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Uniqueness and asymptotic stability properties of the critical solution of the prey-predator retarded equation model

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Abstract

The challenge of modeling time-varying phenomenon using retarded equations is of great interest in mathematical sciences. This is because the non-instantaneous reaction of the state parameters is addressed. In this research, the Volterra prey/predator model system is modified by introducing time-lag functions $f(t-h)$ into the state parameters to account for the non-instantaneous reaction of the state parameters to changes. The compactness, contraction and continuity properties of the functional on the Banach space are utilized to establish the uniqueness of the integral solution of the critical point. The asymptotic stability properties of the critical solution are investigated using the quadratic matrix equation and symmetric linear matrix inequality test. Results obtained shows that the system is asymptotically stable, if the recruitment rate of the prey is kept higher than the recruitment rate of the predator at all time in the system. This is a stronger condition for stability and sustainability of the system, compared to the stability result of the equivalent ordinary differential equation model of the system.

Keywords: Retarded system, prey/predator model, uniqueness of solution, asymptotic stability, and linear matrix inequality test.
Subject classification; 34Kxx.

1.0 Introduction

There is increasing interest in the use of retarded equations in the modeling of dynamic systems. This is because the traditional point-wise modeling assumption lapse of the differential equations is addressed by incorporating response time h>0 in the reacting state parameters. For more on the analysis of retarded models equation refer to the following; [1], [2], [3],[4], [5], [6], [7] and [8].

In this work, the prey-predator model by [9] is linearized and modified by introducing time lag functions $f(t-h)$ into the model equations. These functions account for the past history of the state parameters.

The compactness, contraction and continuity properties of the functional on the Banach space are utilized to establish the uniqueness of the integral solution of the critical point of the modified model equation. The asymptotic stability properties of the critical solution are checked using the quadratic matrix equation and symmetric linear matrix inequality test [7] on asymptotic stability analysis of solutions of retarded systems. This test ensures that the coefficient matrix of the model equation satisfies the quadratic matrix equation and the symmetric linear matrix inequality (LMI), for some choice positive definite symmetric matrices. Also, the eigen-values of the characteristic equation of coefficient matrix should have negative real roots for some chosen values of the rate of the state parameters. These established properties enhanced the predictive capability of the model.

1.1 Notations

A *and* B are nxn matrices such that matrix $L=(A+B)$. P is a positive symmetric matrix, matrix $Q=I$, which is an n×n identity matrix. E^n is the n-dimensional Euclidean space, with $\|\bullet\|$ as the Euclidean vector norm.

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$B_H\left([t-h,t],E^n\right)$ is the Banach space of continuously differentiable functions f, G on $[t-h,t)$. $\varphi(s)$ is a continuously function with norm in $B_H\left([t-h,t],E^n\right)$ defined as $\|\varphi(s)\| = \sup_{t-h\le s<t}|\varphi(s)|$ such that $\varphi(0)=0$. W is a symmetric matrix with diagonal elements $0\le a_{ii}\le 1$, such that each w_i defines the i[th] row of W, $0\le w_{ii}\le 1$, an $\zeta=\{x,y\in E^n; w_i x-(1-w_i)y\}\in W$ is the convex set segment of W.

1.2 Model formulation

Volterra[9] considers $x(t)$ being the population at time t of some animal species known as the prey, and $y(t)$ being the population of predator species which lives on these preys. It is assumed that $x(t)$ increases at a rate proportional to $x(t)$ if the preys were left alone, and $\dot{x}(t)=\eta x(t), \quad \eta>0$. In the presence of a predator which depends on finding prey for food, the growth of the prey is then hindered by how much the predator finds the prey, thus

$$\dot{x}(t)=\eta x(t)-lx(t)z(t), \qquad \eta,\ l>0.$$

The predator is assumed to depend wholly on the prey for its food. Therefore in the absence of the prey, $\dot{y}(t)=-kx(t), \ \ k>0$.

