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A New Method for the Evaluation of Higher Order Derivatives of Three Continuous / Differentiable Functions

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Abstract

This method is based on the well known Leibnitz's and product rule; it does not require the lower order derivatives to generate the higher order derivatives. The emerging method from the combination are easy to compute and less tedious. The need for higher derivatives is appreciated and significant in the computation of some numerical integrators that calculate the numerical solutions of differential equations by making use of the higher derivation of differential equations.

Keywords: Leibnitz's theorem, product rule, derivatives, continuous / differentiable functions.

1.0 Introduction

In implementation of some numerical integrators, the need for the evaluation of the higher derivative of the initial value problems involved arises [1, 2, 3]. For the product of the two differentiable functions, the product rule or Leibnitz's theorem can be used for the evaluation of higher derivative of the initial value problems [2, 4, 5]. But a new need arises when we have a product of three differentiable functions and a product of four or more differentiable functions. This leads to the derivation of this new method which is capable of delivering higher order derivative of three or more differentiable functions. In this paper, the derivation of such method for the higher order derivatives of the function of the form

y = $L_1(x)L_2(x)L_3(x)$ is given:

2.0 The New Method

The derivative of the function $y = L_1(x)L_2(x)L_3(x)$

can be obtained by splitting

$L_1(x)L_2(x)L_3(x)$ into $u = L_1(x)L_2(x)$ and $v = L_3(x)$ such that by using product rule,

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$. If Leibnitz's theorem is used to obtain $\frac{du}{dx}$ then one will have;

$$\frac{dy}{dx} = \left(L_3(x)\right)\left[\sum_{\iota=0}^{n=1} C_i^n \left(\frac{d^{n-\iota}L_1(x)}{dx^{n-\iota}}\right)\left(\frac{d^{\iota}L_2(x)}{dx^{\iota}}\right)\right] + \left(L_1(x)L_2(x)\right)\frac{d\left[L_3(x)\right]}{dx^1} \tag{1}$$

$$=\left(L_3(x)\right)\left(L_1(x)L_2(x)\right)^0\left[\sum_{\iota=0}^{n=1} C_\iota^n \left(\frac{d^{n-\iota}L_1(x)}{dx^{n-\iota}}\right)\left(\frac{d^{\iota}L_2(x)}{dx^{\iota}}\right)\right]\left(\frac{d^{\iota}L_3(x)}{dx}\right)^0 + \left(L_3(x)\right)^0\left(L_1(x)L_2(x)\right)^1\left[\sum_{\iota=0}^{n=1} C_\iota^n \left(\frac{d^{n-\iota}L_1(x)}{dx^{n-\iota}}\right)\left(\frac{d^{\iota}L_2(x)}{dx^{\iota}}\right)^0\right]\left(\frac{d^{\iota}L_3(x)}{dx^1}\right)^1 \tag{2}$$

$$\left(\left[\sum_{\iota=0}^{n=1} C_\iota^n \left[\sum_{\iota=0}^{n=0}\sum_{\iota=0}^{n=1} C_i^n \left(\frac{d^{n-\iota}L_2(x)}{dx^{n-\iota}}\right)\left(\frac{d^{n-\iota}L_3(x)}{dx^{\iota}}\right)\right]\left(\frac{d^{r}L_3(x)}{dx^{r}}\right)\right]\right) \left(L_3(L_1L_2)^{n-n}\ldots L_3^{n-n}(L_1L_2)^1\right)^T \tag{3}$$

The following steps show how to derive the second derivative:

$$\frac{d^2y}{dx^2} = \mathrm{L}_3\left[\sum_{i=0}^{n=2} C_i^n \left(\frac{d^{n-1}L_1(x)}{dx^{n-1}}\right)\left(\frac{d^{i}L_2(x)}{dx^{i}}\right)\right] + 2\left[\sum_{i=0}^{n=1} C_i^n \left(\frac{d^{n-1}L_1(x)}{dx^{n-1}}\right)\left(\frac{d^{i}L_1(x)}{dx^{i}}\right)\right]\left(\frac{d^{1}L_3(x)}{dx^{1}}\right) + [\mathrm{L}_1(\mathrm{x})\mathrm{L}_2(\mathrm{x})]\left(\frac{d^{2}L_3(x)}{dx^{2}}\right) \tag{4}$$

