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A Perturbed Fourth Order Runge-Kutta Method for First Order Ordinary Differential Equations

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Abstract

In this paper a perturbed fourth order Runge-Kutta (PRK) method for solving first-order ordinary differential equations (ODEs) is proposed. The scheme is an improvement over well-known fourth order Runge-kutta (RK) method and the most recent fifth order Runge-Kutta Fehlberg (RKF) method [2]. It has error constants much smaller than Multistep perturbed implicit Runge-Kutta collocation method [1] and 5th Order Runge- Kutta method . Numerical examples are given to test the efficiency of the method.

Keywords: Perturbed, first order ODEs, Implicit, Stiff equations, Stable solutions and non-linear ODEs.

1.0 Introduction

There are many numerical methods for solving first order initial value problems of ordinary differential equations, especially non linear, implicit and stiff problems. The most popular Runge-Kutta fourth order (RK) method has a small error of Ch^5. Also the most recent fifth-order (RK) method is the Runge- Kutta- Fehlberg (RKF) method given below:

$$y_{n+1} = \lambda_1 K_1 + \lambda_2 K_2 + \lambda_3 K_3 \dots\dots + \lambda_6 K_6 \qquad (1.1)$$

with coefficient vectors $K = [\lambda_1, \lambda_2, \lambda_3, \dots . \lambda_6] = \left[\frac{16}{135}, 0, \frac{6656}{12825}, \frac{28561}{56430}, \frac{-9}{50}, \frac{2}{55}\right]$,

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + \frac{1}{2}h\,, y_n + \frac{1}{4}k_1)$$

$$k_3 = hf(x_n + \frac{3}{8}h\,, y_n + \frac{3}{32}k_1 + \frac{9}{32}k_2)$$

$$k_4 = hf(x_n + \frac{12}{13}h\,, y_n + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3)$$

$$k_5 = hf(x_n + h\,, y_n + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4)$$

$$k_6 = hf(x_n + \frac{1}{2}h\,, y_n - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5)$$

Fifth (RKF) method is very tedious to derive, hence Runge-Kutta methods of higher order than fifth are almost impossible to derive.

Other current researchers have tried other means like Efficient Runge-Kutta method [3], Implicit Rational Runge-Kutta schemes [4], Multistep implicit Runge-Kutta method [5].

Unfortunately the above schemes do not give the desired level of accuracy. Thus there is need for further improvement on the Runge-Kutta methods since the current trend in numerical solution of ordinary differential equations is towards efficiency, simple algorithms and high level of accuracy. In this Paper we proposed a new perturbed Runge-Kutta method which gives a very high degree of accuracy.

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1.0 Construction Of The New Method

We consider the initial value problem

$$y' = f(x,y),\ y(x_0) = y_0 \tag{2.1}$$

We define

$$\overset{(h)}{y}(x_n + h) = \overset{(h)}{y_{n+1}},\ \overset{\left(\frac{h}{2}\right)}{y}(x_n + h) = \overset{\left(\frac{h}{2}\right)}{y_{n+1}}$$

to be approximate Runge- Kutta solutions of (2.1) with step size h and two half steps of h respectively. Now the fourth order RK method with step-size h is defined as:

$$\overset{(h)}{y_{n+1}} = y_n + \tfrac{1}{6}[(k_1 + k_4) + 2(k_2 + k_3)]\,, \tag{2.2}$$

with:

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + \tfrac{1}{2}h\,, y_n + \tfrac{1}{2}k_1)$$

$$k_3 = hf(x_n + \tfrac{1}{2}h\,, y_n + \tfrac{1}{2}k_2)$$

$$k_4 = hf(x_n + h\,, y_n + k_3). \tag{2.3}$$

Taking two half steps of h, we define:

$y_{n+\frac{1}{2}}$ and $y_{n+1}^{\left(\frac{h}{2}\right)}$ as:

$$y_{n+\frac{1}{2}} = y_n + \tfrac{1}{6}[(m_1 + m_4) + 2(m_2 + m_3)] \tag{2.4}$$

$$y_{n+1}^{\left(\frac{h}{2}\right)} = y_{n+\frac{1}{2}} + \tfrac{1}{6}[(m_5 + m_8) + 2(m_6 + m_7)] \tag{2.5}$$

where:

