Journal of the Nigerian Association of Mathematical Physics
Volume **19 (November, 2011), pp 175 – 182**

On The Dynamic Analysis of Non-Uniform Beams With Non-Linear Winkler Foundation

[1]I.O.Abiala and [2]J.A. Gbadeyan

[1]Department of Mathematics,
University of Agriculture, Abeokuta, Nigeria.
[2]Department of Mathematics,
University of Ilorin, Ilorin, Nigeria

Abstract

The problem being investigated in this paper is that of the dynamic response of a non-uniform beam subjected to uniformly distributed moving load resting on a horizontal plane. When the beam is loaded in the direction parallel to the horizontal plane, frictional forces resulting from the displacement of the beam under load will act against the direction of the applied load. Therefore, the frictional resistance caused by two surfaces sliding against each other obeys the logarithmic rule which when considered together with the behaviour of the beam produces a non-linear response to the applied load. In the same way, the elastic properties of the beam, the flexural rigidity, and the mass density per unit length, and the elastic modulus parameter are expressed as functions of the spatial variable x. However, the main objectives of this study is to investigate the effect of (i) non-linear constant parameter (ii) velocity of the moving load (iii) load's length, and (iv) the span length of the beam on the dynamic response of beams on non-linear Winkler foundation.

1.0 Introduction

A beam on an elastic foundation is a problem frequently encountered by Structural engineers. The sources of non-linearity in a structural system could be geometrical, material, or both, depending on the elastic nature of the structure. In geometrical non-linearity, where, the structure is still elastic, the effects of large deflections cause the geometry of the structure to change, so that the linear elastic theory breaks down. Typical problems that lies in this category are the elastic instability of structures, such as in the Euler buckling of struts and also the large deflection analysis of beams and plates.

Chau and Seng [1] studied the static response of beams on non-linear elastic foundation where the deformed shape of the structure was represented by a Fourier series, and thereafter, the governing equation is reduced to a set of second-order non-linear simultaneous equations using Galerkin's method. The effect of a non-linear elastic foundation on the mode shapes in stability and vibration problems of uniform beams and columns was investigated by Kanaka and Venkateswara[2]. Coskun and Engin[3] analyzed the non-linear vibrations of an elastic beam resting on a non-linear tensionless Winkler foundation subjected to a concentrated load at the centre. Kargarmovind obtained response of infinite beams supported by nonlinear visco-elastic foundations subjected to harmonic moving loads using a perturbation method[4]. Kang and Tan studied the nonlinear behaviour of a beam under a distributed axial load with time dependent terms by Galerkin discretization and spectral balance method [5]. Santee and Gonalves [6] investigated stability of a beam on nonlinear elastic foundation and obtained the critical boundary of system instability. Zhang and Meng [7] carried out analysis of nonlinear dynamical system of micro-cantilever under combined parametric and forcing excitations. Zhang et al used Galerkin method and numerical integral to research on nonlinear dynamics of a Timoshenko beam with damage on visco-elastic foundation[8]. Borhan and Ahmadian[9] studied the dynamic modeling of geometrically non-linear electrostatically actuated microbeams using a corotational finite element formulation and analysis. Hsiao e tal [10] also investigated a consistent finite element formulation for nonlinear dynamic analysis of planar beam. Li et al [11] studied chaos of a beam on a nonlinear elastic foundation under moving loads where a vibration equation was obtained using Galerkin's method and subsequently, the effects of system parameters on chaotic region were analyzed.

However, these researchers only considered beams with prismatic materials under harmonic and concentrated loads, neglecting investigation on dynamics of a beam on a nonlinear elastic foundation under moving loads, most especially, distributed moving loads. The dynamic response of a non-uniform beam on nonlinear elastic foundation under distributed moving load is investigated in this research work. The nonlinear governing differential equation was transformed into the

Corresponding authors: I.O.Abiala, E-mail: abialaisrael@yahoo.com., *Tel*.: +2348056060214.

finite element equation using Galerkin's method, while the resultant model was solved employing the Newmark's integration method [12].

