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A Linear Multistep Method with Continuous coefficients for Solving First Order Ordinary Differential Equation (ODE)

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Abstract

A linear multistep method (LMM) with continuous coefficients is considered and directly applied to solve first order initial value problems (IVPs). The continuous method is used to obtain Multiple Finite Difference Methods (MFDMs) (each of order 4) which are combined as simultaneous numerical integrators to provide a direct solution to IVPs over sub-intervals which do not overlap. The region of absolute stability is analyzed and the convergence of the MFDMs is discussed by conveniently representing the MFDMs as a block method and verifying that the block methodis zero-stable and consistent. The superiority of the methods over the two stepadamsmoulton method is established numerically

Keywords: Block Method; Linear Multistep Method; Multistep Collocation; Continuous Multistep (CM), Zero-stability.

1.0 Introduction

Lie and Norset[1], Onumanyi *et al*[4], Yahaya and Mohammed [6], Yahaya [5] and Mohammed [3] have all converted conventional linear multistep methods including hybrid ones into continuous forms through the idea of Multistep Collocation (MC). The Continuous Multistep (CM) method, associated with conventional linear multistep methods produces piece-wise polynomial solutions over k steps for the first order differential system.

This research work aims at deriving and plotting a five-step block method for numerical integration of ordinary differential equations. It allows the block formulation and therefore is self starting and for appropriate choice of k, overlap of solution model is eliminated.

2.0 Methodology

Let us first give a general description for the method of multistep collocation (MC) and its link to continuous multistep (CM) method. In equation (1), f is given and y is sought as

$$y = a_1\varphi_1 + a_2\varphi_2 + \dots\dots\dots + a_p\varphi_p \qquad (1)$$

where

$$a = (a_1, a_2 \dots\dots\dots a_p)^T, \qquad \phi = (\phi_1, \phi_2 \dots\dots\dots \phi_p)^T$$

$X_n \le X \le X_{n+k,}$ Where $n = 0, k, \dots, N - k$, and T denote 'Transpose of'
equation (2.1) can be re-written as

$$y = (a_1, a_2 \dots\dots\dots a_p)^T \, (\varphi_1, \varphi_2 \dots\dots\dots \varphi_p)^T \qquad (2)$$

The unknown coefficients $a_1, a_2 \dots\dots\dots a_p$ are determined using respectively the r(0<r<=k) interpolation conditions and the s>0 distinct collocation conditions, p=r+s as follows

$$\sum_{1=1}^{p} a_j\varphi_j(x_i) = y_i, \qquad (i = 1, \dots\dots\dots r)$$

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$$\sum_{1=1}^{p} \beta_j \phi_j'(x_i) = f_i, \qquad (i = 1, ...r) \tag{3}$$

This is a system of p linear equations from which we can compute values for the unknown coefficients provided (3) is assumed non-singular, for the distinct points x_i and c_i the non-singular system is guaranteed see onumanyi et al(1994). We can write (3) as a single set of linear equations of the form

$$\underline{Da} = \underline{F}$$

$$\underline{a} = \underline{\underline{D}}^{-1} \underline{F} \tag{4}$$

Where $\underline{F} = (y_1, y_2, \ldots, y_r, f_1, f_2, \ldots, f_s)^T$ (5)

substituting the vector a, given by (4) and F by (5) into (2) gives

$$y = (y_1, y_2, \ldots, y_r, f_1, f_2, \ldots, f_s) C^T (\varphi_1, \varphi_2 ....\varphi_p)^T \tag{6}$$

equation (6) is the continuous MC Interpol ant C^T known explicitly in the form

$$\underline{\underline{C}}^T \varphi = \begin{pmatrix} C_{11} & C_{21} \ldots\ldots & C_{p1} \\ C_{12} & C_{22} & C_{p2} \\ \vdots & \vdots & \vdots \\ C_{1r} & C_{2r} & C_{pr} \\ \vdots & \vdots & \vdots \\ C_{1p} & C_{2p} & C_{pp} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_r \\ \vdots \\ \varphi_p \end{pmatrix} \tag{7}$$