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$$\frac{d^2y}{dx^2} = L_3(x)[L_1(x)L_2(x)]^0 \left[\sum_{i=0}^{n=2} C_i^n \left(\frac{d^{n-1}L_1(x)}{dx^{n-1}}\right)\left(\frac{d^iL_2(x)}{dx^i}\right)\right]\left(\frac{d^0L_3(x)}{dx^0}\right) +$$

$$2[L_I(x)L_2(x)L_3(x)]^0\left(\left[\sum_{i-0}^{n=1} C_i^n \left(\frac{d^{n-1}L_1(x)}{dx^{n-1}}\right)\left(\frac{d^iL_2(x)}{dx^i}\right)\right]\frac{dL_3(x)}{dx}\right) +$$

$$[L_3(x)]^0[L_1(x)L_2(x)]^1\left[\sum_{i=0}^{n=0} C_i^n \left(\frac{d^{n-1}L_1(x)}{dx^{n-1}}\right)\left(\frac{d^iL_2(x)}{dx^i}\right)\right]\left(\frac{d^2L_3(x)}{dx^2}\right) \tag{5}$$

$$=\left[\sum_{r=0}^{n} C_i^n \left[\sum_{i=o}^{n=o}\sum_{i=0}^{n=1}\sum_{i=0}^{n-(n-2)} C_i^n\left(\frac{d^{n-1}L_1(x)}{dx^{n-1}}\right)\left(\frac{d^iL_2(x)}{dx^i}\right)\right]\left(\frac{d^rL(x)}{dx^r}\right)\right][L_3\,(L_1\,L_2)^{n-n}, L_3(L_1L_2)^{n-n}, L_3^{n-n}(L_IL_2)]^T \tag{6}$$

The following steps show how to derive the third derivative:

$$\frac{d^3y}{dx^3} = L_1\left[\sum_{i=0}^{n=3} C_i^n\left(\frac{d^{n-i}L_1(x)}{dx^{n-i}}\right)\left(\frac{d^iL_2(x)}{dx^i}\right)\right] + 3\left[\sum_{i=0}^{n=2} C_i^n\left(\frac{d^{n-1}L_1(x)}{dx^{n-1}}\right)\left(\frac{d^iL_2(x)}{dx^i}\right)\right]\left(\frac{dL(x)}{dx}\right) + L_1\,L_2\left(\frac{d^3L_3(x)}{dx^3}\right) \tag{7}$$

$$\frac{d^3y}{dx^3} = \begin{bmatrix} \left[\sum_{i=0}^{n=3} C_i^n\left(\frac{d^{n-1}L_1}{dx^{n-1}}\right)\left(\frac{d^iL_2}{dx^i}\right)\right]\left(\frac{d^0L_3}{dx^0}\right) + 3\left[\sum_{i=0}^{n=2} C_i^n\left(\frac{d^{n-1}L_1}{dx^{n-1}}\right)\left(\frac{d^iL_2}{dx^i}\right)\right]\left(\frac{dL_3}{dx}\right) \\ + 3\left[\sum_{i=0}^{n=1} C_i^n\left(\frac{d^{n-1}L_1}{dx^{n-1}}\right)\left(\frac{d^iL_2}{dx^i}\right)\right]\left(\frac{d^2L_3}{dx^3}\right) + \left[\sum_{i=0}^{n=0} C_i^n\left(\frac{d^{n-i}L_1}{dx^{n-i}}\right)\left(\frac{d^iL_2}{dx^i}\right)\right]\left(\frac{d^3L_3}{dx^3}\right) \end{bmatrix} A$$

$$\text{Where}\quad A = \begin{bmatrix} L_1^{\,n-2}(L_2L_3)^{n-n} \\ L_1^{\,n-n}(L_2L_3)^{n-n} \\ L_1^{\,n-n}(L_2L_3)^{n-n} \\ L_1^{\,n-n}(L_2L_3)^{n-2} \end{bmatrix} \tag{9}$$