2. BEAMS ON NON – LINEAR ELASTIC FOUNDATION SUBJECTED TO MOVING LOADS

By considering a beam resting on a horizontal plane, which is loaded in the direction parallel to the horizontal (fig.1). Frictional effects of the surface will produce resistance to deformation of the beam. Since the frictional effect is non-linear, the response of the beam with respect to load will be non-linear. This problem can be treated in the same manner as that of the beam resting on a non-linear elastic foundation provided a suitable function can be found to describe the frictional resistance in which the displacement y is produced by a force q acting parallel to the surface(fig.2). This problem had been treated by Chau and Seng [1]. If the problem in [1] is modified to include the inertial term, then, the governing equation of the non-uniform beams on non-linear elastic foundation subjected to moving loads is

$$\frac{\partial^2}{\partial x^2}[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}]+\rho A(x)\frac{\partial^2 y(x,t)}{\partial t^2}+k(x)y(\alpha+y)^{-1}=q(x,t) \tag{1}$$

Where EI is modulus of elasticity, and $\rho A(x)$, the mass density per area of the beam element are functions of the spatial coordinate x, y is the deflection term, q is the applied force while k and α are constants dependent on the surface of the object.

The possible boundary conditions for simply supported beam are:

$$y(0)=y(l)=0$$
$$\left.\frac{\partial^2 y}{\partial x^2}\right|_{x=0}=0 \tag{2}$$

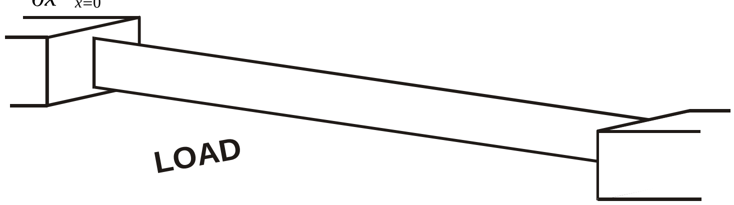

Fig.1 : Beam loaded against frictional resistance.

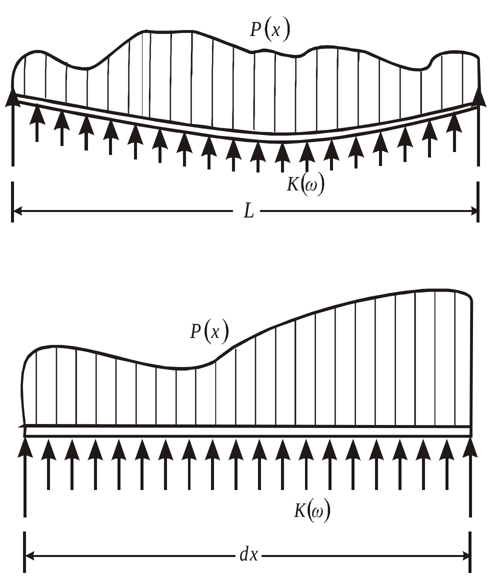

Fig.2 : Force components acting on beam.

From the third term in equation (1), we have, using Binomial expansion;

$$(\alpha+y)^{-1}=\alpha^{-1}[\ 1+\frac{y}{\alpha}\ ]^{-1}=\frac{1}{\alpha}-\frac{y}{\alpha^2}+\frac{y^2}{\alpha^3}-\frac{y^3}{\alpha^4}+\cdots\cong\frac{1}{\alpha}-\frac{y}{\alpha^2} \tag{3}$$

By using equation (3) in (1), and noting that for moving load problem:

$$q(x,t) = {}^{1}\!/_{\varepsilon}[-pg - p(\frac{\partial^2 y}{\partial t^2} + 2v\frac{\partial^2 y}{\partial x \partial t} + v^2\frac{\partial^2 y}{\partial x^2}][H(x-\xi+{}^{\varepsilon}\!/_{2}) - H(x-\xi-{}^{\varepsilon}\!/_{2})]$$

we obtained

$$\frac{\partial^2}{\partial x^2}[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}] + \rho A(x)\frac{\partial^2 y(x,t)}{\partial t^2} + \frac{1}{\alpha}k(x)y - \frac{1}{\alpha^2}k(x)y^2$$

$$= {}^{1}\!/_{\varepsilon}[-pg - p(\frac{\partial^2 y}{\partial t^2} + 2v\frac{\partial^2 y}{\partial x \partial t} + v^2\frac{\partial^2 y}{\partial x^2}][H(x-\xi+{}^{\varepsilon}\!/_{2}) - H(x-\xi-{}^{\varepsilon}\!/_{2})] \qquad (4)$$

