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The divergence of Hansen-Sengupta method applied on Trapezoidal - Newton operator for nonlinear interval system of equations

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Abstract

The Hansen-Sengupta operator is discussed in the light of circular interval arithmetic for the algebraic inclusion of zeros of nonlinear interval systems of equations which is known to be efficient for handling such problems. It was the aim of this paper to extend such good convergence behavior possessed by Hansen-Sengupta operator on the well known Trapezoidal-Newton functional iterative method. It was discovered that the Hansen-Sengupta method applied on Trapezoidal-Newton method will produce not only overestimated results but also results that are not finitely bounded. This was demonstrated by numerical example wherein we compared notes with results obtained from Uwamusi [16] and concluded that Hansen-Sengupta method applied on the Trapezoidal-Newton method indeed, diverges.

Keywords: interval nonlinear systems of equations, Hansen –Sengupta operator, Trapezoidal-Newton method, circular interval arithmetic

Subject Classification (MSC 2000): 65G20, 65G30, 65G40.

1.0 Introduction

We are interested in the solution of nonlinear interval system of equations

$$F(x) = 0 \tag{1.1}$$

where

$F : D \subset IR^n \to IR^n$, and $[x] = \{[\underline{x}_1, \overline{x}_1] \ldots [\underline{x}_n, \overline{x}_n]\} \subset IR^n$ is a parallelepiped parallel to the axes often called a box for each $\underline{x}_i \leq x \leq \overline{x}_i$.

We assume that F is a smooth homeomorphism mapping with $F \in C^1(D) \subseteq IR^n$.

We are interested in bounding the solution of (1.1) or establish their that no such results exist by some efficient interval methods. This means that given a box : $[X] = [\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2] \times \ldots \times [\underline{x}_n, \overline{x}_n] \subset IR^n$, evaluating F_i of the function F with desired interval based methods will produce intervals $[F_i]$ of the function F which are guaranteed to validate and enclose the zeros of F even in the presence of nonlinearities and round off errors.

Rump[12] proposed a method for reducing the width of co-multiplication of intervals. The main motive behind this paper is to present a class of interval based algorithms formulated in such a way that Rump's interval operations are applicable in the contexts of Hansen-Sengupta[5] which are able to either guarantee that the system has no solution or to yield sharp bounds of the results computed.

By repeatedly solving the linear interval system

$$J(x)(\hat{x} - x) \ni -f(x), x \in X, x \subseteq D_i, \tag{1.2}$$

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from

$$f(\hat{x}) \in f(x) + J(x)(\hat{x} - x), \tag{1.3}$$

one obtains an equivalent system(1.1) where J(x) is denoted by A(x) in the form:

$$A(x)(\hat{x} - x) \ni -b(x). \tag{1.4}$$

Method (1.4) does not only have the capacity of taking into consideration the problem of dependencies but also has a simpler structure and its hull is also straight forward. Thus, it follows that

$$x^{(k+1)} = N([x^{(k)}]) \bigcap (x^{(k)}), (k = 0,1,...,), \tag{1.5}$$

where $N([x^{(k)}])$ is the interval Newton method.

Now suppose instead of solving method (1.4), we consider

$$f(\hat{x}) = f(x^{(k)}) + \int_o^1 J(x^{(k)} + t(\hat{x} - x^{(k)}))(\hat{x} - x^{(k)})dt \tag{1.5}$$

where $x, \hat{x} \in ID \text{ and } t \in [0,1]$.

We can estimate $J(x^{(k)} + t(\hat{x} - x^{(k)}))$ in the interval [0,1] at the point t=0.

As discussed in Shokri[13], we estimate (1.5) by the trapezoidal rule in the form:

$$0 = f(x^{(k)}) + \frac{1}{2}[J(x^{(k)}) + J(\hat{x})](\hat{x} - x^{(k)}), \tag{1.6}$$

where from,

we obtain an iterative formula

$$x^{(k+1)} = x^{(k)} - 2[J(x^{(k)}) + J(\hat{x}^{(k+1)}]^{-1} f(x^{(k)} \bigcap x^{(k)} \tag{1.7}$$

and,

$$\hat{x}^{(k+1)} = x^{(k)} - J(x^{(k)}) f(x^{(k)}) \quad , \tag{1.8}$$

is the Newton step length which is a predictor.