$$\begin{aligned}
&m_1 = \tfrac{1}{2}hf(x_n, y_n),\quad m_2 = \tfrac{1}{2}hf(x_n + \tfrac{1}{4}h\,, y_n + \tfrac{1}{2}m_1)\\
&m_3 = \tfrac{1}{2}hf\left(x_n + \tfrac{1}{4}h\,, y_n + \tfrac{1}{2}m_2\right),\ m_4 = \tfrac{1}{2}hf(x_n + \tfrac{1}{2}h\,, y_n + m_3)\\
&m_5 = \tfrac{1}{2}hf\left(x_n + \tfrac{1}{2}h\,, y_{n+\frac{1}{2}}\right)\ ,\ m_6 = \tfrac{1}{2}hf\left(x_n + \tfrac{3}{4}h\,, y_{n+\frac{1}{2}} + \tfrac{1}{2}m_5\right)\\
&m_7 = \tfrac{1}{2}hf\left(x_n + \tfrac{3}{4}h\,, y_{n+\frac{1}{2}} + \tfrac{1}{2}m_6\right),\quad m_8 = \tfrac{1}{2}hf\left(x_n + h\,, y_{n+\frac{1}{2}} + m_7\right)
\end{aligned} \tag{2.6}$$

General any Runge-Kutta method of order K can be expanded into Taylor's series; and as $K \to \infty$ the series converges to exact solution y$(x_n + h)$.

Thus K-order Runge –Kutta for

$\overset{(h)}{y_{n+1}}$ and $y_{n+1}^{\left(\frac{h}{2}\right)}$ can be written in infinite series as:

$$y_{n+1} = \overset{(h)}{y_{n+1}} + C\mathrm{h}^{k+1} \tag{2.7}$$

$$y_{n+1} = y_{n+1}^{\left(\frac{h}{2}\right)} + C\left(\frac{h}{2}\right)^{k+1} \tag{2.8}$$

$C\mathrm{h}^{k+1}$ and $C\left(\frac{h}{2}\right)^{k+1}$ are the remainder terms of the series (2.7) & (2.8) respectively, C is independent of the choice of h.

Subtracting (2.8) from (2.7), we have:

$$0 = y_{n+1}^{(h)} - y_{n+1}^{\left(\frac{h}{2}\right)} + [C\mathrm{h}^{k+1} - \frac{Ch^{k+1}}{2^{k+1}}]$$

$$\Rightarrow C\mathrm{h}^{k+1} = \frac{2^{k+1}}{2^{k+1}-1}\ [\ \overset{\left(\frac{h}{2}\right)}{y_{n+1}} - \overset{(h)}{y_{n+1}}\ ] \tag{2.9}$$

we call (2.9) a local bound for the reminder terms.

Since $\frac{2^{k+1}}{2^{k+1}-1} \leq \frac{2^{k+1}}{2^{k+1}-\frac{13}{8}} = \frac{2^{k+4}}{2^{k+4}-13}$

We take $$\frac{2^{k+4}}{2^{k+4}-13}\left[y_{n+1}^{\left(\frac{h}{2}\right)} - y_{n+1}^{(h)} \right] \quad (2.10)$$

as our global minimum bound for the remaining Ch^{k+1} term.

Thus our proposed scheme for solving (2.1) is:

$$y_{n+1} = y_{n+1}^{(h)} + \frac{256}{243}\ \phi\left(y_{(x_{n+1})}^{\left(\frac{h}{2}\right)},\ y_{(x_{n+1})}^{(h)}\right) \quad (2.11)$$

we put k = 4 in (2.10) since we are perturbing 4th order RK method,

$$\phi\left(y_{(x_{n+1})}^{\left(\frac{h}{2}\right)},\ y_{(x_{n+1})}^{(h)}\right) = \left[y_{n+1}^{\left(\frac{h}{2}\right)} - y_{n+1}^{(h)} \right]$$

$\phi\left(y_{n+1}^{\left(\frac{h}{2}\right)},\ y_{n+1}^{(h)}\right)$ is the perturbation on $y_{n+1}^{(h)}$

$y_{n+1}^{\left(\frac{h}{2}\right)}$ *and* $y_{n+1}^{(h)}$ are the fourth order Runge- kutta methods with step size h and two- half step of h respectively already defined in equations (2.2) to (2.3) and (2.4) to (2.6).

2.0 Numerical Experiment

To test the desirability and efficiency of the new proposed scheme, we solve some examples whose exact solutions are known on the interval $0 \leq x \leq 0.5$. The errors arising from computed values at mesh points are shown tables I, II, III & IV.