Since, $EI(x)$,the flexural rigidity , $A(x)$,the beam's area and $k(x)$,the foundation modulus varies from element to element ,we have:

$$EI(x) = \sum_{r=1}^{nspan} EI_r(x - \sum_{i=1}^{r-1} L_i)[H(x - \sum_{i=1}^{r-1} L_i) - H(x - \sum_{i=1}^{r} L_i)] \qquad (5)$$

$$A(x) = \sum_{r=1}^{nspan} A_r(x - \sum_{i=1}^{r-1} L_i)[H(x - \sum_{i=1}^{r-1} L_i) - H(x - \sum_{i=1}^{r} L_i)] \qquad (6)$$

$$k(x) = k(4x - 3x^2 + x^3) \qquad (7)$$

In order to solve equation (4) using finite element method, we employed Galerkin's weighted Residual procedure to obtain the weak formulation of the problem [13].

3. THE WEAK FORMULATION OF THE BEAM EQUATION ON NON-LINEAR ELASTIC FOUNDATION:

The weak formulation procedure of non-linear problem is similar to that of linear problems [13], therefore (4) becomes:

$$\int_0^l [\ \frac{\partial^2}{\partial x^2}[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}] + \rho A(x)\frac{\partial^2 y(x,t)}{\partial t^2} + \frac{1}{\alpha}k(x)y - \frac{1}{\alpha^2}k(x)y^2$$

$$- {}^{1}\!/_{\varepsilon}[-pg - p(\frac{\partial^2 y}{\partial x^2} + 2v\frac{\partial^2 y}{\partial x \partial t} + v^2\frac{\partial^2 y}{\partial x^2}][H(x-\xi+{}^{\varepsilon}\!/_{2}) - H(x-\xi-{}^{\varepsilon}\!/_{2})]\ ]Rdx = 0 \qquad (8)$$

Where R is the Galerkin's weight or test function.

Rearranging equation (8), integrating twice the first term on the left-hand side with respect to x and using the method in [14], we obtain:

$$\int_0^l EI(x)\frac{\partial^2 y}{\partial x^2}\frac{\partial^2 R}{\partial x^2}dx + Q + \int_0^l \rho A(x)\frac{\partial^2 y(x,t)}{\partial t^2}Rdx + \frac{1}{\alpha}\int_0^l k(x)yRdx - \frac{1}{\alpha^2}\int_0^l k(x)y^2 Rdx =$$

$$- {}^{pg}\!/_{\varepsilon}\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2} Rdx - {}^{p}\!/_{\varepsilon}\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2}\frac{\partial^2 y}{\partial t^2}Rdx - {}^{2pv}\!/_{\varepsilon}\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2}\frac{\partial^2 y}{\partial x \partial t}Rdx - {}^{pv^2}\!/_{\varepsilon}\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2}\frac{\partial^2 y}{\partial x^2}Rdx \qquad (9)$$

where

$$Q = \varphi R - \phi\frac{\partial R}{\partial x}\Big|_0^l$$

$$\varphi = EI(\frac{\partial^3 y}{\partial x^3}), -the-shear-force$$

$$\phi = EI(\frac{\partial^2 y}{\partial x^2}), -the-bending-moment$$

4. DISCRETIZATION OF THE PROBLEM:

The finite element model of the problem is obtained from equation (9) by using standard mathematical discretizations [15] of the beam element into a number of finite elements employed in the earlier problems , which yields:

$$\sum_{i=1}^{n} \left[\int_\Omega EI(x)\frac{\partial^2 y}{\partial x^2}\frac{\partial^2 R}{\partial x^2}dx + Q + \int_\Omega \rho A(x)\frac{\partial^2 y(x,t)}{\partial t^2}Rdx + \frac{1}{\alpha}\int_\Omega k(x)yRdx - \frac{1}{\alpha^2}\int_\Omega k(x)y^2Rdx \right.$$
$$\left. + \frac{pg}{\varepsilon}\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2}Rdx + \frac{p}{\varepsilon}\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2}\frac{\partial^2 y}{\partial t^2}Rdx + \frac{2pv}{\varepsilon}\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2}\frac{\partial^2 y}{\partial x\partial t}Rdx + \frac{pv^2}{\varepsilon}\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2}\frac{\partial^2 y}{\partial x^2}Rdx \right] = 0 \quad (10)$$

Where, $\Omega = l_e$,the domain of the beam element.

