvement over the logarithmic equation, $\frac{\bar{\varepsilon}}{v} \rightarrow \infty$ in Fig. 1., which can satisfy only one of these four boundary conditions, and over the power function, which violates the boundary conditions for $\frac{dU}{dy}$.

The concepts and solutions presented herein should be helpful not only for their academic value but also as aids for modelling and interpreting velocity distributions in open channel and pipe flows. Their applicability should be limited only by the requirements of fully developed uniform flow and the assumption of a parabolic distribution of turbulent diffusivity.

**REFERENCES**

[1] Cebecci, T. and Smith, A.M.O., *Analysis of Turbulent Boundary Layers,* Academic Press, Inc., New York, N.Y., 1974, p. 117.
[2] Deissler, R.G., “Analysis of Turbulent Heat Transfer, Mass Transfer and Friction in Smooth Tubes at High Prandtl and Schmidt Numbers” *Technical Report* **1210, National Advisory Committee for Aeronautics, 1955.**
[3] Karman, T. von, “On Laminar and Turbulent Friction” Translation by National Advisory Committee for Aeronauticcs, T.M. 1092, (Originally published in German in 1921).
[4] Prandtl, L., “Recent Results of Turbulence Research, “Translation by National Advisory Committee for Aeronautics, T.M. 720, (Originally published in German in 1933).
[5] Schliching, H. *Boundary Layer Theory,* 6th ed., McGraw-Hill Book Co., Inc., New York, N.Y, 1968.
[6] Vanoni, V.A., “Transportation of Suspended Sediment by Water”, Transactions, ASCE, Vol. III, Paper No. 2257, 1946, pp. 67-133.