In the meantime, we are inspired by the following well known definitions which may be useful in this paper.

Definition (1.1) Wozniakowski[15].

An iteration is said to be numerically stable if it produces a sequence $\{x^{(k)}\}$ of approximations of the solution x^* such that for large k the relative error $\frac{\|x^{(k)} - x^*\|}{\|x^*\|}$ is of order $\eta(1 + cond(F,d))$ and η is the relative machine precision and d is a data vector.

Definition (1.2), Wozniaskowski[15].

An iteration is said to be well behaved if a slightly perturbed $x^{(k)}$ is an almost exact solution of a slightly perturbed problem. This implies that

$$F(x^{(k)} + \delta x^{(k)}, d + \delta d_k) = 0(\eta^2) \text{ where } \frac{\|\delta x^{(k)}\|}{\|x^{(k)}\|} \text{ and } \frac{\|\delta d_k\|}{\|d\|} \text{ are of order } \eta.$$

The remaining parts of the papers are arranged as follows:

Section 2 is a review of terminologies used in the paper as well as the interval operation due Rump[12]. In section 3 we discuss the method of Trapezoidal-Newton operator in the sense of Shokri[13] wherein we incorporate Hansen-Sengupta[5] method. Basic properties of convergence characteristic of Hansen Sengupta method are described. We implemented the methods in section 4 using sample numerical problem taken from Uwamusi[16]. It is shown that Trapezoidal Newton method with interval approach has no rational map with finite intersection property. Finally we ended the paper with conclusion drawn from the numerical results obtained from the given example.

2.0 Notation

We denote $x \in IR^n \; and \; A \in IR^{nxn}$ to signify the set of interval vectors and respectively interval matrices.

An interval vector x is said to be thick if there exist $x_1 \in X$ and $x_2 \in X$ with $x_1 \neq x_2$ such that the width $w(x) > 0$. An interval vector is said to be thin if for all $x_1 \in X$ and $x_2 \in X$ with $x_1 = x_2$, then the width $w(x) = x_2 - x_1$.

We hereby introduce interval operation due to Rump[12] as follows: Let $(a,r) = \left[x \in R \| x - a| < r \right]$ where a, is the centre and r the radius. The basic interval operations $(+,-,o,/)$ such that for intervals

$a = (a_1, r_1) \; and \; b = (b_1, r_2) \in IR \; and \; \circ \in \{+,-,o,/\}$,

there follows $\{ xoy | x \in a, y \in b \} \subseteq aob$

With these, we have that:

$$< a_1, r_1 > \pm < b_1, r_2 > = < a_1 \pm b_1, r_1 + r_2 >, \qquad (2.1)$$

$$< a_1, r_1 > \cdot < b_1, r_2 > = < a_1 b_1, |a_1| r_2 + |b_1| r_1 + r_1 r_2, \qquad (2.2)$$

$$< a_1, r_1 > / < b_1, r_2 > = < a_1, r_2 > . < b_1, r_2 >^{-1}$$

where $0 \notin < b_1, r_2 >$. (2.3)

Inclusion isotonicity for intervals is implied by

$$< a_1, r_1 > \subseteq < b_1, r_2 > \text{ if and only if } |b_1 - a_1| \le r_2 - r_1 - < a_1, r_1 > = < -a_1, r_1 >.$$

Let note that these operations hold for commutativity and associativity but fail woefully for distributivity except for its subdistributivity, i.e. $(a \pm b)c = ac \pm bc$ for $a, b, c \in IR$. A disk inversion due to Carstensen and Petkovic[4] in the form of complex plane is adopted for our purpose as follows

$$[a_{ij}]^{-1} = \{a, r\}^{-1} =$$

$$\left\{ \frac{1}{a(1 - \frac{r^2}{|a|^2})}, \frac{r}{|a|^2 - r^2} \right\}, \qquad (2.4)$$

$$\left\{ \frac{1}{a}, \frac{r}{|a|(|a| - r)} \right\}, |a| > r \qquad (2.5)$$

$$\left\{ \frac{-a}{r^2 - |a|^2}, \frac{r}{r^2 - |a|^2} \right\}, |a| < r \qquad (2.6)$$

The aim of using of using these is to compute rigorous bounds on the solution of system (1.1) with computable overestimation factor that is supposedly small. Such bounds are expected to enclose truncation, rounding and often modeling errors. The use of different specific interval machinery as inner enclosures to check the validity of the quality of such bounds obtained dictates our interest in this paper.

