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Some Uniform Order Block Methods for the Solution of First Order Ordinary Differential Equations.

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Abstract

In this Paper, we consider the derivation of a continuous formulation of a linear multistep method for ordinary differential equations by collocation methods without the use of predictor-corrector approach. All the discrete schemes used in each of the block method at k=2 and k=3 derived, come from a single continuous formulation and its derivative. The block suggested approach is self-starting and produce parallel solution of the ordinary differential equations (ODES) which minimizes the cost of computation compared to other variants. Both block methods at k=3 and k=2 converges to the exact solutions with the two Numerical examples tested with this approach.

Keywords: Uniform order, Block methods, first order odes, initial value problem, self starting and parallel solutions.

1.0 Introduction

The Traditional multistep methods can be made continuous through the idea of multistep collocation method (MC) see [1], [2], and [3] .

These earlier works focused on construction of continuous multistep (CM) methods by employing the Collocation method. The continuous multistep methods produce piece wise polynomial solution over k-step $[x_n, x_{n+k}]$ for the first order system of ODES of the form

$$y' = f(x\,, y) \qquad a < x < b\,, \quad y(a) = y_0 \qquad (`1)$$

The aim of this paper is to demonstrate using a single continuous formulation and its derivative to derive some discrete schemes which form the block scheme to solve (1) directly without requiring a starting value and speed up of computational process, since all the solutions were obtained at once.

2.0 The Multistep Collocation Method

Following [3], Consider the collocation method defined for the step $[x_n, x_{n+k}]$ by

$$y(x) = \sum_{j=0}^{t-1} \alpha_j\,(x)\, y_{n+j} + h \sum_{j=0}^{m-1} \beta_j\, f(x_i\,, y\,(x_i)) \qquad (2)$$

Where t and m denotes respectively the number of interpolation and collocation points used. This method is typically expressed as

$$\rho(E) y_n = h\delta(E) f_n \qquad (3)$$

Where E is the shift operator specified by

$E^j\, y_n = y_{n+j}$ while $\boldsymbol{\rho}$ and $\boldsymbol{\delta}$ are the characteristic polynomials and are given as

$$\rho(r) = \sum_{j=0}^{t-1} \alpha_j\, x^i\,, \;\; \delta(r) = \sum_{j=0}^{m-1} \beta_j\, f x^i \qquad (4)$$

y_n is the numerical approximation to the Exact solution y($\boldsymbol{x_n}$) and $f_n = f(x_n, y_n)$.

3.0 Derivation of the Present Methods

(a) Block method of order $[\,\mathbf{3}, \mathbf{3}\,]^{\boldsymbol{T}}$ at k=2

We approximate solution to (1) in the form

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$$y(x) = \sum_{j=0}^{m+t} a_j\, x^j \quad , \quad j = 0,1,(k+1) \tag{5}$$

$$y'(x) = \sum_{j=1}^{m+t} j a_j\, x^j \tag{6}$$

Where, a_j are the parameters to be determined, t and m are points of interpolation and collocation. Specifically, for this method at k= **2**, when equation (6) is collocated at $x = x_{n_{+j}}$ $j = 0,(k-1)$ and also (5) is interpolated at $x = x_{n_{+j}}$ $j = 0,(k-1)$, we get the following system of non Linear equations.

$$\begin{aligned}
&a_0 + a_1x_n + a_2x_n^2 + a_3x_n^3 = y_n\\
&a_0 + a_1x_{n+1} + a_2x_{n+1}^2 + a_3x_{n+1}^3 = y_{n+1}\\
&a_1 + 2a_2x_n + 3a_3x_n^2 = f_n\\
&a_1 + 2a_2x_{n+1} + 3a_3x_{n+1}^2 = f_{n+1}
\end{aligned} \tag{7}$$

When equation (7) is arranged in matrix equation form we have

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 \\ 0 & 1 & 2x_n & 3x_n^2 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 \end{bmatrix} \begin{vmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{vmatrix} = \begin{vmatrix} y_n \\ y_{n+1} \\ f_n \\ f_{n+1} \end{vmatrix} \tag{8}$$

Where $\propto_j (x)$ and $\beta_j(x)$ are obtained as a continuous coefficients. Specifically the proposed solution takes the form

$$y(x) = \propto_0 (x)y_n + \propto_1 (x)y_{n+1} + h\{\beta_0(x)f_n + \beta_1(x)f_{n+1}\} \tag{9}$$

We used Maple 11 Software to invert the matrix D in equation (8) to determine values a_j , $j = 0,1,k+1$ and finally obtained the continuous formulation of the form

$$y(x) = \left[\frac{2(x-x_{n+1})^3 + 3h(x-x_{n+1})^2}{h^3}\right] y_n + \left[\frac{h^3 - 3h(x-x_{n+1})^2 - 2(x-x_{n+1})^3}{h^3}\right] y_{n+1} + \frac{[(x-x_{n+1})^3 + h(x-x_{n+1})^2]f_n}{h^2} + [(x-x_{n+1})^3 + 2h(x-x_{n+1})^2 + h^2]\frac{f_{n+1}}{h^2} \tag{10}$$