$$\underline{\underline{C}}^T \varphi = \begin{pmatrix} \sum_{j=1}^{p} C_{ji\varphi_j} \\ \vdots \\ \sum_{j=1}^{p} C_{jr\varphi_j} \\ \sum_{j=1}^{p} C_{jr+1\varphi_j} \\ \vdots \\ \sum_{j=1}^{p} C_{jp\varphi_j} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_2 \\ \beta_1 \\ \vdots \\ \beta_s \end{pmatrix} \tag{8}$$

$$F^T C^T \phi = (\alpha_1 y_1 + \alpha_2 y_2 + ... + \alpha_r y_r + \beta_1 f_1 + \beta_2 f_2 + ... + \beta_s f_s)$$

or

$$F^T C^T \phi = \sum_{j=1}^{r} \alpha_j y_j + h_j \left(\sum_{j=1}^{s} \beta_j / h_j \; f_j \right)$$

where from (8)

$$\alpha_j = \sum_{q=i}^{p} C_{qi} \phi_j, \qquad j = 1, ...r$$

$$\beta_j / h_j = \sum_{q=i}^{p} \left[\frac{C_{qi+r}}{h_i} \right] \phi_j \qquad j = 1, ...s$$

Therefore

$$y = \sum_{j=1}^{r} \alpha_j y_j + h_j \left(\sum_{j=1}^{s} \beta_j / h_j \right) f_j \tag{9}$$

3.0 Derivation of the Continuous Formula

We seek to derive a continuous form of the following 5-step block method
We propose an approximate solution to (1) in the form

$$y_p(x) = \sum_{j=0}^{m+t-1} a_j x^i, i = 0,(1)(m+t-1) \tag{10}$$

Where m=4, t=1, p=m+t-1. Also $\alpha_j,\ j = 0,1,.........(m+t-1)$ are the parameters to be determined, p is the degree of the polynomial interpolant of our choice. Specifically, we interpolate equation (10) at x_{n+2} and collocate at $x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}$ using the method described above.

The general form of the method is expressed as;

$$y(x) = \alpha_0 y_{n+2} + h[\beta_0 f_{n+2} + \beta_1 f_{n+3} + \beta_2 f_{n+4} + \beta_3 f_{n+5}] \tag{11}$$

The matrix D of the method is expressed as:

$$\begin{pmatrix} 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 \\ 0 & 1 & 2x_{n+5} & 3x_{n+5}^2 & 4x_{n+5}^3 \end{pmatrix}$$

Using maple software package gives the columns of D^{-1}which are the elements of the matrix C. The elements of C are then used to generate the values of continuous coefficient:

$$\alpha_0(x), \beta_0(x), \beta_1(x), \beta_2(x), \beta_3(x) \tag{12}$$

The values of the continuous coefficient (12) are substituted into (11) to give the continuous form of the five step block methods as;

$$\begin{aligned} y(x) = y_{n+2} &+ \left[\frac{-(x-x_n)^4 + 16(x-x_n)^3 h - 94(x-x_n)^2 h^2 + 240(x-x_n)h^3 - 216h^4}{24h^3}\right] f_{n+2} \\ &+ \left[\frac{3(x-x_n)^4 - 44(x-x_n)^3 h + 228(x-x_n)^2 h^2 - 480(x-x_n)h^3 + 352h^4}{24h^3}\right] f_{n+3} \\ &+ \left[\frac{-3(x-x_n)^4 + 40(x-x_n)^3 h - 188(x-x_n)^2 h^2 + 360(x-x_n)h^3 - 248h^4}{24h^3}\right] f_{n+4} \\ &+ \left[\frac{(x-x_n)^4 - 12(x-x_n)^3 h + 52(x-x_n)^2 h^2 - 96(x-x_n)h^3 + 64h^4}{24h^3}\right] f_{n+5} \end{aligned} \tag{13}$$