Finally, equation (10), in matrix form, becomes:

$$[K]\{y\} + [C]\{\dot{y}\} + [M]\{\ddot{y}\} = \{F\} \quad (11)$$

Where,

$$[K] = \sum_{i=1}^{n} \left[\int_\Omega EI(x)\frac{\partial^2 y}{\partial x^2}\frac{\partial^2 R}{\partial x^2}dx + \frac{1}{\alpha}\int_\Omega k(x)yRdx + \frac{pv^2}{\varepsilon}\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2}\frac{\partial^2 y}{\partial x^2}Rdx \right] \quad (12)$$

$$[M] = \sum_{i=1}^{n} \left\{ \int_\Omega \rho A(x)\frac{\partial^2 y}{\partial t^2}Rdx + \frac{p}{\varepsilon}\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2}\frac{\partial^2 y}{\partial t^2}Rdx \right\} \quad (13)$$

$$[C] = \sum_{i=1}^{n} \left\{ \frac{2pv}{\varepsilon}\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2}\frac{\partial^2 y}{\partial x\partial t}Rdx \right\} \quad (14)$$

$$\{F\} = \sum_{i=1}^{n} \left\{ -\frac{pg}{\varepsilon}\int_{\xi-\varepsilon/2}^{\xi+\varepsilon/2}Rdx + \frac{1}{\alpha^2}\int_\Omega k(x)y^2Rdx + Q^e \right\} \quad (15)$$

5.0 Derivation of the Element Equations of The Problem:

Once again, we use the Hermitian polynomial [16] to interpolate the equations (12) to (15) in order to obtain the element equations, such that the complete stiffness matrix for the problem becomes

$$[K_{ij}^e] = [K_{(1)ij}^e] + [K_{(2)ij}^e] + [K_{(3)ij}^e] = \int_{\sum_{e=1}^{r-1}L_e}^{\sum_{e=1}^{r}L_e} \alpha_{ij}\{\sum_{r=1}^{nspan} EI_r(x - \sum_{e=1}^{r-1} L_e)\}dx + [K_{(2)ij}^e] + [K_{(3)ij}^e] \quad (16)$$

while the element mass matrix $[M_{ij}^e]$ and centripetal acceleration matrix $[C_{ij}^e]$ respectively are

$$[M_{ij}^e] = [M_{(1)ij}^e] + [M_{(2)ij}^e] = \int_{\sum_{e=1}^{r-1}L_e}^{\sum_{e=1}^{r}L_e} \beta_{ij}\{\sum_{r=1}^{nspan} \rho A_r(x - \sum_{e=1}^{r-1} L_e)\}dx + [M_{(2)ij}^e] \quad (17)$$

and

$$[C_{ij}^e] = [C_{(1)ij}^e] + [C_{(2)ij}^e] \quad (18)$$

Finally, from the equations (15), we obtained the element force vector:

$$[F] = \begin{Bmatrix} f_1^e \\ f_2^e \\ f_3^e \\ f_4^e \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \\ Q_4^e \end{Bmatrix} \quad (19)$$

The specification of Q_1^e, Q_2^e, Q_3^e and Q_4^e in equation (19) depends on the associated boundary conditions for a particular problem.

6.0 Assembly, Derivation, and Solutions of the Element:

EQUATIONS.

In order to obtain the complete element system of equations of the problem, equations (16),(17),(18) and (19) are assembled, depending on the number of elements under consideration. Finally, the assembled equations are then used in

(11),while the resultant system of equations is solved using the Newmark's method[12] after imposing the associated boundary conditions to obtain the dynamic responses of the non- uniform beams resting on non-linear elastic foundation subjected to moving loads.

7.0 NUMERICAL EXAMPLES: The non-uniform simply supported beam resting on nonlinear Winkler foundation is used. The total length of the beam L=10m,the mass density per beam length $\rho = 7.04gm^3$,the beam's element area $A = 20m^2$, and the load's length $\varepsilon = 0.5m$ and the foundation elastic modulus is 500,while the nonlinear constant parameter α is 0.5 .