The first derivative of equation (10) gives

$$y' = \left[\frac{6(x-x_{n+1})^2 + 6h(x-x_{n+1})}{h^3}\right] y_n + \left[\frac{-6h(x-x_{n+1}) - 6(x-x_{n+1})^2}{h^3}\right] y_{n+1} + \frac{[3(x-x_{n+1})^2 + 2h(x-x_{n+1})]fn}{h^2} + \left[\frac{3(x-x_{n+1})^2 + 4h(x-x_{n+1})}{h^2} + h^2\right] f_{n+1} \tag{11}$$

Evaluating both (10) and (11) at $x = x_{n+2}$, gives the following discrete schemes.

$$\begin{aligned}
&y_{n+2} + 4y_{n+1} - 5y_n = 2h(f_n + 2f_{n+1})\\
&y_{n+1} - y_n = \frac{h}{12}(f_n + 8f_{n+1} - f_{n+2})
\end{aligned} \tag{12}$$

Equation (12) has Order $[\mathbf{3},\ \mathbf{3}]^{\boldsymbol{T}}$ with Error constants $[\frac{1}{6},\frac{1}{24}]^T$.

(b) Block method of order $[\,4,4,4\,]^T$ at $k = 3$

We interpolate equation (5) at $x = x_{n+j}$, $j = 0,(k-2)$ and collocate equation (6) at $x = x_{n+j}$, $j = o,(k-1)$, we get the following system of non Linear equations.

$$\begin{aligned}
&a_0 + a_1x_n + a_2x_n^2 + a_3x_n^3 + a_4x_n^4 = y_n\\
&a_0 + a_1x_{n+1} + a_2x_{n+1}^2 + a_3x_{n+1}^3 + a_4x_n^4 = y_{n+1}\\
&a_1 + 2a_2x_n + 3a_3x_n^2 + 4a_4x_n^3 = f_n\\
&a_1 + 2a_2x_{n+1} + 3a_3x_{n+1}^2 + 4a_4x_{n+1}^3 = f_{n+1}\\
&a_1 + 2a_2x_{n+2} + 3a_3x_{n+2}^2 + 4a_4x_{n+2}^3 = f_{n+2}
\end{aligned} \tag{13}$$

When equation (13) is arranged in matrix equation form we have

$$\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 \end{bmatrix} \begin{vmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{vmatrix} = \begin{vmatrix} y_n \\ y_{n+1} \\ f_n \\ f_{n+1} \\ f_{n+2} \end{vmatrix} \tag{14}$$

The proposed continuous formulation takes the form.

$$y(x) = \alpha_0(x) y_n + \alpha_1(x) y_{n+1} + h\{\beta_0(x) f_n + \beta_1(x) f_{n+1} + \beta_2(x) f_{n+2}\} \tag{15}$$

We used Maple 11 Software to invert the matrix D in equation (14) and obtained values for a_j , $j = 0,1,k+1$. We finally obtained the continuous formulation of form.

$$y(x) = \left[\frac{h^4 - 4h^2(x-x_n)^2 + 4h(x-x_n)^3 - (x-x_n)^4}{h^4}\right] y_n + \left[\frac{4h^2(x-x_n)^2 - 4h(x-x_n)^3 + (x-x_n)^4}{h^4}\right] y_{n+1} + \frac{[12h^3(x-x_n) - 29h^2(x-x_n)^2 + 22h(x-x_n)^3 - 5(x-x_n)^4] f_n}{12h^3} + [-5h^2(x-x_n)^2 + 7h(x-x_n)^3 - 2(x-x_n)^4] \frac{f_{n+1}}{3h^3} + \frac{[h^2(x-x_n)^2 - 2h(x-x_n)^3 + (x-x_n)^4] f_{n+2}}{12h^3} \tag{16}$$

The first derivative of equation (16) gives

$$y'(x) = \left[\frac{-8h^2(x-x_n) + 12h(x-x_n)^2 - 4(x-x_n)^3}{h^4}\right] y_n + \left[\frac{8h^2(x-x_n) - 12h(x-x_n)^2 + 4(x-x_n)^3}{h^4}\right] y_{n+1} + \frac{[12h^3 - 58h^2(x-x_n) + 66h(x-x_n)^2 - 20(x-x_n)^3] f_n}{12h^3} + [-10h^2(x-x_n) + 21h(x-x_n)^2 - 8(x-x_n)^3] \frac{f_{n+1}}{3h^3} + \frac{[2h^2(x-x_n) - 6h(x-x_n)^2 + 4(x-x_n)^3] f_{n+2}}{12h^3} \tag{17}$$

