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A Fifth Order Hybrid Linear Multistep method For the Direct Solution Of $y''' = f(x, y)$

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Abstract

A linear multistep hybrid method (LMHM)with continuous coefficients isconsidered and directly applied to solve third order initial and boundary value problems (IBVPs). The continuous method is used to obtain Multiple Finite Difference Methods (MFDMs) (each of order 5) which are combined as simultaneous numerical integrators to provide a direct solution to IVPs over sub-intervals which do not overlap. The convergence of the MFDMs is discussed by convenientlyrepresenting theMFDMs as a block method and verifying that the block method is zero-stable and consistent. The superiority of the MFDMs over the methods in Olabode and Yusuph [12] is established numerically.

Keywords: Multiple finite difference methods, third order, boundary value problem, block methods, multistep methods.

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1.0 Introduction

The mathematical formulation of physical phenomena in science and engineering often leads to initial value problems of the form:

$$y''' = f(x, y), \; y(a) = y_0 \; , \; y'(a) = \eta_0 \; , \; y''(a) = \eta_1 \qquad (1)$$

However, only a limited number of analyticalmethods are available for solving (1) directly without reducing to a first order system of initial value problems. Some authors have proposed solution to higher order initial value problems of ordinary differential equations using different approaches[1-5]. In particular [2] developed a class of hybrid collocation method for third order ordinary differential equations. Awoyemi[1] derived a p-stable linear multistep method for general third order initial value problems of ordinary differential equations which is to be used in form of predictor-corrector forms and like most linear multistep methods, they require starting values from Runge-Kutta methods or any other one-step methods. The predictors are also developed in the same way as correctors. Moreover, the block methods in [3] are discrete and are proposed for non-stiff special second order ordinary differential equations in form of a predictor- corrector integration process. Also like other linear multistep methodsthey are usually applied to the initial value problems as a single formula but they are not self-starting; and they advance the numerical integration of the ordinary differential equations in one-step at a time, which leads to overlapping of the piecewise polynomials solution Model. There is the need to develop a method which is self-starting, eliminating the use of predictors with better accuracy and efficiency. This study, therefore propose a block hybrid multistep method for the direct solution of third order initial value problems of ordinary differential equations.

Recently,several researches [6-10]proposed LMMs for the direct solution of the general second and third order IVPs, which were showed to be zero stable and were implemented without the need for either predictors or starting values from other methods. Jator [11] used the LMMs developed for IVPs and additional methods obtained from the same continuous k-step LMM to solve third order BVPs with Dirichlet and Neumann boundary conditions and also [12] developed a linear multistep method for the direct solution of initial value problems of ordinary differential equations for special third order initial value problem. We extended their methods into hybrid form by adding one off-step point at collocation. The 3-step block hybridmethod is P-stable, consistent and more accurate than the existing one. Experimental results confirmed the superiority of the new scheme over the existing method.

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The paper is organized as follows. In Section 2, we derive a continuous approximation Y (x) for the exact solution y(x). Section 3 is devoted to the specification of the methods and how the MFDMs are obtained. Analysis, stability region and implementation of MFDM are discussed in section 4. Numerical examples are given in Section 5 to show the efficiency of the MFDMs. Finally, the conclusion of the paper is discussed in Section 6.

2.0 Development of Methods.

In this section, our objective is to derive hybrid linear multi-step method(HLMM) of the form

$$\sum_{j=0}^{r-1}\alpha_j y_{n+j} = h^3\sum_{j=0}^{s-1}\beta_j f_{n+j} + h^3\beta_v f_{n+v} \tag{2}$$

Where α_j, β_j and β_v are unknown constants and v_j is not an integer. We note that α_k =1, $\beta_j \neq 0, \alpha_0$ and β_0 do not both vanish. In order to obtain (2), we proceed by seeking to approximate the exact solution y(x) of the form

$$Y(x) = \sum_{j=0}^{r+s-1} l_j x^j , \tag{3}$$

Where $x \in [a,b], l_j$ are unknown coefficients to be determined and $1 \le r \prec k \; s \succ 0$ are the number of interpolation and collocation points respectively. We then construct our continuous approximation by imposing the following conditions.