We can nowEvaluate (13) at the points $x = x_{n+5}, x = x_{n+4}, x = x_{n+3}\ x = x_{n+1} and\ x = x_n$ to obtain the following five discrete methods which are used as a block integrator.

$$y_{n+5} - y_{n+2} = \frac{h}{8}[3f_{n+2} + 9f_{n+3} + 9f_{n+4} + 3f_{n+5}] \tag{14}$$

$$y_{n+4} - y_{n+2} = \frac{h}{3}[f_{n+2} + 4 f_{n+3} + f_{n+4}]$$

$$y_{n+3} - y_{n+2} = \frac{h}{24}[9 f_{n+2} + 19 f_{n+3} - 5 f_{n+4} + f_{n+5}]$$

$$y_{n+2} - y_n = \frac{h}{3}[27 f_{n+2} - 44 f_{n+3} + 31 f_{n+4} - 8 f_{n+5}]$$

$$y_{n+1} - y_{n+2} = \frac{h}{24}[-55 f_{n+2} + 59 f_{n+3} - 37 f_{n+4} + 9 f_{n+5}] \qquad (15)$$

Equations (14) and (15) constitute the members of a zero-stable block integrator of order $(4,4,4,4,4)^T$ with $C_5 = \left[-\frac{3}{8}, -\frac{1}{90}, -\frac{19}{720}, -\frac{251`}{720}, -\frac{269}{90} \right]^T$ .The application of the block integrator with n=0 give the accurate values of y_1, y_2, y_3, y_4 along with y_5 as shown in Tables 1 , 2 and 3. To start the IVP integration on the sub-interval $[x_0, x_5]$, we combine (14) and (15), when n=0 i.e the 1-block 5 point methods as given in equation (16) produces its unknown simultaneously without recourse to any starting method (predictor) to generate y_1, y_2, y_3, y_4 before computing y_5 using equation (14).

4.0 Convergence Analysis of the Block Method

We can put the five integrator represented by equations (14) and (15) into the matrix-equation form and for easy analysis the result was normalized to obtain;

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} + h \begin{pmatrix} 0 & \frac{161}{24} & -\frac{293}{24} & \frac{211}{24} & -\frac{55}{24} \\ 0 & 9 & -\frac{44}{3} & \frac{31}{3} & -\frac{8}{3} \\ 0 & \frac{75}{8} & -\frac{111}{8} & \frac{81}{8} & -\frac{21}{8} \\ 0 & \frac{28}{3} & -\frac{40}{3} & \frac{32}{3} & -\frac{8}{3} \\ 0 & \frac{75}{8} & -\frac{325}{24} & \frac{275}{24} & -\frac{55}{24} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix} \qquad (16)$$

The first characteristic polynomial of the block hybrid method (16) is given by

$$\rho(R) = \det\left(RA^0 - A^1\right)$$

Substituting the value of A^0 and A^1 into the function above gives

$$\rho(R) = \det\left[R \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right]$$

$$=\det\begin{pmatrix} R & 0 & 0 & 0 & -1 \\ 0 & R & 0 & 0 & -1 \\ 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & R-1 \end{pmatrix}$$

$$=\left[R^4(R-1)\right] \tag{17}$$

Therefore, R=0, R=1. The block method is zero stable and consistent since the order of the method p=4>1, and by Henrici (1962), the block method is convergent.

Stability Region of Block Method

To compute and plot absolute stability region of the block methods, the method of section three are reformulated as general linear methods expressed as

$$\begin{bmatrix} Y \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} A & U \\ B & V \end{bmatrix} \begin{bmatrix} hf(y) \\ y_{i-1} \end{bmatrix}$$

Substituting the value of A, B, U and V into the stability matrix $M(z) = V + zU(I - zA)^{-1}B$ and stability function $\rho(\eta, z) = \det(\eta I - M(z))$ and using maple software yields the stability polynomial of the block method , using a mat lab program stability produces the absolute stability region of the block method as shown in fig 4.1