Evaluating equation (16) at $x = x_{n+2}$ *and* $x = x_{n+3}$. Also equation (17) at $\boldsymbol{x = x_{n+3}}$ gives

$$\begin{aligned} y_{n+2} - y_n &= \frac{h}{3}[f_{n+2} + 4f_{n+1} + f_n] \\ y_{n+3} - 9y_{n+1} + 8y_n &= 3h[f_{n+2} - 2f_{n+1} - f_n] \\ y_{n+1} - y_n &= \frac{h}{24}[f_{n+3} - 5f_{n+2} + 19f_{n+1} + 9f_n] \end{aligned} \tag{18}$$

The Block Scheme of (18) are of Orders $[4, 4, 4]^T$, with error constants $[-\frac{1}{90}, \frac{1}{5}, -\frac{19}{720}]^T$

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4.0 Implementation Strategies

The proposed Block method at $k = 2$, $n = 0, 2, 4, 6, 8, 10$ …. and Block method at $k = 3$, $n = 0, 3, 6, 9, \ldots \ldots$, when applied it with tested problems gives all the required solutions at once. As such with these new Block methods proposed, we expect to gain in terms efficiency, accuracy and cost effectiveness.

Example 1

$$y' = x\, y,$$
$$y(0) = 1, \; h = 0.1$$

Analytic solution is $y(x) = e^{\frac{1}{2}x^2}$

Example 2

$$y' - 2y = e^{-x},$$
$$y(0) = \frac{3}{4}, \qquad h = 0.1$$

Analytic solution is $y(x) = (e^x - \frac{1}{4})e^{-2x}$

Table 1: Approximate Solution Of Example 1, $h = 0.1$

x	Theoretical Solution	Block Method k=2	Block Method k=3
0.1	1.005012521	1.004999662	1.005013050
0.2	1.02020134	1.020201337	1.020201517
0.3	1.04602786	1.046012304	1.046028759
0.4	1.083287086	1.083286999	1.083290177
0.5	1.133148453	1.133126935	1.133150363
0.6	1.197217363	1.197216935	1.197221844
0.7	1.277621313	1.277587256	1.277631600
0.8	1.377127764	1.377126114	1.377135511
0.9	1.4993025	1.499245761	1.499316976
1.0	1.648721271	1.648716177	1.648750627

Table 2: Comparison of Exact Errors of Example 1

x	Block Method k=2	Block Method k=3
0.1	1.2859 E -05	5.29 E-07
0.2	0.3 E-08	1.77 E -07
0.3	1.5556 E-05	8.99 E -07
0.4	8.7 E -08	3.091 E -06
0.5	2.1869 E -05	1.91 E -06
0.6	4.28 E-07	4.481 E-06
0.7	3.4057 E-05	1.0287 E-05
0.8	1.65 E-05	7.749 E-06
0.9	5.6739 E-05	1.4476 E-05
1.0	5.094 E-06	2.9356 E-05