Example 3.1 [See Yakubu D G [1]]

$y' = 2xy\,,\ y(0) = 1,\quad 0 \leq x \leq 0.5,\ h = 0.1$

Exact solution: $y(x) = e^{x^2}$

Table 1: Numerical Results of example 3.1

Mesh Values(x)	Theoretical Solutions	Yakubu D G Perturbed method [1]	5th Order RKF Method [2]	New Perturbed RK Method. (This study)
0.1	1.010050167084170	1.0100060698	1.01005017261	1.010050167089093
0.2	1.040810774192390	1.04035365545	1.04081078693	1.040810774263558
0.3	1.094174283705210	1.0925059290	1.09417430745	1.094174283934795
0.4	1.173510870991810	1.1692589465	1.17351091246	1.173510871393305
0.5	1.284025416687740	1.1936278646	1.28402548677	1.284025416885589

Table 11: Exact errors of example 3.1

Mesh (values)	Yakubu D G Perturbed method [1]	5th Order RKF method [2]	New Perturbed RK method (This study)
0.1	4.40 E (-05)	5.52 E (-09)	4.92 E (-12)
0.2	4.57 E (-04)	1.27 E (-08)	7.11 E (-11)
0.3	1.66 E (-03)	2.37 E (-08)	2.29 E (-10)
0.4	4.25 E (-03)	4.14 E (-08)	4.01 E (-10)
0.5	9.03 E (-02)	7.00 E (-08)	1.97 E (-10)

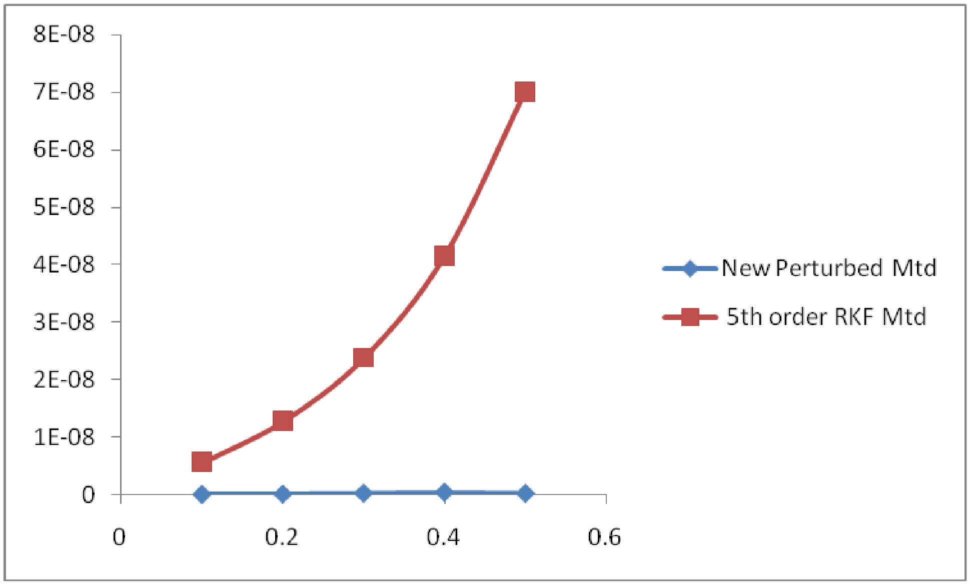

Fig. 1: Error graph of Example (3.1)

Example 3.2: $y' = x + y$, $y(0) = -1$, $h = 0.1$

Exact solution: $y(x) = -(x + 1)$

Table 111: Numerical Results of example 3.2 with exact errors

Mesh (values)	Theoretical Solutions	5th Order RKF method [1]	New Perturbed RK method (This study)	Error of RK Method [2]	Error of New RK Method (This study)
0.1	-1.10	-1.0999999999999998	-1.10	2.00 E (-15)	00
0.2	-1.20	-1.1999999999999996	-1.20	4.00 E (-15)	00
0.3	-1.30	-1.2999999999999994	-1.30	6.00 E (-15)	00
0.4	-1.40	-1.3999999999999992	-1.40	8.00 E (-15)	00
0.5	-1.50	-1.4999999999999990	-1.50	-1.00 E (-14)	00

Table IV: Numerical Results of example 3.3 with exact errors

Mesh (values)	Theoretical Solutions	5th Order RKF method [2]	New Perturbed RK method. (This study)	Error of RK Method [2]	Error of New RK Method (This study)
1.1	0.43745861168	0.43745887727	0.43745862652	2.65 E (-07)	1.48 E (-08)
1.2	0.39262421335	0.39262448596	0.3926242288	2.72 E (-07)	1.54 E (-08)
1.3	0.35879680881	0.35879705307	0.35879682265	2.44 E (-07)	1.38 E (-08)
1.4	0.33229030618	0.33229053136	0.33229031838	2.25 E (-07)	1.22 E (-08)
1.5	0.31090705565	0.31090725491	0.31090706636	-1.99E (-07)	1.07 E (-08)