3.0 The Method

Central to our discussion, we review the following definition.

Definition 3.1 Neumaier[9]. We say that a sequence of interval matrices $[A^k]$ converges if the lower and upper bounds converge, or equivalently, if the midpoints and radii converge. This means that

$$\lim_{k\to\infty} A^k = \left[\lim_{k\to\infty} \underline{A}^{(k)}, \lim_{k\to\infty} \overline{A}^{(k)}\right],$$

$$mid\left(\lim_{k\to\infty} A^{(k)}\right) = \lim_{k\to\infty} mid(A^{(k)})$$

$$rad\left(\lim_{k\to\infty} A^{(k)}\right) = \lim_{k\to\infty} rad(A^{(k)})$$

Thus the sequence

$A^{(o)} \supseteq A^{(1)} \supseteq ..... \supseteq A^{(k)} \supseteq A^{(k+1)} \supseteq$ of nested interval matrices converges to the limit $\lim_{k\to\infty} A^{(k)} = \bigcap_{k\geq o} A^{(k)}$

Theorem 3.2, Alefeld[1].

Let $F: D \subset IR^n \to IR^n$ be continuously differentiable and assume that $(IGA(f'([x]^o))$ exists for some interval vector

$[x]^o \subseteq D$

Assume that $f'([x]^o)$ exists

(a) if $N([x]) \subseteq [x]$ (3.1)

for some $[x] \subseteq [x]^o$ then f has a zero x* in [x] which is unique even in $[x]^o$.

Assuming further that

$$\rho(Q) < 1 \text{ where } Q = \left| I - IGA(f'([\bar{x}]^o f^{/}([x]^o) \right|; \tag{3.2}$$

(b) if f has a zero x* in $[x]^o$ then the sequence $\{[x]^k\}_{k=o}^{\infty}$ defined by

$$x^{(k+1)} = N([x^{(k)}]) \bigcap [x^{(k)}], (k = 0,1,....), \tag{3.3}$$

is well behaved, $x^* \in [x]^k$ *and* $\lim it_{k\to\infty} [x]^k = x^*$.

In particular, $\{[x]^{(k)}\}_{k=0}^{\infty}$ is monotonically decreasing and x* is unique in $[x]^o$.

Moreover, if

$$df'([x]_{ij}) \leq \alpha \|d[x]\|_\infty, \alpha \geq 0, \mid 1 \leq i, j \leq n \quad for \quad \forall [x] \subseteq [x]^o,$$

then

$$\left\| d[x]^{(k+1)} \right\| \leq \eta d[x]^{(k)} \|_\infty^2, \eta \geq 0,$$

(c) $N[x]^{ko} \bigcap [x]^{ko} = 0$ for some $k \geq 0$ *if* and only if $f(x) \neq 0 \;\; \forall x \in [x^0]$.

Proof: See Alefeld[1 and 2] for example.

Theorem 3.3, Neumaier[9].

Let $A \in IR^{nxn}$ be an H-matrix with positive diagonal elements and let $IR^n \to IR^n$ be continuous, diagonal and isotone. Then the function $f: IR^n \to IR^n$ defined by $\overline{F}(\overline{x}) = A\overline{x} + Q(x)$ has a unique zero $x^* \in R^n$. Moreover the inequality

$$\left| x^* - \overline{x} \right| \leq r, \tag{3.5}$$

holds for every nonnegative vector $r \in R^n$ satisfying

$$< A > r \geq \left| f(\overline{x}) \right|. \tag{3.6}$$

The Hansen-Sengupta [5] method is defined by

$$H(x, \overline{x}) = x^{(k+1)} = \Gamma(R\overline{A}, RD \cdot rad(A) abs(x^{(k)} + Rb_D, x^{(k)}), \tag{3.7}$$

where abs(x) denotes interval extension of the absolute value function that should not be confused with $|x|$ and that in the

limit $x^{\infty} = \Gamma(R\bar{A}, a, x^{\infty})$, (3.8)

where it is set that:

$$a = RD.rad(A)abs(x^{\infty}) + Rb_D \,, \tag{3.9}$$

$$x_i^{\infty} \subseteq \frac{\left(a_i - \sum_{j \neq i}\left((R\bar{A})_{ij}\right)\right)}{(R\bar{A})_{ii}} \tag{3.10}$$

and

$$rad(x_i^{\infty}) \leq \frac{1}{(R\bar{A})_{ii}}(Rad(a_i) + \sum_{j \neq i}\left|R\bar{A}\right|_{ij} rad(x_k^{\infty}) \,. \tag{3.11}$$