Volterra[9] presented the differential equation model as,

$$\begin{aligned}\dot{x}(t) &= \eta x(t)-lx(t)z(t)\\ \dot{z}(t) &= lx(t)z(t)-kz(t),\\ x(t_0) &= x_0,, \quad z(t_0)=z_0,\end{aligned} \tag{1.0}$$

Linearizing system (1.0) at the critical points $(\alpha,\ \Gamma)=(0,0),\ and\ (\frac{k}{l},\ \frac{\eta}{l})$ by using the definition of the deviation of the state parameters from their equilibrium points as; $\frac{d}{dt}(\alpha+\delta x(t))=\delta\,\dot{x}(t)$ and $\frac{d}{dt}(\Gamma+\delta z(t))=\delta\,\dot{z}(t)$, then

$$\delta\,\dot{x}(t)=\eta\,(\alpha+\delta x(t))\ -\ l[(\alpha+\delta x(t))(\Gamma+\delta z(t))]$$

$$\delta\,\dot{z}(t)=l[(\alpha+\delta x(t))(\Gamma+\delta z(t))]-\eta\,(\Gamma+\delta x(t)).$$

And the linearized system of (1.0) is

$$\begin{aligned}\delta\,\dot{x}(t) &= \eta\,\delta x(t)\ -l\alpha\delta z(t)-\ l\Gamma\delta x(t)\\ \delta\,\dot{z}(t) &= l\alpha\delta z(t)+\ l\Gamma\delta x(t)-k\,\delta z(t),\\ x(t_0) &= x_0\ \ and\ \ z(t_0)=z_0.\end{aligned} \tag{1.1}$$

For the critical point $(\alpha,\ \Gamma)=(0,0)$, system (1.1) yield

$$\begin{aligned}\delta\,\dot{x}(t) &= \eta\,\delta x(t)\\ \delta\,\dot{z}(t) &= -\,k\,\delta z(t),\\ x(t_0) &= x_0\ \ and\ \ z(t_0)=z_0,\end{aligned} \tag{1.1a}$$

and this has a trivial solution ($x=0, \quad z=0$) which is not stable.

Next, consider linearization at the critical point $(\alpha,\ \Gamma)=(\frac{k}{l},\ \frac{\eta}{l})$, and system (1.1) yield

$$\begin{aligned}\delta\,\dot{x}(t) &= -\,k\delta z(t)\\ \delta\,\dot{z}(t) &= \eta\delta x(t),\\ x(t_0) &= x_0\ \ and\ \ z(t_0)=z_0,\end{aligned} \tag{1.1b}$$

which has characteristic root $\lambda=\pm\,i\sqrt{\eta k}$, so the critical point is stable but not asymptotically stable.

1.3 Model modification

Assume that the prey and the predator take average reacting time $h>0$ to respond to changes in the system; that is after the predator have killed many of the prey; the predators begin to die out after a time-lag $t-h,\ h>0$ due to food shortage. This in turn gives the prey population a chance to recover within the same time-lag. Thus the retarded equation model from (1.1) is

$$\delta \dot{x}(t) = \eta\,\delta x(t-h) - l\alpha\delta z(t) - l\Gamma\,\delta x(t)$$
$$\delta \dot{z}(t) = l\alpha\delta z(t) + l\Gamma\,\delta x(t) - k\,\delta z(t-h),$$
$$x(t_0) = x_0 \quad and \quad z(t_0) = z_0. \tag{1.2}$$

1.5 Uniqueness of solution of the modified model equation

Consider system (1.2) in the form,

$$\dot{V}(t) = AV(t) + BV(t-h),$$
$$V(t_0) = \phi(s), \qquad t-h \le s < t, \tag{1.3}$$

where,

$$\dot{V}(t) = \begin{pmatrix} \delta\dot{x}(t) \\ \delta\dot{z}(t) \end{pmatrix}, \quad A = \begin{pmatrix} -l\Gamma & -l\alpha \\ l\Gamma & l\alpha \end{pmatrix}, \quad B = \begin{pmatrix} \eta & 0 \\ 0 & -k \end{pmatrix}, \quad V(t) = \begin{pmatrix} \delta x(t) \\ \delta z(t) \end{pmatrix}$$