$$Y(x_{n+j}) = y_{n+j}, \quad j = 0,1.2,......, r-1 \tag{4}$$

$$Y'''(x_{n+\mu}) = f_{n+\mu} \tag{5}$$

Equation (4) and (5) lead to a system of (r+s) equations which is solved by Cramer's rule to obtain l_j. Our continuous approximation is constructed by substituting the values of l_j into equation (3). After some manipulation, the continuous method is expressed as

$$Y(x) = \sum_{j=0}^{r-1}\alpha_j(x) y_{n+j} + h^3\sum_{j=0}^{s-1}\beta_j(x) f_{n+j} + h^3\beta_v(x) f_{n+v} \tag{6}$$

where $\alpha_j(x), \beta_j(x)$ and $\beta_v(x)$ are continuous coefficients. We note that since equation (1) involves first and second derivatives, the first and second derivative formula

$$\begin{aligned} Y'(x) &= \frac{1}{h}\left(\sum_{j=0}^{r-1}\alpha_j'(x) y_{n+j} + h^3\sum_{j=0}^{s-1}\beta_j'(x) f_{n+j} + h^3\beta_v'(x) f_{n+v}\right) \\ Y''(x) &= \frac{1}{h^2}\left(\sum_{j=0}^{r-1}\alpha_j''(x) y_{n+j} + h^3\sum_{j=0}^{s-1}\beta_j''(x) f_{n+j} + h^3\beta_v''(x) f_{n+v}\right) \end{aligned} \tag{7}$$

equation (7) is easily obtained from (6) and is then used to provide the first and second derivatives for the methods by imposing the condition

$$Y'(x) = \delta(x), \; Y''(x) = \gamma(x) \tag{8}$$

$$Y'(a) = \delta_0, \; Y''(a) = \gamma_0 \tag{9}$$

3.0 Specification of the methods

Our methods are obtained from section two and expressed in the formof (2) given by

$$\bar{y}(x) = \sum_{j=0}^{r-1}\alpha_j(x) y_{n+j} + h^3\sum_{j=0}^{s-1}\beta_j(x) f_{n+j} + h^3\beta_v(x) f_{n+v} \tag{10}$$

with the following specification r=3, s=5, k=3, $\gamma_i(x) = x^i, \; i = 0,1,....,8$ we also express

$\alpha_j(x), \beta_j(x)$ and $\beta_v(x)$ as functions of $t = \dfrac{(x - x_n)}{h}$ as follows:

$$\alpha_0 = \left(1-\frac{3}{2}t+\frac{1}{2}t^2\right),\ \alpha_1 = \left(2t-t^2\right),\ \alpha_2 = \left(-\frac{1}{2}t+\frac{1}{2}t^2\right)$$

$$\beta_0(x) = \frac{h^3}{25200}\left(1448t-3920t^2+4200t^3-2345t^4+728t^5-119t^6+5t^7\right)$$

$$\beta_1(x) = \frac{h^3}{5040}\left(1702t-2331t^2+1050t^4-518t^5+105t^6-8t^7\right)$$

$$\beta_2(x) = \frac{h^3}{1680}\left(-288t+532t^2-525t^4+364t^5-91t^6+8t^7\right)$$

$$\beta_{\frac{5}{2}}(x) = \frac{h^3}{1575}\left(232t-420t^2+420t^4-308t^5+84t^6-8t^7\right)$$

$$\beta_3(x) = \frac{h^3}{5040}\left(-190t+343t^2-350t^4+266t^5-77t^6+8t^7\right)$$

The MFDMs are obtained by evaluating (10) at $x = \left\{x_{n+3}, x_{n+\frac{5}{2}}\right\}$ to obtain the following

$$y_{n+3} = y_n - 3y_{n+1} + 3y_{n+2} + \frac{h^3}{150}\left[f_n + 70f_{n+1} + 90f_{n+2} - 16f_{n+\frac{5}{2}} + 5f_{n+3}\right] \quad (11)$$

$$y_{n+\frac{5}{2}} = \frac{3}{8}y_n - \frac{5}{4}y_{n+1} + \frac{15}{8}y_{n+2} + \frac{h^3}{15360}\left[41f_n + 2645f_{n+1} + 2835f_{n+2} - 896f_{n+\frac{5}{2}} + 175f_{n+3}\right] \quad (12)$$

In particular, to start the initial value problem for n = 0, we obtain the following equations from (9):

$$h\delta_0 = -\frac{3}{2}y_0 + 2y_1 - \frac{1}{2}y_2 + \frac{h^3}{12600}\left[724f_0 + 4255f_1 - 2160f_2 + 1856f_{\frac{5}{2}} - 475f_3\right] \quad (13)$$

$$h^2\gamma_0 = y_0 - 2y_1 + y_2 + \frac{h^3}{360}\left[-112f_0 - 333f_1 + 228f_2 - 192f_{\frac{5}{2}} + 49f_3\right] \quad (14)$$