The numerator in equation (3.10) may contain zero in some cases. By the definition of optimal preconditioner in Kearfott[6] and due to analysis given in Kearfott and Xing[7], we are able to conclude that $\inf\left((R\bar{A})_{ii}\right) = 1.$

Thus the width w of the solution vector is then guided by

$$w(x^{(k+1)} - x^{(k)}) = w\left[RF(x^{(k)}) + \sum_{\substack{j=2 \\ j \neq i}}\left[R_{ij}A_{ij}\right](x_k - X_k)\right].$$

The preconditioned system (3.7) has an M- matrix centered about the identity matrix I with right hand vector in the form

$$M[x] = r \,, \tag{3.12}$$

Where $M \in IR^{n \times n}, and\, r \in IR^n$.

That means $\bar{M}_{ij} = \underline{M}_{ij} > 0$ for $i \neq k$ and that $\bar{M}_{ii} + \underline{M}_{ii} = 2$.

The solution set of method (3.7) is bounded by the inequalities:

$$\underline{M}\,|x| \leq D\,mid(r) + rad(r)$$

$$\bar{M}\,|x| \geq D\,mid(r) + rad(r)$$

where D is the diagonal matrix defined by

$$D_{ij} = \begin{cases} 0 & if \;\; i \neq j \\ 1 & if \;\; i = j \;\; and \;\; x_i \geq 0 \\ -1 & if \;\; i = j \;\; and \;\; x_i \leq 0 \end{cases}$$

Shi and Tian[14] obtained inequalities for method (3.7) in the form:

$$\underline{M}_{ii}|x_i| + \sum_{j \neq i}\underline{M}_{ij}|x_j| = |mid\;(r_i) + rad\;(r_i)|\,, \tag{3.13}$$

and

$$\underline{M}_{jj}|x_j| + \sum_{k \neq j}\underline{M}_{jk}|x_k| = |mid\;(r_j) + rad\;(r_j)|\;\forall j \neq i\,, \tag{3.14}$$

$$\bar{M}_{ii}\,D_{ii}\,x_i + \sum_{j \neq i}\bar{M}_{ij}|x_j| \geq -\bar{r}_i\,, \tag{3.15}$$

$$\underline{M}_{ii}\,D_{ii}\,x_i + \sum_{j \neq i}\underline{M}_{ij}|x_j| \leq -\underline{r}_i\,, \tag{3.16}$$

where from the following are valid:

$$\underline{M}_{ii}|x_i| + \sum_{j\neq i} \underline{M}_{ij}|x_j| \leq \frac{(D_{ii}-1)\underline{r}_i}{2} + \frac{(D_{ii}+1)\underline{r}_i}{2},$$

$$\bar{M}_{ii}|x_i| + \sum_{j\neq i} M_{ij}|x_j| \geq \frac{(D_{ii}-1)\bar{r}_i}{2} + \frac{(D_{ii}+1)\bar{r}_i}{2}.$$

Setting $x' = \left(|x_1|,|x_2|,....,|x_{i-1}|,x_i,|x_{i+1}|,...,|x_n|\right)^T$ we can infer from the analysis presented in Rohn [11], that

$$M(x'-|x|)+|x| \leq Mr.$$

Thus

$x' = x_i = b_i + ((I-A)x)_i \leq b^{/} radb)_i + (A''|x|)_i = (r + A''|x|)_i$ where A'' is a matrix of perturbation bound. With the above exposition we infer that Hansen-Sengupta method converges for any starting point for the interval nonlinear systems of equations.