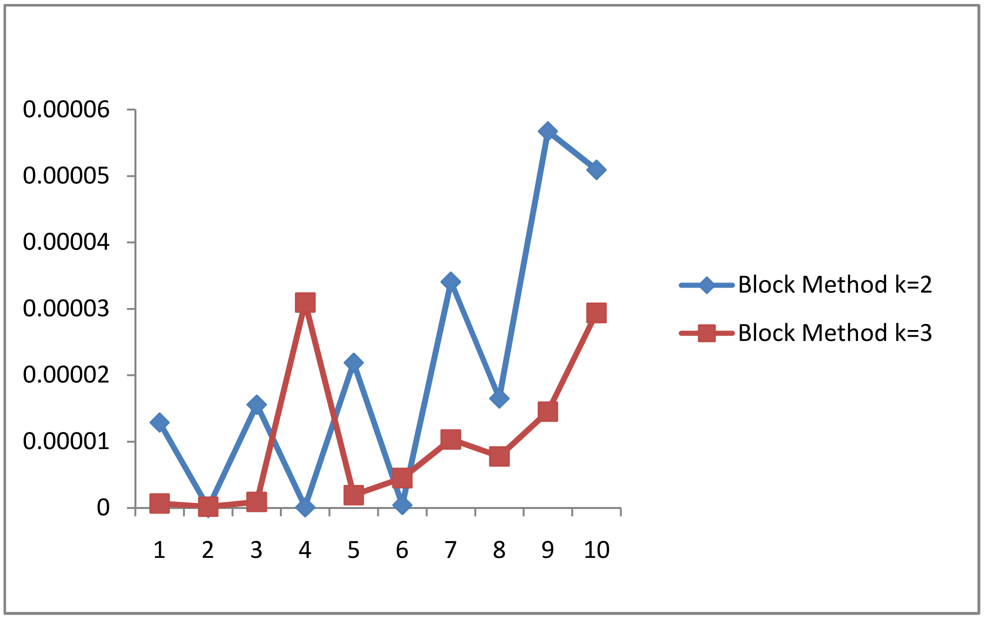

Fig.1: Error graph of example1

Table 3: Approximate Solution of Example 2, $h = 0.1$

x	Theoretical Solution	Block Method k=2	Block Method k=3
0.1	0.7001547298	0.7001637263	0.7001559430
0.2	0.6511507416	0.6511490804	0.6511510272
0.3	0.6036153117	0.6036194834	0.6036167761
0.4	0.557987805	0.5579856884	0.5579896313
0.5	0.5145607994	0.5145623661	0.5145619294
0.6	0.4735130831	0.4735110823	0.4735146427
0.7	0.4349360628	0.4349363021	0.4349376555
0.8	0.3988548346	0.398831777	0.3988559551
0.9	0.36522449377	0.3652445657	0.3652461726
1.0	0.3340456204	0.3340443581	0.3340467836

Table 4: Comparison of Exact Errors of Example 2

x	Block Method k=2	Block Method k=3
0.1	8.9965 E-06	1.2132 E-06
0.2	1.6612 E-06	2.856 E-07
0.3	4.1717 E-06	1.4644 E-06
0.4	2.1166 E-06	1.8263 E-06
0.5	1.5667 E-06	1.13 E-06
0.6	2.0008 E-06	1.5596 E-06
0.7	2.393 E-07	1.5927 E-06
0.8	1.6569 E-06	1.1205 E-06
0.9	2.0072 E-05	2.16789 E-05
1.0	1.2623 E-06	1.1632 E-05

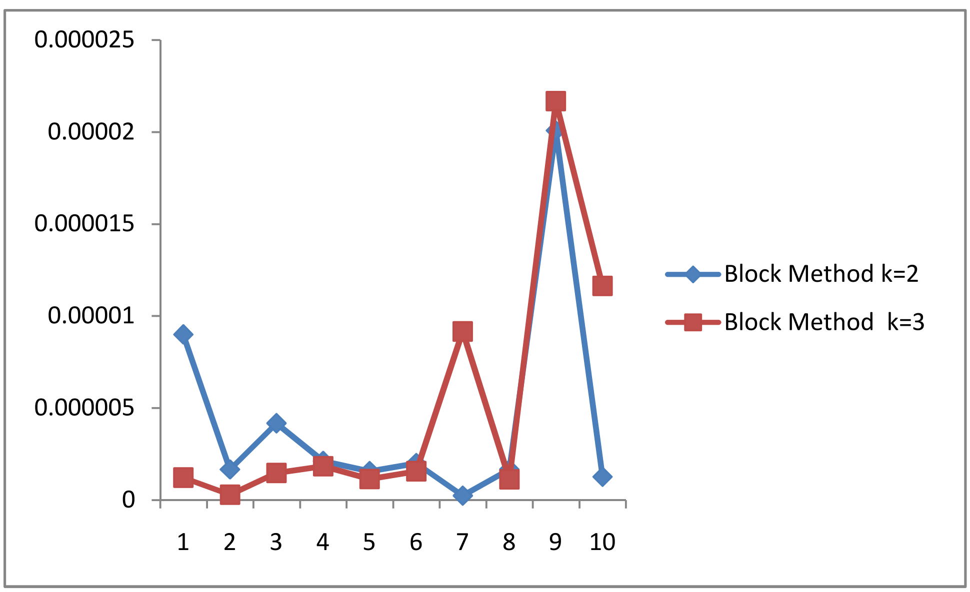

Fig.11: Error graph of example 2

5.0 Discussion of Results

Table 1 shows the numerical solutions of example 1 at $k = 2, k = 3$ and the theoretical solution is also given. Table 2 shows the comparison of exact errors of example 1. It has been observed that block method at $k = 3$ performed well by converging to the exact solution (See error graph of figure 1)

Table 3 shows the approximate shows the approximate solution of example 2 at $k = 2$ and $k = 3$ with the Theoretical solutions.

Table 4 shows the comparison of exact errors of example 2. It has been observed that block method at $k = 2$ performed well by converging closed to exact solutions (See error graph of figure 11).

Conclusion

All the discrete Schemes used in each of the block method were all derived from a single continuous formula and its derivative which are of uniform order of accuracy. The efficiency of the two block schemes were tested with the two numerical examples solved. Both results converges to the exact solution with smaller error difference and also obtained in block form which speed up the computational process and as such gain the efficiency, accuracy and cost effective in the implementations. Based on the error graphs, block method at $k = 3$ performed better in the problem1 while block method k=2 performed well with second problem tested.

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