It is worth noting that the derivatives are provided by

$\delta(x_{n+\tau}) = \delta_{n+\tau}$ *and* $\gamma(x_{n+\tau}) = \gamma_{n+\tau},\ \tau = 1, 2, \frac{5}{2}$ *and* 3 as follows:

$$h\delta_{n+1} = -\frac{1}{2}y_n + \frac{1}{2}y_{n+2} - \frac{h^3}{2520}\left[19f_n + 388f_n - 9f_{n+2} + 32f_{n+\frac{5}{2}} - 10f_{n+3}\right]$$

$$h\delta_{n+2} = \frac{1}{2}y_n - 2y_{n+1} + \frac{3}{2}y_{n+2} + \frac{h^3}{2520}\left[52f_n + 2785f_n + 2040f_{n+2} - 832f_{n+\frac{5}{2}} + 155f_{n+3}\right]$$

$$h\delta_{n+\frac{5}{2}} = y_n - 3y_{n+1} + 2y_{n+2} + \frac{h^3}{2520}\left[2693f_n + 188135f_{n+1} + 241155f_{n+2} - 58208f_{n+\frac{5}{2}} + 12625f_{n+3}\right]$$

$$h\delta_{n+3} = \frac{3}{2}y_n - 4y_{n+1} + \frac{5}{2}y_{n+2} + \frac{h^3}{2520}\left[23f_n + 1796f_{n+1} + 2655f_{n+2} - 32f_{n+\frac{5}{2}} + 178f_{n+3}\right]$$

$$h^2\gamma_{n+1} = y_n - 2y_{n+1} + y_{n+2} + \frac{h^3}{1800}\left[39 f_n + 140 f_{n+1} - 375 f_{n+2} + 256 f_{n+\frac{5}{2}} - 60 f_{n+3}\right]$$

$$h^2\gamma_{n+2} = y_n - 2y_{n+1} + y_{n+2} + \frac{h^3}{1800}\left[8 f_n + 895 f_{n+1} + 1260 f_{n+2} - 448 f_{n+\frac{5}{2}} + 85 f_{n+3}\right]$$

$$h^2\gamma_{n+\frac{5}{2}} = y_n - 2y_{n+1} + y_{n+2} + \frac{h^3}{5760}\left[33 f_n + 2797 f_{n+1} + 5523 f_{n+2} + 128 f_{n+\frac{5}{2}} + 159 f_{n+3}\right]$$

$$h^2\gamma_{n+3} = y_n - 2y_{n+1} + y_{n+2} + \frac{h^3}{1800}\left[7 f_n + 900 f_{n+1} + 1545 f_{n+2} + 768 f_{n+\frac{5}{2}} + 380 f_{n+3}\right]$$

4. Analysis and Implementation of the Method

Following [13] and [4] we define the local truncation error associated with the conventional form of (2) to be the linear difference operator

$$L\left[y(x);h\right] = \sum_{j=0}^{k}\left\{\alpha_j y(x+jh) - h^3\beta_j y'''(x+jh)\right\} + h^3\beta_v f_{n+v} \tag{15}$$

Assuming that y(x) is sufficiently differentiable, we can expand the terms in (15) as a Taylor series about the point x to obtain the expression

$$L\left[y(x);h\right] = C_0 y(x) + C_1 hy' + ..., + C_q h^q y^q(x) + ..., \tag{16}$$

Where the constant coefficients $C_q, \quad q = 0,1,...$ are given as follows: $C_q, \quad q = 0,1,...$

$$C_0 = \sum_{j=0}^{k}\alpha_j,$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j,$$

$$..C_q = \left[\frac{1}{q!}\sum_{j=1}^{k} j^q\alpha_j - q(q-1)\sum_{j=1}^{k} j^{q-2}\beta_j\right].$$

According to [14], we say that the method (5) has order p if

$$C_0 = C_1 = ... = C_P = C_{P+1} = C_{P+2} = 0, \quad C_{P+3} \neq 0$$

Our calculations reveal that the methods (11) to (14) have order p = 5 and error constants given by the vector

$$C_8 = \left(-\frac{1}{1200}, -\frac{139}{491520}, \frac{251}{8}, \frac{183}{56}\right)^T$$