4. Example:

We illustrate our discussion with the following problem discussed in Uwamusi[16]

$$F(x) = \begin{cases} x_2^2 - 3x_1^2 \\ x_1^2 + x_1x_3 + x_3^2 - 3x_2^2 \\ x_2^2 + x_2 + 1 - 3x_3^2 \end{cases}$$

$$x^{(o)} = (1/4, 1/2, 3/4), \varepsilon = 0.01$$

Table 1: Results Using Trapezoidal Newton Method (1.7)

Iterations	Results in midpoint-radius vector	$\left\Vert F(X^{(k)})\right\Vert_\infty$
1	[0.33169161,0.012680616] [0.5362032,0.01241961] [0.794301886,0.006144126]	2.0118213×10^{-2}
2	[0.337703974,0.000558556] [0.585171538,0.0060343] [0.801821044,0.00038168]	1.153693×10^{-3}
3	[0.338826244,0.000003618] [0.586294836 0.0000038] [0.799870224,0.000001674]	1.0659344×10^{-2}
4	[0.341908146,0.000000038] [0.591894964,0.000000096] [0.804670818,0.000000808	1.1499132×10^{-2}
5	?	?

Table 2: Results for Hansen-Sengupta method with error bounds

Iterations	Results Mid(X_k), Rad(X_k)	$\left\Vert F(\bar{X}_k)\right\Vert_\infty$
1	0.358937022, 1.7095978×10^{-2} 0.600208287, 1.6381989×10^{-2} 0.808328540, 8.809388×10^{-3}	2.6257369×10^{-2}
2	0.336461223, 3.350353×10^{-3} 0.585636548, 2.417562×10^{-3} 0.801816087, 8.112458×10^{-3}	3.351703×10^{-3}

3	0.337953381, 3.4978×10^{-5} 0.585355878, 3.4033×10^{-5} 0.801709854, 1.5718×10^{-5}	2.18688×10^{-4}
4	0.337917117, 3.6264×10^{-5} 0.585289640, 6.6708×10^{-5} 0.801634504, 4.15×10^{-7}	3.1×10^{-8}
5	0.337917117, 4×10^{-9} 0.585289640, 7.305×10^{-12} 0.801634504, 1.9341×10^{-12}	0
6	0.337917117,0 0.585289640,0 0.801634504,0	0

Using a result due to Schafer[10] which is a version of Miranda's theorem [8] we are able to show that the so called Trapezoidal- Newton method with Hansen-Sengupta approach has no rational map with a fixed point solution to the given problem. Further insight into this regard can be found in Rohn[11], Kearfort and Xing[7]. Again Trapezoidal Newton method is also not a H-continuous map since Baire category , see e.g., Anguelov et al, [3] failed to hold as we found in the given problem. This means that the graph completion operator is not inclusion isotone for this type of method.

5.0 Conclusion

The paper reported a defect which is common with multiple applications of a preconditioner in the interval based Hansen-Sengupta method for finding solution to interval nonlinear system of equations. In particular we studied this effect on Trapezoidal -interval Newton method following the idea of Shokri [13] and the cited references there in. The inherent problems encountered arose as a result, where there are many paths near some points, the Trapezoidal –Newton method based on Hansen-Sengupta approach may produce not only overly an overestimated results but also results that are not finitely bounded as shown in Table 1.

On the other hand computed values using original Hansen-Sengupta method (3.7) as discussed in Uwamusi[16] produced quite satisfactory results on the same sample test problem1, in the sense that monotonic and inclusion isotonicity property of interval arithmetic with order preserving were obtained. From Table 1 , one can see that divergence started at the third iteration with Trapezoidal-Newton method whereas results obtained from Uwamusi[16] provided numerically good solution as sequence of iterates tend to infinity.

Termination criterion for the iteration is $\frac{\left\|F(x^{(k)})\right\|}{\left\|F(x^{(0)})\right\|}<10^{-12}$ or $\left\|s^{(k)}\right\|<10^{-10}$ and $s^{(k)}=x^{(k+1)}-x^{(k)}$.

In the case of Table 1 we stopped the iteration when there was no longer any reason to continue with the iteration after four successive iterations as the solution obtained diverged from the true solution.
Thus the way an iterative method is written or evaluated will greatly harm the quality of interval solutions. This suggests that it is strongly recommended in interval based iteration to reduce as much as possible correlations among intervals.

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