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A bi-diagonal method for finding the determinant of a matrix

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Abstract

The determinant of a matrix always depends on the concept of row or column. That is to evaluate the determinant of a matrix using several existing methods we use rows and column. In this paper we introduce the concept of false-determinant which is the determinant obtained using the diagonal elements of a matrix instead of rows or columns and present a method which uses this concept to find the determinant of a 3×3 ***matrix.***

Keywords: matrix; determinant; diagonal; false-determinant.

1.0 Introduction

The mathematical theory of matrix has its origin in the theory of determinant. Matrix plays an important role in many branches of Mathematics, Physics, Engineering, Statistics, and Economics. The origin of the determinants lies in the solution of linear equations. Most historians attributed the invention of determinant to Leibnitz. The theory of determinants is part of matrix theory, which in turn is a branch of algebra. Determinants are important in solving system of linear equations. They are also useful tools in areas of analytic geometric, calculus and differential equations. More on the determinant can be found in [2] and [3].

Determinant of a matrix has several properties. We will state here the properties that are of interest in this paper and the reader is referred to [2] and [3] for more properties and applications of determinants. To the best of our knowledge, the concept of false-determinant was not discussed in the literature not even to use them in finding determinant of a matrix. It is the aim of this paper to introduce the concept of false-determinant which is the determinant obtained using the diagonal elements of a matrix instead of rows or columns and presents a method which uses this concept to find the determinant of a 3×3 matrix. This result is motivated by a similar result obtained in [1] and a question: can we obtained the determinant of a matrix without the concept of row and column

Property1: Suppose $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ then $\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Property 2: If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ then

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{21}a_{13}a_{32} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22} - a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23}$$

Property 3: The determinant is independent of the row used to evaluate it.

2.0 Existing methods

There are several methods for finding the determinant of a matrix. We will mention here only two methods basket weaving method (Sarus method) and cofactor expansion method. For more methods of finding the determinant of a matrix the reader is referred to [2, 3, 4] and [6].

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2.1 Basket weaving method

Basket weaving method is used to compute the determinant of 3×3 matrices. It is considered as a classical 3×3 trick. In obtaining the determinant using this method, one proceeds by rewriting the first two columns of the matrix say A to the right of it and then looking at products of elements on the same ‘’slant’’ i.e. the elements on the same slope . The algebraic sign of each product of the elements which slant down to the right is taken to be positive. While for the product of those elements slanting down to the left is taken to be negative. For the slant which has less than three elements the product is considered being zero, that is to say, slant with less than three elements are to be neglected.

If $$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$ then

$$\det(A)=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$=a_{11}a_{22}a_{33}+a_{21}a_{13}a_{32}+a_{31}a_{12}a_{23}-a_{31}a_{13}a_{22}-a_{21}a_{12}a_{33}-a_{11}a_{32}a_{23}$$

2.2 The cofactor expansion method

Let $A=(a_{ij})\in\Re^{n\times n}$ for $i,\ j\in\{1,2,...,n\}$ the ijth **minor** corresponding to a_{ij} is defined by $m_{ij}=\det A(\{i\},\{j\})$. The ijth **cofactor** of a_{ij} is similarly defined as $c_{ij}=(-1)^{i+j}m_{ij}$. The determinant of *A* using the cofactor expansion method or Laplace expansion of the determinant by minors along the ith row is defined as

$$\det A=\sum_{j=1}^{n}(-1)^{i+j}a_{ij}m_{ij}=\sum_{j=1}^{n}(-1)^{i+j}a_{ij}c_{ij}\ \text{ for } i\in\{1,2,...,n\}.$$

3. The bi-diagonal method

In this section we are going to present the concept of false-determinant and relate it to the determinant of 3×3 matrix. Before we present the method we need to define some terms to be used throughout the paper. Throughout the paper by A we mean a 3×3 matrix with entries as follows:

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The product of principal and secondary diagonal elements will be denoted by P_P and P_S respectively. That is $P_P=a_{11}a_{22}a_{33}$ and $P_S=a_{13}a_{22}a_{31}$.

Definition 1 (False-determinant)**.** This is a determinant of a square matrix obtained by expansion using one of its diagonals. That is either the principal or the secondary diagonal instead of expansion by row or column and will be denoted by $\text{f}\det$.

Definition 2 (Principal false-determinant)**.** This is a false-determinant obtained by expansion using the expansion by principal diagonal and will be denoted by $|\ |_P$.

Definition 3 (Secondary false-determinant)**.** This is a false-determinant obtained by expansion using the expansion by secondary diagonal denoted by $|\ |_S$.