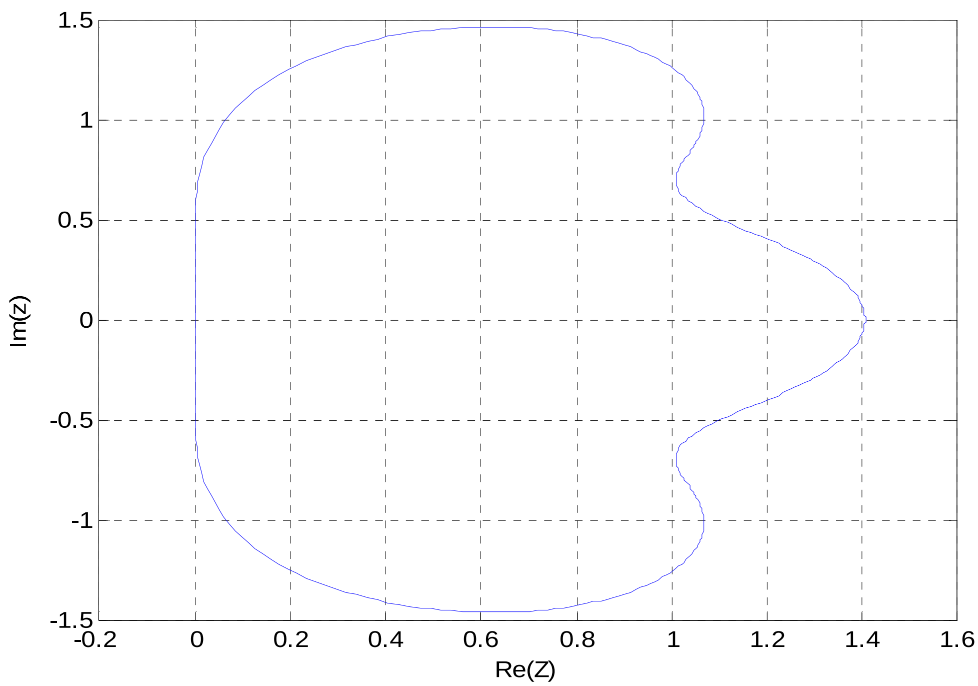

Fig 4.1

5.0 Numerical Examples

In this paper we use newly constructed block method and the standard Adams-moulton method for k = 5 to solve stiff and non-stiff initial value problems (IVP), in order to test for efficiency of the schemes derived.

Example 5.1
Consider the initial value problem

$$\begin{aligned} &y' = -y, && y(0) = 1 \\ &0 \le x \le 1 && h = 0.1 \\ &exact\ solution: && y(x) = e^{-x} \end{aligned} \tag{18}$$

Example 5.2
Consider the initial value problem

$$\begin{aligned} &y' = -9y, && y(0) = e^{1} \\ &0 \le x \le 1 && h = 0.1 \\ &exact\ solution: && y(x) = e^{1-9x} \end{aligned} \tag{19}$$

Example 5.3

$$\begin{aligned} &y' = x + y, && y(0) = 1 \\ &0 \le x \le 1 && h = 0.1 \\ &exact\ solution: && y(x) = 2e^{x} - x - 1 \end{aligned} \tag{20}$$

Firstly we transform the schemes by substitution, to get a recurrence relation. Substituting n = 0, 5 and 10 and solving simultaneously using maple software package we obtain the required results displayed in Tables 1 - 3.

Table1. Example 5.1 for k=5

X	**Exact Solution**	**Numerical Solution**		**Absolute Error**	
		K=5 Proposed Method	**K=5 Standard adams - moulton method**	**K=5 Proposed Method**	**K=5 Standard adams-moulton method**
0.1	0.9048374180	0.9048549405	0.900000000	1.75225E-05	4.837418000E-03
0.2	0.8187307531	0.8187488967	0.810000000	1.81436E-05	8.730753100E-03
0.3	0.7408182207	0.7408344615	0.729000000	1.62408E-05	1.181822070E-02
0.4	0.6703200460	0.6703348438	0.656100000	1.47978E-05	1.422004600E-02
0.5	0.6065306597	0.6065438712	0.561242622	1.32115E-05	4.528803810E-02
0.6	0.5488116361	0.5488342186	0.475575000	2.25825E-05	7.323663600E-02
0.7	0.4965853038	0.4966071254	0.399152307	2.18216E-05	9.743299680E-02
0.8	0.4493289641	0.4493486023	0.329449042	1.96382E-05	1.198799219E-01
0.9	0.4065696597	0.4065874913	0.268637805	1.78316E-05	1.379318550E-01
1.0	0.3678794412	0.3678954677	0.216909410	1.60265E-05	1.509700310E-01