In order to analyze the methods for zero-stability, we normalize (11) to (14) and write them as a block method given by the matrix difference equation

$$A^0 Y_{\mu+1} = A^1 Y_\mu + h^2\left[B^0 F_{\mu+1} + B^1 F_\mu\right] \tag{17}$$

where

$Y_{\mu+1} = (y_{n+1}, ..., y_{n+3})^T$, $Y_\mu = (y_{n-3}..., y_n)^T$, $F_{\mu+1} = (f_{n+1}, ..., f_{n+3})^T$, $F_\mu = (f_{n-3}..., f_n)^T$ *and* $n = 0,3,...$ and matrices A^0, A^1, B^0 and B^1 are defined as follows:

A^0 is an identity matrix of dimension 4

$$A^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B^{1}=\begin{pmatrix} \frac{629}{5040} & -\frac{61}{420} & \frac{188}{1575} & -\frac{17}{560} \\ \frac{74}{63} & -\frac{97}{105} & \frac{1216}{1575} & -\frac{62}{315} \\ \frac{143125}{64512} & -\frac{29375}{21504} & \frac{625}{504} & -\frac{6875}{21504} \\ \frac{404}{112} & -\frac{243}{140} & \frac{324}{175} & -\frac{261}{560} \end{pmatrix}$$

$$B^{0}=\begin{pmatrix} 0 & 0 & 0 & \frac{103}{1050} \\ 0 & 0 & 0 & \frac{799}{1575} \\ 0 & 0 & 0 & \frac{17875}{21504} \\ 0 & 0 & 0 & \frac{261}{175} \end{pmatrix}$$

It is worth noting that zero-stability is concerned with the stability of thedifference system in the limit as htends to zero. Thus, as $h \to 0$, the method (17) tends to the difference system

$A^0Y_{\mu+1} - A^1Y_{\mu} = 0$ whose first characteristic polynomial $\rho(R)$ is given by

$$\rho(R) = \det\left(RA^0 - A^1\right) = R^3(R-1) \tag{18}$$

Following Fatunla [13], the block method (17) is zero-stable, since from (18),

$\rho(R) = 0$ *Satisfy* $|R_j| \le 1 \ j = 1...,k$ and for those roots with $|R_j|$=1, the multiplicity does not exceed 2. The block method (17) is consistent as it has order $P \succ 1$. According to [14], we can safely assert the convergence of the block method (17).

It is vital to note that the main method given by (10) can be used as anumerical integrator directly and singly in the conventional way on overlapping sub-intervals. However, our method is implemented more efficiently by combining methods (11) to (14), each of order five with relatively small error constants, as simultaneous integrators for IVPs without looking for any other methods toprovide the starting values. We proceed by explicitly obtaining initial conditions at $x_{n+3}, \ n = 0,3,...,N-5$ using the computed values $y(x_{n+3}) = y_{n+3}, \delta(x_{n+3}) = \delta_{n+3}$ *and* $\lambda(x_{n+3}) = \lambda_{n+3}$ over sub-intervals $[x_0, x_3],...[x_{n-3}, \ x_N]$ which do not overlap(see [10]). For instance, $n = 0, (y_1, y_2, y_3)^T$ are simultaneously obtainedover the sub-interval $[x_0, x_3]$ as y_0 is known from the IVP, for $n = 3, (y_4, y_5, y_6)^T$ are simultaneously obtained over the sub-interval $[x_3, x_6]$, as $y_3$is known form the previous block, and so on. Hence, the sub-intervals do notover-lap and the solutions obtained in this manner are more accurate that thoseobtained in the conventional way.

4.1 Stability Region of Block Method

To compute and plot absolute stability region of the block methods, the method of section three are reformulated as general linear methods expressed as

$$\begin{bmatrix} Y \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} A & U \\ B & V \end{bmatrix} \begin{bmatrix} hf(y) \\ y_{i-1} \end{bmatrix}$$

Substituting the values of A, B, U and V into the stability matrix $M(z) = V + zU(I - zA)^{-1}B$ and stability function

$\rho(\eta, z) = \det(\eta I - M(z))$ and using maple software yields the stability polynomial of the block method as;

$$M(z) = \frac{\begin{aligned}&(-450\eta^2z^4 + 1125\eta^3z^4 + 73979\eta z^3 - 466210\eta^2z^3 + 22811\eta^3z^3 \\ &+ 331812\eta z^2 - 13949630\eta^2z^2 + 824780\eta^3z^2 - 316470z + 9188160\eta z - 72530400 \\ &\eta^2z + 20011200\eta^3z - 7106400 - 114912000 - 139104000\eta^2 + 24192000\eta^3 \\ &+ 22982400\eta)\end{aligned}}{(1125z^4 + 22811z^3 + 824780z^2 + 2001200z + 24192000)}$$