The following steps show how to derive the fourth derivative:

$$\frac{d^4y}{dx^4} = \begin{pmatrix} \left[\sum_{i=0}^{n=4} C_i^n\left(\frac{d^{n-i}L_1}{dx^{n-i}}\right)\left(\frac{d^iL_2}{dx^i}\right)\right]\left(\frac{d^0L_3}{dx^0}\right) + 4\left[\sum_{i=0}^{n=3} C_i^n\left(\frac{d^{n-i}L_1}{dx^{n-i}}\right)\left(\frac{d^iL_2}{dx^i}\right)\right]\left(\frac{dL_3}{dx}\right) \\ 6\left[\sum_{i=0}^{n=2} C_i^n\left(\frac{d^{n-i}L_1}{dx^{n-i}}\right)\left(\frac{d^iL_2}{dx^i}\right)\right]\left(\frac{d^2L_3}{dx^2}\right) + 4\left[\sum_{i=0}^{n=1} C_i^n\left(\frac{d^{n-i}L_1}{dx^{n-i}}\right)\left(\frac{d^iL_2}{dx^i}\right)\right]\left(\frac{d^3L_3}{dx^3}\right) \\ + \\ \left[\sum_{i=0}^{n=0} C_i^n\left(\frac{d^{n-i}L_1}{dx^{n-i}}\right)\left(\frac{d^iL_2}{dx^i}\right)\left(\frac{d^4L_3}{dx^4}\right)\right] \end{pmatrix} \begin{bmatrix} L_1^{n-3}(L_2L_3)^{n-n} \\ L_1^{n-n}(L_2L_3)^{n-n} \\ L_1^{n-n}(L_2L_3)^{n-n} \\ L_1^{n-n}(L_2L_3)^{n-n} \\ L_1^{n-n}(L_2L_3)^{n-3} \end{bmatrix} \tag{10}$$

$$\frac{d^4y}{dx^4} = \left[\sum_{r=0}^{n} C_i^n\left[\sum_{i=0}^{n-n}\sum_{i=0}^{n-(n-1)}\sum_{i=0}^{n-(n-2)}\sum_{i=0}^{n-(n-3)}\sum_{i=0}^{n-(n-4)} C_i^n\left(\frac{d^{n-i}L_1}{dx^{n-i}}\right)\left(\frac{d^iL_2}{dx^i}\right)\right]\left(\frac{d^rL_3}{dx^r}\right)\right] \begin{bmatrix} L_1^{n-3}(L_2L_3)^{n-n} \\ L_1^{n-n}(L_2L_3)^{n-n} \\ L_1^{n-n}(L_2L_3)^{n-n} \\ L_1^{n-n}(L_2L_3)^{n-n} \\ L_1^{n-n}(L_2L_3)^{n-3} \end{bmatrix} \tag{11}$$