Table 2: Example 5.2 for k=5

X	Exact Solution	Numerical Solution		Absolute Error	
		Proposed Method	Standard adams -moulton method	Proposed Method	Standard adams - moulton method
0.1	1.105170918E+00	1.252501337E+00	1.116169023	1.473304190E-01	1.10E-02
0.2	4.493289640E-01	5.267040462E-01	0.469773633	7.737508220E-02	2.04E-02
0.3	1.826835240E-01	2.125875480E-01	0.196943367	2.990402400E-02	1.43E-02
0.4	7.427357800E-02	8.737521120E-02	0.082916632	1.310163320E-02	8.64E-03
0.5	3.019738300E-02	3.381617705E-02	0.034863568	3.618794050E-03	4.67E-03
0.6	1.227734000E-02	1.558146272E-02	0.014664854	3.304122720E-03	1.89E-03
0.7	4.991594000E-03	6.552343872E-03	0.006167789	1.560749872E-03	1.18E-03
0.8	2.029431000E-03	2.644647840E-03	0.002594167	6.152168400E-04	5.74E-04
0.9	8.251050000E-04	1.086971770E-03	0.001091092	2.618667700E-04	2.66E-04
1.0	3.335463000E-04	4.206825865E-04	0.000458908	8.713628650E-05	1.23E-04

Table 3: Example 5.3 for k=5

X	Exact Solution	Numerical Solution		Absolute Error	
		K=5 Proposed Method	K=5 Standard adams method	K=5 Proposed Method	K=5 Standard adams method
0.1	1.110341836	1.110261878	1.1103	7.9958000E-05	4.1836E-05
0.2	1.242805516	1.242706481	1.2428	9.9035000E-05	5.516E-06
0.3	1.399717616	1.399608957	1.398705	1.0865900E-04	1.012615E-03
0.4	1.583649396	1.583528852	1.58137075	1.2054400E-04	2.278645E-03
0.5	1.797442542	1.797310105	1.795919758	1.3243700E-04	1.522784E-03
0.6	2.044237600	2.043959411	2.041239184	2.7818900E-04	2.998418E-03
0.7	2.327505414	2.327180378	2.322629073	3.2503600E-04	1.214683E-03
0.8	2.651081856	2.650723944	2.643961475	3.5791200E-04	7.120382E-03
0.9	3.019206222	3.018809913	3.009424243	3.9630900E-04	9.837978E-03
1.0	3.436564	3.436126961	3.423639805	4.3703900E-04	1.292385E02

Consider the absolute errors of the two methods above. A close observation of Table 1 and 3 shows that the discrete schemes of the newly constructed block method performs far better than the standard Adams-moulton method when applied to non-stiff equations. A close observation of Table 2 also shows that the discrete schemes of the newly constructed block method performs a little better than the standard Adams-moulton method when applied to stiff equations

Conclusions

A Collocation technique which yields a method with Continuous Coefficients has been presented for the approximate Solution of first Order ODEs with initial conditions. Three test examples have been solved to demonstrate the efficiency of the proposed method and the results compare favorably with the exact Solution, a desirable feature of good numerical methods. Interestingly, all the discrete schemes used in the Block formulation were from a single continuous formulation (CF). We also recover from our continuous approximate form k=5 the well known Milne`s scheme. The continuous formula allows variation and is immediately employed as block method for direct solution of the class of problem in (1.1).

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