$$V(t-h) = \begin{pmatrix} \delta x(t-h) \\ \delta z(t-h) \end{pmatrix} \quad and \quad \varphi(t) = \begin{pmatrix} x(t_0) \\ z(t_0) \end{pmatrix},$$

so that the integrodifferential equation of (1.3) is defined as

$$\frac{d}{dt}\left(\varphi(t) + B\int_{t-h}^{t}\varphi(s)ds\right) = (A+B)\varphi(t). \tag{1.4}$$

Definition (1.0)

Let $\varphi(s) \in B_H$ for $t-h \le s < t$, then a function $V(s,\varphi(s))$ is an integral solution of the critical point (α, Γ) of system (1.3) on $[t-h,t)$ if the following conditions hold:

i. $V(s,\varphi(s))$ is continuous on $[t-h,t)$,

ii. $V(s,\varphi(s)) = V(t),\ for\ t-h \le s < t$,

iii $\int_{t-h}^{t}\varphi(s)ds \in B_H$,

iv and from (1.4),

$$V(s,\varphi(s)) = \varphi(0)L + B\int_{t-h}^{t}\left(\int_0^s \varphi(\varpi)d\varpi\right)ds - L\int_0^t \varphi(s)ds, \text{ for } t-h \le s < t,$$

so that, the integral solution of (1.3) at the critical point $\Omega = (\alpha, \Gamma)$ is defined as

$$V_\Omega(s,\varphi(s)) = \varphi(0)L + B\int_{t-h}^{t}\left(\int_0^s f_\Omega(\varpi,\varphi(\varpi))d\varpi\right)ds - L\int_0^t G_\Omega(s,\varphi(s))ds. \tag{1.5}$$

By definition (1.0), assume that the following hypothesis hold for $V(s,\varphi(s)) \in B_H$;

a. the functions $f_\Omega(\varpi,\varphi(\varpi)),\ G_\Omega(s,\varphi(s))$, form compacts set in $D \subset B_H$, where $D = \left\{\left(V(x(t),z(t),s) : \|V(s,\varphi(s)) - \varphi(s)\| < \varepsilon,\ s \in [t-nh,t),\ n \ge 1\right)\right\}$. Then for all $\{f_i\}_{i=1}^{\infty}$, $\{G_i\}_{i=1}^{\infty}$, convergent subsequences can be selected such that $\{f_i\}_{i=1}^{\infty} \to \varphi\ and\ \{G_i\}_{i=1}^{\infty} \to \varphi$ in $D \subset B_H$.

b. $f_\Omega : [t-h,t) \times D \to E^n$ and $G_\Omega : [t-h,t) \times D \to E^n$ are continuously differentiable and satisfies the contraction condition on B_H, then there exist constants k_0, m_0 for $0 < k_0 < 1$ and $0 < m_0 < 1$ such that $\|f_\Omega(\varpi,\varphi(\varpi)) - f_\Omega(\varpi,\theta(\varpi))\| \le m_0\|\varphi - \theta\|$ and $\|G_\Omega(s,\varphi(s)) - G_\Omega(s,\theta(s))\| \le k_0\|\varphi - \theta\|,\quad for\ \varphi,\theta \in B_H$

Theorem 1.1

Let B and L be stable matrix, and assume hypothesis a, b are satisfied, then for a given $\varphi(s) \in B_H$, there exists a unique solution of the critical point $\Omega = (\alpha,\ \Gamma)$ of system (1.3) defined as $V_\Omega(\ s, \varphi(s)\ ) \in B_H$.

Proof:

Let $V(s, \varphi(s))$ be a continuous differentiable function on $[t-h, t)$, with vector norm defined as $\|\bullet\|$ on B_H. Since $V(s, \varphi(s))$ satisfies conditions i – ii, then equation (1.5) defined the integral solution of the critical point of system (1.3).