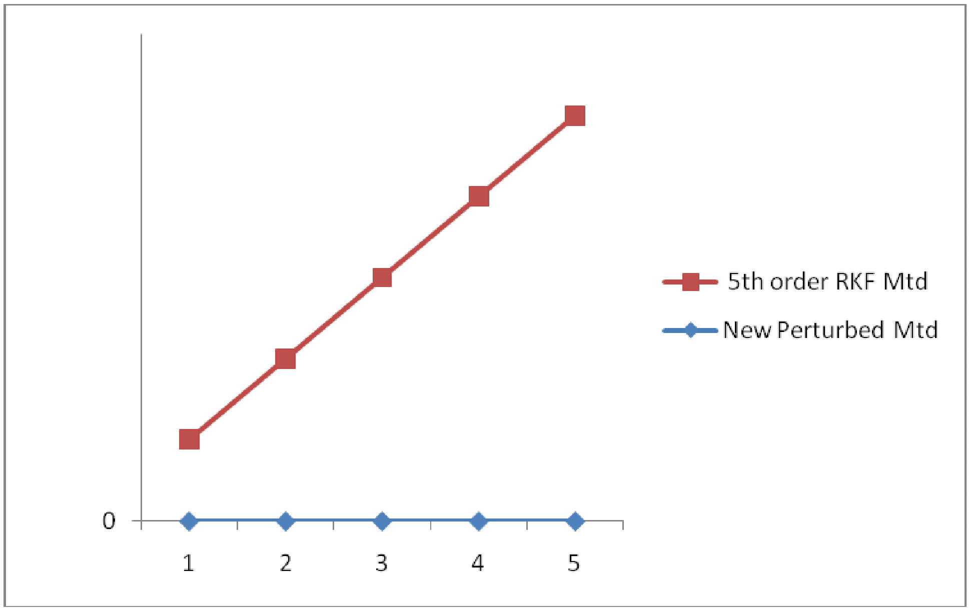

Fig. II: Error graph of Example 3.2

Example 3.3: $y' = \frac{-3y^2}{x}$, $y(1) = 0.5$, $h = 0.1$ $for\ 1 \le x \le 1.5$

Exact solution: $y(x) = \frac{1}{3In(x)+2}$

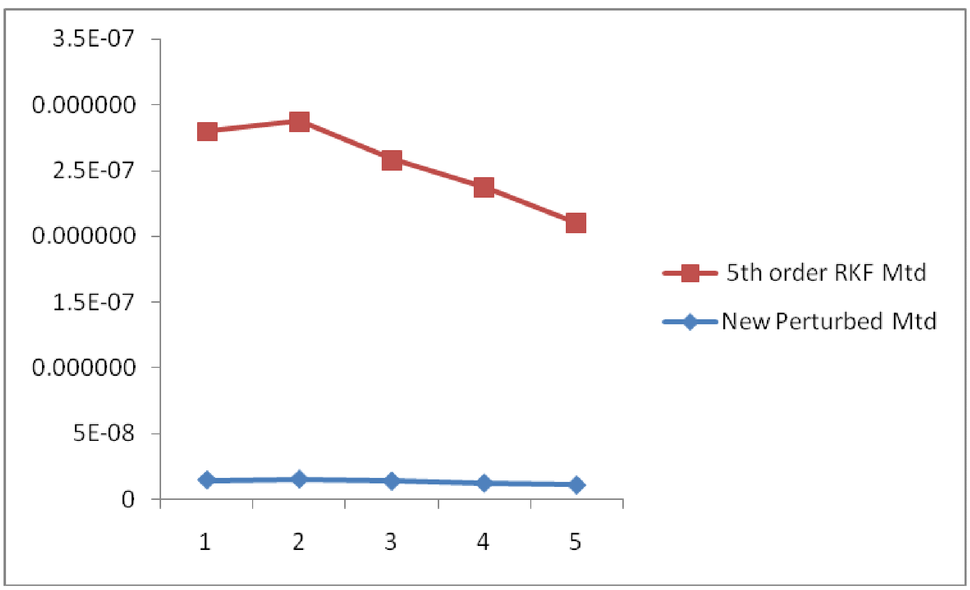

Fig. III: Error graph of Example (3.3)

3.0 Conclusion

The main advantages of this proposed perturbed method are its simplicity and efficient algorithms, higher level of accuracy. (See tables of results and Error graphs).
The method can be used to solve any initial value problem of first order ODEs and other special classes of ODEs like Implicit, Stiff and Non-linear problems with very high degree of accuracy. Since higher order RK methods give better level of accuracy, but very tedious and almost impossible to derive, this proposed new scheme gives alternative method for obtaining maximum order of RK method for solving first order ODEs.

Also the proposed method can be extended to determine solutions within off grid points, i.e. at $y_{n+\frac{1}{2}},\ n=0,1,2,...$, which is similar to hybrid block methods of the same order.

The coefficients of new schemes are smaller and simpler than the fifth order RKF method.

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