And

$$\frac{d^5y}{dx^5}=\left\{\begin{matrix}\left[\sum_{i=0}^{n=5}C_i^n\left(\frac{d^{n-i}L_1}{dx^{n-i}}\right)\left(\frac{d^iL_2}{dx^i}\right)\right]\left(\frac{d^0L_3}{dx^0}\right)+5\left[\sum_{i=0}^{n=4}C_i^n\left(\frac{d^{n-i}L_1}{dx^{n-i}}\right)\left(\frac{d^iL_2}{dx^i}\right)\right]\left(\frac{dL_3}{dx}\right)\\ +10\left[\sum_{i=0}^{n=3}C_i^n\left(\frac{d^{n-i}L_1}{dx^{n-i}}\right)\left(\frac{d^iL_2}{dx^i}\right)\right]\left(\frac{d^2L_3}{dx^2}\right)+10\left[\sum_{i=0}^{n=2}C_i^n\left(\frac{d^{n-i}L_1}{dx^{n-i}}\right)\left(\frac{d^iL_2}{dx^i}\right)\right]\left(\frac{d^3L_3}{dx^3}\right)\\ 5\left[\sum_{i=0}^{n=1}C_i^n\left(\frac{d^{n-i}L_1}{dx^{n-i}}\right)\left(\frac{d^iL_2}{dx^i}\right)\right]\left(\frac{d^4L_3}{dx^4}\right)+\left[\sum_{i=0}^{n=0}C_i^n\left(\frac{d^{n-i}L_1}{dx^{n-i}}\right)\left(\frac{d^iL_2}{dx^i}\right)\right]\left(\frac{d^5L}{dx^5}\right)\end{matrix}\right\}\begin{bmatrix}L_1^{n-4}(L_2L_3)^{n-n}\\ L_1^{n-n}(L_2L_3)^{n-n}\\ L_1^{n-n}(L_2L_3)^{n-n}\\ L_1^{n-n}(L_2L_3)^{n-n}\\ L_1^{n-n}(L_2L_3)^{n-n}\\ L_1^{n-n}(L_2L_3)^{n-4}\end{bmatrix} \qquad (12)$$

In summary

$$\frac{d^ny}{dx^n}=\left\{\sum_{r=0}^{n}C_i^n\left(\sum\sum\ ,\ldots\sum C_i^n\left(\frac{d^{n-i}L_1}{dx^{n-i}}\right)\left(\frac{d^iL_2}{dx^i}\right)\right)\left[\frac{d^rL_3}{dx^r}\right]\right\}\times\left[L_1(L_2L_3)^0,1,1,1,\ldots,1,L_1^0(L_2L_3)\right]^T \qquad (13)$$

Conclusively, $\frac{d^ny}{dx^n}$ of y(x), which depends on three differentiable functions;$L_1(x)L_2(x)L_3(x)$ is given as :

$$\frac{d^ny}{dx^n}=\left\{\sum_{r=0}^{n}C_i^n\left[\sum_{i=0}^{n-i}C_i^n\left(\frac{d^{n-i}L_1}{dx^{n-i}}\right)\left(\frac{d^iL_2}{dx^i}\right)\right]\left[\frac{d^rL_3}{dx^r}\right]\right\}\begin{bmatrix}L_1(x)(L_2L_3)^0\\ \vdots\\ L_1^0(x)(L_2L_3)^1\end{bmatrix} \qquad (14)$$

Where $C_i^n=\frac{n!}{i!(n-i)!}$ and $\begin{bmatrix}L_1(x)(L_2L_3)^0\\ \vdots\\ L_1^0(x)(L_2L_3)^1\end{bmatrix}$ is a (n+1)$\times$1 matrix (15)

3.0 Implementation of the New Method

Consider the function: $y = x^2\cos x\sin 2x$.

The first derivative using the new method:

$$\frac{dy}{dx}=L_3\left[\sum_{i=0}^{n=1}C_i^n\left(\frac{d^{n-i}L_1}{dx^{n-i}}\right)\left(\frac{d^iL_2}{dx^i}\right)\right]+(L_1L_2)\left[\frac{dL_3}{dx}\right]$$

$$=\sin 2x[(1)(2x)(\cos x)+(1)(x^2)(-\sin x)]+(x^2)(\cos x)[2\cos 2x]$$
$$=2x\sin 2x\cos x-x^2\sin 2x\sin x+2x^2\cos x\cos 2x$$

The 2nd derivative using the new method is:

$$\frac{d^2(x^2\cos x\sin 2x)}{dx^2}=2\sin2x\cos x - 4x\sin x\sin2x - 5x2\cos x\sin2x + 8x\cos2x\cos x - 4x^2\cos2x\sin x$$