By the hypothesis of compactness of $f_\Omega(\varpi, \varphi(\varpi)), G_\Omega(s, \varphi(s))$ in D, for $t_o < t_1 < \ldots < t_n$ there exists convergence subsequence solution $\{V_\Omega(s, \varphi_i(s))\}_{i=1}^{n} \in D$ which converges in D. Also using hypothesis a ,b and defining $\Lambda_{\max\lambda}$, $\nu_{\max\lambda}$, as the maximum characteristic roots of matrices B and L respectively, and $e^{t\Lambda_{\max\lambda}}$, $e^{t\nu_{\max\lambda}}$ as the corresponding eigen-vectors, then

$$\|V_\Omega(s, \varphi(s)) - V_\Omega(s, \theta(s))\|$$

$$= \left\| \left(B\int_{t-h}^{t} \left(\int_0^s f_\Omega(\varpi, \varphi(\varpi))\, d\varpi \right) ds + L\int_0^t G_\Omega(s, \varphi(s)) ds \right) - \left(B\int_{t-h}^{t} \left(\int_0^s f_\Omega(\varpi, \theta(\varpi))\, d\varpi \right) ds + L\int_0^t G_\Omega(s, \theta(s)) ds \right) \right\|$$

$$\le \|B\| \left(\left\| \int_{t-h}^{t} \left(\int_0^s f_\Omega(\varpi, \varphi(\varpi))\, d\varpi \right) ds - \int_{t-h}^{t} \left(\int_0^s f_\Omega(\varpi, \theta(\varpi))\, d\varpi \right) ds \right\| \right) + \|L\| \left(\left\| \int_0^t G_\Omega(s, \varphi(s)) ds - \int_0^t G_\Omega(s, \theta(s)) ds \right\| \right)$$

$$= \|B\| \left(\left\| \int_{t-h}^{t} f_\Omega(\varphi(s) ds - \int_{t-h}^{t} f_\Omega(s, \theta(s) ds \right\| \right) + \|L\| k_0 \| v_\Omega(t, \varphi(t)) - v_\Omega(t, \varphi_\Omega(t)) \|$$

$$= m_0 e^{t\Lambda_{\max\lambda}} \|w_\Omega(s, \varphi(s) - w_\Omega(s, \theta(s)\|_{t-h\le s<t} + k_0 e^{t\nu_{\max\lambda}} \|v_\Omega(t, \varphi(t)) - v_\Omega(t, \varphi_\Omega(t))\|$$

$$\le m_0 e^{t\Lambda_{\max\lambda}} \sup_{\varphi\in B_H} |w_\Omega(s, \varphi(s) - w_\Omega(s, \theta(s)|_{t-h\le s<t} + k_0 e^{t\nu_{\max\lambda}} \sup_{\varphi\in B_H} |v_\Omega(t, \varphi(t)) - v_\Omega(t, \varphi_\Omega(t))|$$

Since $0 < m_0 < 1$, and $0 < k_0 < 1$ then

$$m_0 e^{t\Lambda_{\max\lambda}} \sup_{\varphi\in B_H} |w_\Omega(s, \varphi(s) - w_\Omega(s, \theta(s)|_{t-h\le s<t} + k_0 e^{t\nu_{\max\lambda}} \sup_{\varphi\in B_H} |v_\Omega(t, \varphi(t)) - v_\Omega(t, \varphi_\Omega(t))| < 1.$$

Then $V_\Omega(s, \varphi(s))$ is a strict contraction in D and the fixed point of $V_\Omega(s, \varphi_i(s))$ for $t - h \le s < t$ which is defined as $V_\Omega(t)$ is the unique integral solution of the critical point of system (1.3).