There are six non-uniform elements in the problem with the length of each element given as $L_1 = 1m$, $L_2 = 1.4m$, $L_3 = 1.5m$, $L_4 = 1.6m$, $L_5 = 2m$, $L_6 = 2.5m$, and the flexural rigidities $EI_1 = 2.7728 \times 10^5\, Nm$, $EI_2 = 3.9947 \times 10^5\, Nm$, $EI_3 = 8.2858 \times 10^5\, Nm$, $EI_4 = 2.6179 \times 10^6\, Nm$, $EI_5 = 6.3936 \times 10^6\, Nm$, $EI_6 = 9.3936 \times 10^6\, Nm$, while $A_1 = 2m^2$, $A_2 = 2.8m^2$, $A_3 = 3m^2$, $A_4 = 3.2m^2$, $A_5 = 4m^2$, $A_6 = 5m^2$ .the elastic modulus parameter for each element are $k_1 = 50$, $k_2 = 70$, $k_3 = 75$, $k_4 = 80$, $k_5 = 100$, and $k_6 = 125$. The main objective of this research work is to study the effect of the nonlinearity of the Winkler foundation on the dynamic response of non-uniform beam elements to distributed moving loads. The value of the nonlinear constant parameter is varied to show its effect on the responses. However, the following observations were made from the analysis:

(a) Effects of nonlinear constant parameter: Three different values of the nonlinear constant parameter $\alpha = 0.5, 0.8, 1.1$ were used in order to study its effect on the dynamic response of non-uniform simply supported beam with nonlinear Winkler foundation under distributed moving load. It is observed, that the response amplitude decreases with increasing in nonlinear constant parameter (figure 3).

(b) Effects of velocity: In order to study the effect of the velocity on the dynamic response of non-uniform simply supported beam resting on nonlinear Winkler foundation under moving load, different values of the velocity were used with $V = 3m/s$, $3.5m/s$, $4m/s$ with k=500, $\alpha = 0.5$. As the velocity increases ,the amplitude also increases,(figure 4), but after attained the critical value of the velocity , it is observed that as V increases, the amplitude decreases,(figure 5). However, these changes are more drastic in nature than when the one in the linear case.

(c) Effects of the load's length: To investigate the effect of the length of the load on the dynamic response of non-uniform beam resting on nonlinear Winkler foundation while other properties remain unchanged, but with $\varepsilon = 0.5$, $\varepsilon = 0.7$, $\varepsilon = 0.9$ respectively were studied. It is observed that as the load's length increases, the amplitude decreases,(figure6). This is in contradiction with a situation in linear problem.

(d) *Effects of changing in boundary conditions*: For the cantilever beam, the behavioural pattern of the responses is in other way round. It is observed that unlike ,in simply supported type, the response amplitude decreases as the velocity increases(figure 7), and reverses after exceeding the critical value of the velocity(figure8). The critical value of the velocity here is about 6m/s, which is higher than that of the simply supported beam. However, just like in the simply supported beam, the response amplitude decreases as the load's length increases(figure 9). In addition, the response amplitude decreases with increases in the nonlinear constant parameter α (figure10), which is similar to the one in simply supported case. This is as a result of breaking down in linearity properties of the beam's foundation.