Using thematlab package we were able to plot the region of absolute stability of the block method as shown in Figure4.1. Absolute Stability Region of Block Method

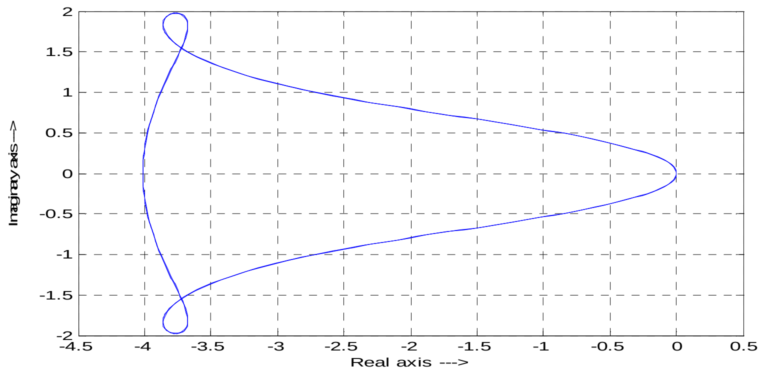

Fig 4.1: Absolute Stability Region

The stability region for the methods has large interval of absolute stability (-4, 0) since its region of absolute stability contains the left half- plane $\{Z \ \varepsilon \ C \ |\mathrm{Re}(Z) \prec 0\}$

5.0 NumericalExperiment

In this paper we use proposed block hybrid methods and compare their result with three step block methods proposed by [12] to solve special third order (IVPs) , in order to test for efficiency of the schemes derived.

Example 5.1

$y''' = -y, \quad with \quad initial \quad condition \quad y(0) = 1,$

$y'(0) = -1, \quad y''(0) = 1 \quad and \quad exact \quad solution$

$y(x) = e^{-x}$

Table 5.1: Comparison of errors for problem 5.1

x	**Exact solution** $y(x)=e^{-x}$	**Computed Method**	**Error**	**Solution by Olabode and Yusuph [12]**
0.1	0.904837418	0.904837418	0.00000E+00	6.35960E-10
0.2	0.818730753	0.818730753	3.00000E-10	2.37798E-09
0.3	0.74081822	0.740818221	4.00000E-10	4.18172E-09
0.4	0.670320046	0.670320046	0.00000E+00	5.63564E-09
0.5	0.60653066	0.60653066	2.00000E-10	7.01263E-09
0.6	0.548811636	0.548811637	5.00000E-10	7.59403E-09
0.7	0.496585304	0.496585304	4.00000E-10	6.39141E-09
0.8	0.449328964	0.449328964	1.00000E-10	2.41722E-09
0.9	0.40656966	0.406569659	7.00000E-10	4.75940E-09
1.0	0.367879441	0.367879441	1.00000E-10	1.52286E-08

Example 5.2

$y''' = 3\sin x, \quad with \ initial \quad condition \quad y(0)=1, \quad y'(0)=0 \quad y''(0)=-2 \quad and \quad exact \ solution$

$y(x) = 3\cos x + \frac{x^2}{2} - 2$

Table 5.2: Comparison of errors for problem 5.2

x	**Exact solution** $y(x)=3\cos x+\frac{x^2}{2}-2$	**3-step block hybrid Method y-computed**	**Error in 3-step block hybrid Method**	**Solution by Olabode and Yusuph [12]**
0.1	0.990012496	0.990012496	2.00000E-10	1.65922E-10
0.2	0.960199733	0.960199733	9.99999E-11	4.76275E-10
0.3	0.911009467	0.911009466	6.00000E-10	6.23182E-10
0.4	0.843182982	0.843182981	1.00000E-09	2.91345E-10
0.5	0.757747686	0.757747685	1.00000E-09	8.71118E-10
0.6	0.656006845	0.656006843	2.00000E-09	3.92904E-09
0.7	0.539526562	0.53952656	2.00000E-09	9.55347E-09
0.8	0.410120128	0.410120126	2.00000E-09	1.80415E-08
0.9	0.269829905	0.269829902	2.70000E-09	3.03120E-08
1.0	0.120906918	0.12090692	2.00000E-09	4.73044E-08

Example 5.3

$y''' = e^x$, *with initial condition* $y(0) = 3$, $y'(0) = 1$, $y''(0) = 5$ *and exact solution* $y(x) = 2 + 2x^2 + e^x$