The 3rd derivative using the new method is:

$$\frac{d^3y}{dx^3}= 13x^2\sin x\sin2x - 14x^2\cos x\cos2x - 30x\cos x\sin2x + 12\cos x\cos2x - 6\sin x\sin2x - 24x\sin x\cos2x$$

The 4th derivative using the new method is:

$$\frac{d^4y}{dx^4}= 16x^2\cos x\sin2x - 12\cos x\sin2x + 8x\sin x\sin2x + x^2\cos x\sin2x - 48\sin x\cos2x48x\cos x\cos2x +$$
$$8x^2\sin x\cos2x - 48\cos x\sin2x + 64x\sin x\sin2x + 24x^2\cos x\sin2x - 64x\cos x\cos2x - 32x^2\sin x\cos2x + 16x^2\cos x\sin2x$$

The 5th derivative using the new method is:

$$\frac{d^5(x^2\cos x\sin 2x)}{dx^5}= 260\sin x\sin2x - 121x^2\sin x\sin2x + 400x\sin x\cos2x - 190\cos x\cos2x -$$
$$10x\cos x\sin2x + 240\cos x\sin2x + 160x\cos x\sin2x + 32x^2\cos x\cos2x$$

4.0 Conclusion and Recommendation

This method is preferred over the known Liebnitz's and product rule, in that the Liebnitz's method can only be used for a function which depends on two differentiable functions and the product rule requires the preceding derivatives before it can give the higher derivatives[6,7,8]. But this new method does not require the lower order derivatives to generate the higher order derivatives. Moreover, if the functions to be differentiated can be presented in their series form, the higher derivatives of entire functions can be generated recursively.

The following points are obvious concerning the new method:

(i) The superscript n decreases regularly by 1
(ii) The superscript i increases regularly by 1
(iii) The numerical coefficients are the normal binomial coefficients.

For increased accuracy in most numerical methods that involve the use of higher order derivatives, this new method can be used to obtain higher order derivatives of the functions involved. The labour involved in calculating and evaluating higher derivatives through the use of this new method is very minimal, since you can jump the process of obtaining the preceding derivatives to the point of obtaining desired derivative (order).

When it comes to the computational implementation, the following steps will give a very easy approach:

(i) obtain the series expression of the component ($L_1L_2L_3$) of the function to be differentiated
(ii) obtain by recursion, the derivatives of the component of the function by using differentiating the series expressions mentioned above
(iii) *n* in the new formula represents the desired order of the derivative. This can be manipulated to obtain higher order derivatives as desired .i.e. If one desires the second derivative, n will be 2; for the third derivative n will be 3 etc.

THEOREM: Let L_1L_2 L_3 be three differentiable and continuous functions then the nth derivative of $Y(x) = L_1(x) L_2(x) L_3(x)$ is obtained as

$$\frac{d^n}{dx^n} = \left\{ \sum_{r=0}^{n} C_r^n \left(\sum_{i=0}^{n-(n-2)} \sum_{i=0}^{n-(n-1)}, \ldots, \sum_{i=0}^{n-(n-n)} C_i^n \left(\frac{d^{n-i} L_1}{dx^{n-i}} \right) \left(\frac{d^i L_2}{dx^i} \right) \right) \left[\frac{d^r L_3}{dx^r} \right] \right\} \left[L_1 (L_2 L_3)^0, 1,1,1, \ldots, 1, L_1^0 (L_2 L_3) \right]^T \quad \text{Where}$$

$$C_i^n = \frac{n!}{i!(n-r)!}$$ and $\left[L_1 (L_2 L_3)^0, 1,1,1, \ldots, 1, L_1^0 (L_2 L_3) \right]^T$

is a transpose of the column matrix $\begin{bmatrix} L_1 (L_2 L_3)^0 \\ \vdots \\ L_1^0 (L_2 L_3)^1 \end{bmatrix}$

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