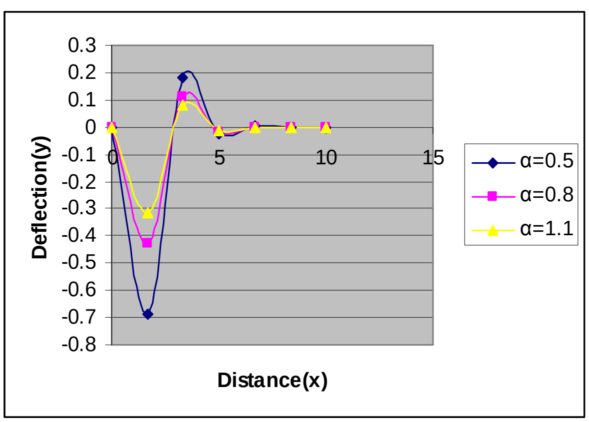

Fig 3: Effect of increasing in non-linear constant co-efficient

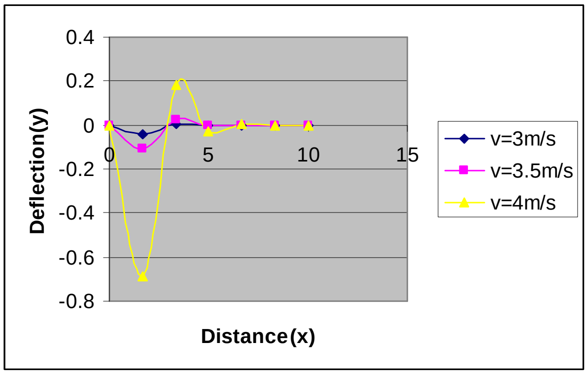

Fig 4: Effect of increasing in velocity on the response of beams

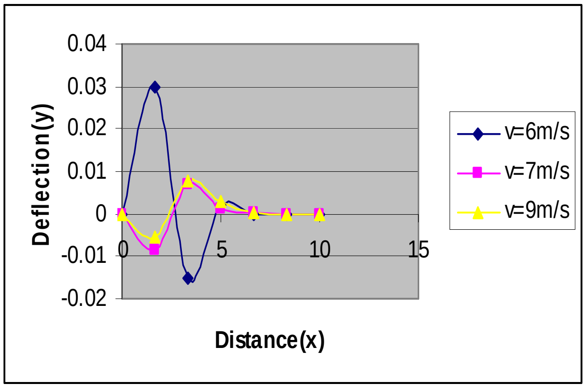

Fig 5: Effect of exceeding critical value of the velocity

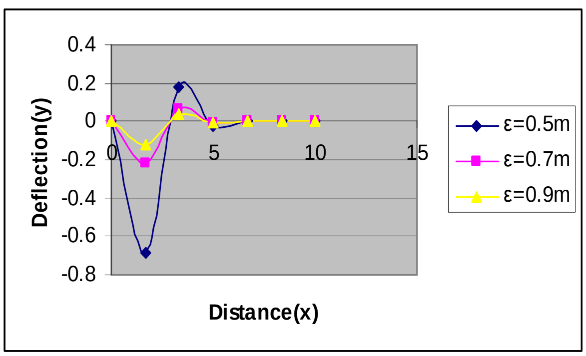

Fig 6: Effect of increases in load's length

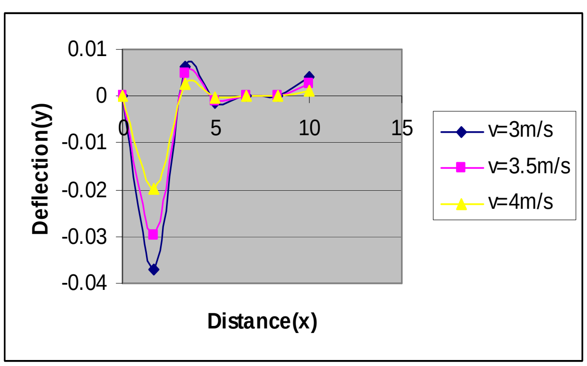

Fig 7: Effect of increasing in the velocity on the dynamic response in cantilever beam

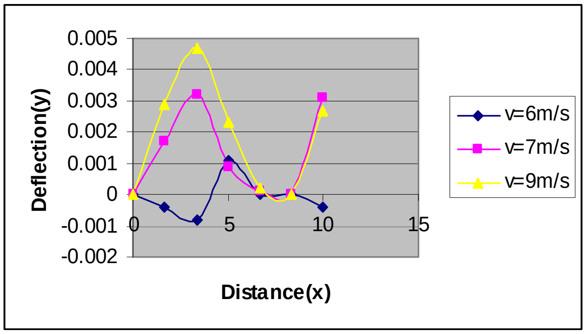

Fig 8: Effect of exceeding critical value of the velocity in cantilever beam

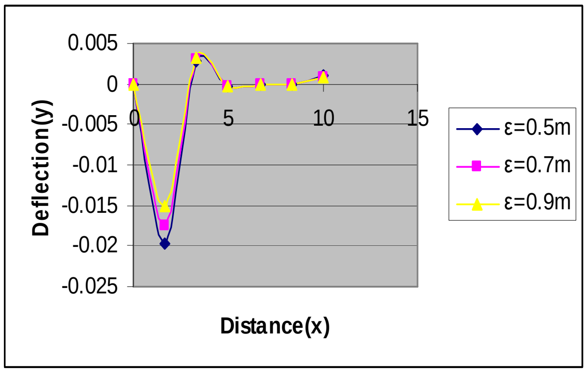

Fig 9: Effect of increases in load's length in cantilever beam

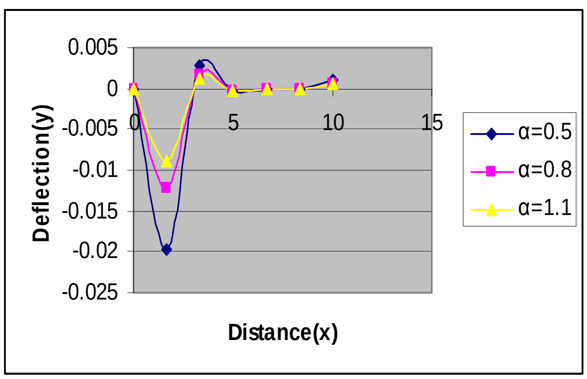

Fig 10: Effect of non-linear constant co-efficient on the response of cantilever beam