Table 5.3: Comparison of errors for problem 5.3

x	Exact solution $y(x) = 2 + 2x^2 + e^x$	3-step block hybrid Method y-computed	Error in 3-step block hybrid Method	Solution by Olabode and Yusuph [12]
0.1	3.125170918	3.125170918	0.00000E+00	-7.5647E-11
0.2	3.301402758	3.301402758	0.00000E+00	1.83983E-09
0.3	3.529858808	3.529858807	1.00000E-09	4.42400E-09
0.4	3.811824698	3.811824697	1.00000E-09	1.03587E-08
0.5	4.148721271	4.14872127	1.00000E-09	1.12999E-08
0.6	4.5421188	4.542118799	1.00000E-09	1.46095E-08
0.7	4.993752707	4.993752706	9.99999E-10	2.05295E-08
0.8	5.505540928	5.505540927	1.00000E-09	1.95075E-08
0.9	6.079603111	6.079603109	2.00000E-09	1.08431E-08
1.0	6.718281828	6.718281822	6.00000E-09	1.54095E-08

Example 5.4

$y''' = -e^x$, *with initial condition* $y(0) = 1$, $y'(0) = -1$, $y''(0) = 3$ *and exact solution* $y(x) = 2 + 2x^2 - e^x$

Table 5.4: Comparison of errors for problem 5.4

x	Exact solution $y(x) = 2 + 2x^2 - e^x$	3-step block hybrid Method y-computed	Error in 3-step block hybrid Method	Solution by Olabode and Yusuph [12]
0.1	0.914829082	0.914829082	1.00000E-10	7.24352E-10
0.2	0.858597242	0.858597242	0.00000E+00	3.83983E-09
0.3	0.830141192	0.830141193	8.00000E-10	9.32400E-09
0.4	0.828175302	0.828175303	7.00000E-10	1.69587E-08
0.5	0.851278729	0.85127873	8.00000E-10	2.60999E-08
0.6	0.8978812	0.8978812	3.00000E-10	3.55095E-08
0.7	0.966247293	0.966247293	3.00000E-10	4.51295E-08
0.8	1.054459072	1.054459073	1.00000E-09	5.45075E-08
0.9	1.160396889	1.16039689	1.00000E-09	6.28431E-08
1.0	1.281718172	1.281718168	4.00000E-09	6.95410E-08

Example 5.5 (BVP)

$y''' = -y$, *with boundary condition* $y(0) = 0$, $y'(0) = 0$, $y(1) = 1$

Exact solution is given by

$$y(x) = 0.6779319384\mathrm{e}^{-x} + 1.174212561\mathrm{e}^{\frac{1}{2}x}\sin\left(\frac{1}{2}\sqrt{3}\,x\right) - 0.6779319384\mathrm{e}^{\frac{1}{2}x}\cos\left(\frac{1}{2}\sqrt{3}\,x\right)$$

Table 5.5: Numerical Results of Example 5.5

x	Exact Solution y(x)	3-step block hybrid Method y-computed	Error in 3-step block hybrid Method
0	0	0	0
0.1	0.0101694033	0.01016880955	$5.9375\,10^{-7}$
0.2	0.0406703794	0.04067049268	$1.1328\,10^{-7}$
0.3	0.0914802071	0.09147962996	$5.7714\,10^{-7}$
0.4	0.1625298840	0.1625301465	$2.625\,10^{-7}$
0.5	0.2536955573	0.2536950379	$5.194\,10^{-7}$
0.6	0.3647657501	0.3647661921	$4.420\,10^{-7}$

0.7	0.4954347975	0.4954343790	$4.185\ 10^{-7}$
0.8	0.6452688564	0.6452694991	$6.427\ 10^{-7}$
0.9	0.8137014809	0.8137012013	$2.796\ 10^{-7}$
1	1	1	0

Conclusion

We have derived a three-step continuous HLMM from which MFDMs are obtainedand applied to solve $y''' = f(x, y)$ without first adapting the ODE to an equivalent first order system or reducing it to an initial-value problem. The MFDMs are applied as simultaneous numerical integrators over sub-intervals which do not overlap and hence they are more accurate than SFDMs which are generally applied as single formulas over overlapping intervals. We have shown that the methods are convergent and have large intervals of absolute stability, which make them suitable candidates for computing solutions on wider intervals. In addition to providing additional methods and derivatives, the continuous HLMMcan be used to obtain global error estimates. Our future research will be focused on adapting the MFDMs to solve third order partial differential equations.

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