Definition 4 (Common diagonal element)**.** This is the element which is common to both principal and secondary diagonal. That is the intersection of the diagonals, which is the element “ a_{22} ” of the matrix A .

Definition 5 (Signed false-determinant). This is a false-determinant in which the common diagonal is taken with negative algebraic sign denoted by $\overline{\text{fdet}}$.

Definition 6 (Principal-signed false determinant). This is a signed determinant obtained by using principal diagonal denoted by $|\ |_{-P}$. Therefore, we have

$$\begin{aligned}|A|_{-P} &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{22}\left(a_{11}a_{33} - a_{13}a_{31}\right) + a_{33}\left(a_{11}a_{22} - a_{12}a_{21}\right) \\ &= a_{11}a_{22}a_{33} + a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}\end{aligned}$$

Definition 7 (Secondary-signed false determinant). This is a signed determinant obtained by using secondary diagonal and will be denoted by $|\ |_{-S}$. For our matrix A we have

$$\begin{aligned}|A|_{-S} &= a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ &= a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right) - a_{22}\left(a_{11}a_{33} - a_{13}a_{31}\right) + a_{31}\left(a_{12}a_{23} - a_{13}a_{22}\right) \\ &= a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{11}a_{22}a_{33}\end{aligned}$$

Definition 8 (Unsigned false-determinant). This is a false-determinant in which the common diagonal is taken with positive algebraic sign denoted by f+det.

Definition 9 (Principal-unsigned false-determinant). This is a unsigned determinant obtained by using principal diagonal and will be denoted by $|\ |_{+P}$. Therefore

$$\begin{aligned}|A|_{+P} &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) + a_{22}\left(a_{11}a_{33} - a_{13}a_{31}\right) + a_{33}\left(a_{11}a_{22} - a_{12}a_{21}\right) \\ &= 3a_{11}a_{22}a_{33} - \left(a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{12}a_{22}a_{31}\right) \\ &= 3P_P - \left(a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{12}a_{22}a_{31}\right)\end{aligned}$$

Definition 10 (Secondary-unsigned false-determinant). This is an unsigned determinant obtained by using secondary diagonal denoted by $|\ |_{+S}$. It follows that

$$\begin{aligned}|A|_{+S} &= a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ &= a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right) + a_{22}\left(a_{11}a_{33} - a_{13}a_{31}\right) + a_{31}\left(a_{12}a_{23} - a_{13}a_{22}\right) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - 3a_{13}a_{22}a_{31} \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - 3P_S\end{aligned}$$

Theorem 1. *Let* $A = (a_{ij}) \in \Re^{3\times 3}$ *then* $\det(A) = |A|_{-P} + |A|_{-S} + P_P - P_S$.

Proof. The statement follows directly from Definitions 6 and 7 and Property 2.

Theorem 2. *Let* $A=(a_{ij})\in\Re^{3\times3}$ *then* $\det(A)=|A|_{+P}+|A|_{+S}-3P_P+3P_S$.

Proof. The statement follows from Definitions 9 and 10 and Property 2.

4.0 A numerical example

Consider the following matrix

$$A=\begin{pmatrix}1 & -3 & 0\\ 1 & 4 & 2\\ 2 & -4 & 2\end{pmatrix}$$

The determinant of A is found using other methods to be 10. We shall now try to use the bi-diagonal method presented in Theorem 1 and 2 to find the determinant of A. Starting with Theorem 1, it was shown that $\det(A)=|A|_{-P}+|A|_{-S}+P_P-P_S$. Now for our matrix $|A|_{-P}=22, |A|_{-S}=-20, P_P=8$ and $P_S=0.$ Therefore,

$$\det(A)=|A|_{-P}+|A|_{-S}+P_P-P_S$$
$$=22-20+8+0=10$$

Using the second theorem we know that $\det(A)=|A|_{+P}+|A|_{+S}-3P_P+3P_S$. From our matrix we have $|A|_{+P}=10,$ $|A|_{+S}=-4, P_P=8$ and $P_S=0$. Therefore, we have

$$\det(A)=|A|_{+P}+|A|_{+S}-3P_P+3P_S$$
$$=38-4-24+0=10.$$

5.0 Conclusion

In this paper we have introduced the concept of false-determinant which is the determinant obtained using the diagonal element of a matrix instead of rows or columns and presented a method which uses this concept to find the determinant of a 3×3 matrix. Further research may concentrate on how this method can be extended to $n\times n$ matrices where $n>3$.

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