8.0 Conclusion

The dynamic analysis of non-uniform beams with non-linear Winkler foundation using finite element method is studied in this paper. The non-linear term on the foundation of the modeled governing equation of the problem was transformed analytically using binomial expansion series. In order to obtain the weak formulation of the problem, We employed the Galerkin's Weighted Residual Method (GWRM) which was used by the Authors in [1] and [13]. The resulting equation equations were interpolated using Hermitian interpolation polynomial to derive the element equations for the stiffness, mass, centripetal matrices and load vectors respectively. The assembled element equations were solved using the Newmark's method with the aid of a computer program written in Visual Basic codes. Apart from the confirmation of the claims in [1], [2], [10] and [11] among others, the effects of increasing in velocity, exceeding the critical value of velocity and the load's length on the dynamic response of simply supported and cantilever beams were presented.

References

[1] F.W.Chau and O.L.Seng: "Beams on nonlinear elastic foundation", Journal of applied solid Mechanics, vol. 2,587-600, 1988.

[2] R.K.Kanaka and R.G.Venkateswara: "Effect of nonlinear elastic foundation on the mode shapes in stability and vibration problems of uniform beams/columns", Journal of sound and vibration, vol.160 (2), 369-371, 1993.

[3] I.Coskun and H.Engin:" Nonlinear vibrations of a beam on an elastic foundation" Journal of sound and vibration, vol. 223(3), 335-354, 1999.

[4] M.H.Kargarnovind, D.Younesian and D.J.Thompson: "Response of infinite beams supported by nonlinear visco-elastic foundations subjected to harmonic moving loads", Computers and Structures, vol.83, 1865-1870, 2005.

[5] B.Kang and C.A.Tan: "Nonlinear behaviour of a beam under a distributed axial load" Commun Nonlinear Sci., vol.11, 203, 2006.

[6] D.M.Santee and P.B.Gonalves: "Stability of a beam on nonlinear elastic foundation" Shock Vib., vol.13, 273, 2006.

[7] W.Zhang and G.Meng: "Nonlinear dynamical system of micro-cantilever under combined parametric and forcing excitations in MEMS Journal Sensor and Actuators A, Physical vol; 119, 2003.

[8] Y.Zhang, D.F.Sheng and C.J.Cheng: Nonlinear dynamics of a Timoshenko beam with damage visco-elastic foundation, Chinese Quarterly of mechanics vol.25, 230, 2004.

[9] H.Borhan and M.T.Ahmadian: "Dynamic modeling of geometrically nonlinear electrostatically actuated microbeams" Journal of Physics, vol.34, 606-613, 2006.

[10] K.M.Hsiao, R.T.Yang and A.C.Lee: "A consistent finite element formulation for nonlinear dynamic analysis of planar beam," International Journal of numerical Methods in Eng., vol.37, 75-89, 1994.

[11] S.Li,S.Yang, B.Xu,and H.Xing: "Chaos of a beam on a nonlinear elastic foundation under moving loads ", Journal of Physics(conference series), vol.96,012116,2008.

[12] N.M. Newmark: A Method of computation for structural Dynamics. J.Eng.Mech.Div.ASCE:67-94, 1959.

[13] J.N.Reddy: An introduction to the finite element method, 2nd Ed., McGraw-Hill, NewYork, 1993.

[14] M.S.Dada: Transverse vibration of Euler-Bernoulli beams on elastic foundation under mobile distributed masses. Journal of the Nigerian Association of Mathematical physics, vol.7, 225-233, 2003.

[15] Y.W.Kwon and H. Bang: The finite element method using MATLAB, CRC Press, Boca Raton, New York, London, Tokyo ;1996.

[16] Y.K.Cheung and M.F.Yeo: A Practical introduction to finite element analysis, PITMAN: 1978.

[17] M.A.Crisfield: "Nonlinear Finite element analysis of solids and structures, Advanced Topics, vol.2, Wiley, Chichester.