1.4 Conditions for asymptotic stability

Consider system (1.3) with the coefficient matrix as

$$A = \begin{pmatrix} -l\Gamma & -l\alpha \\ l\Gamma & l\alpha \end{pmatrix}, \quad B = \begin{pmatrix} \eta & 0 \\ 0 & -k \end{pmatrix},$$

Han[4] defined the characteristic polynomial equation of the coefficient matrix of (1.3) as defined in (1.5) at the critical point $\alpha = \frac{k}{l}$ *and* $\Gamma = \frac{\eta}{l}$ (which is dependent on the time lag $h > 0$) as

$$f(e^{-h}, \lambda) = \begin{vmatrix} -\eta + \eta e^{-h} - \lambda & -\eta \\ k & k - ke^{-h} - \lambda \end{vmatrix} = 0,$$

$$f(e^{-h}, \lambda) = \lambda^2 + \lambda((\eta - k) - (\eta - k)e^{-h}) + \eta k(2e^{-h} - e^{-2h}) = 0 \qquad (1.6).$$

Igobi et al[7] stated the necessary and sufficient conditions for system (1.3) to assume asymptotic stability as:

(i) all real roots of the characteristic polynomial equation of (1.6) must have negative real value.

(ii) for any positive definite symmetric matrix P, the fundamental matrix $L = A + Be^{-h}$ must satisfy the quadratic matrix equation

$$L^T P + PL = -Q \qquad (1.7)$$

for Q being an identity matrix and $L = \begin{bmatrix} -\eta + \eta e^{-h} & -\eta \\ k & k - ke^{-h} \end{bmatrix}$.

(iii) given matrices Q, P, L as stated in (ii) above and matrix B as in (1.3), then the exists a symmetric matrix W which ensures that the symmetric linear matrix inequality (LMI)

$$\begin{bmatrix} \sum_{11} & \sum_{12} & \sum_{13} \\ \sum_{21} & \sum_{22} & \sum_{23} \\ \sum_{31} & \sum_{32} & \sum_{33} \end{bmatrix} \leq 0, \tag{1.8}$$

for $\sum_{11} = \left| -Q + BW^{T}P^{T} \right|, \quad \sum_{22} = |WB|, \quad \sum_{33} = \left|PBM^{T}\right| \; and \; \sum_{ij} = 0, \; for \; i \neq j$, is satisfied (see proof in [7]).

1.5 Asymptotic stability analysis

The asymptotic stability conditions of section (1.4) are satisfied if the following hold;

a . the rate of recruitments in the system must be such that $\eta > k$ (that is, the rate of recruitment of the prey η must at all time be greater than the rate of recruitment of the predator k), then the characteristic polynomial equation of (1.6) will have negative real roots and this satisfies condition (i) for system (1.3) to be asymptotically stable.

b. Resolving the matrix equation (1.7) into

$$\begin{pmatrix} 2m_{11} & 2m_{21} & 0 \\ m_{12} & (m_{11} + m_{22}) & m_{21} \\ 0 & 2m_{12} & 2m_{22} \end{pmatrix} \begin{pmatrix} p_{11} \\ p_{12} \\ p_{22} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix},$$

for m_{11}, m_{12}, m_{21}, m_{22} being the elements of matrix L, and substituting the values to have

$$\begin{pmatrix} 2(-\eta + \eta e^{-h}) & 2\eta & 0 \\ -k & -\eta + \eta e^{-h} + k - ke^{-h} & \eta \\ 0 & -2k & 2(k - ke^{-h}) \end{pmatrix} \begin{pmatrix} p_{11} \\ p_{12} \\ p_{22} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}.$$

Therefore, the elements of the positive definite symmetric matrix P are defined as follow,

$$p_{11} = -\left[(k^2 - 4\eta k) - (k^2 - \eta k)e^{-h}\right],$$

$$p_{12} = \left[(2k^2 - \eta k) + (-2\eta^2 - \eta k)e^{-h}\right],$$

$$p_{22} = -\left[\eta k + (-\eta^2 - 2\eta - 4\eta k)e^{-h} + (2\eta^2 + 2\eta k)e^{-2h}\right]$$

$$p_{00} = 2\left[3\eta^2 k + (2\eta k^2 - 5\eta^2 k)e^{-h} + (3\eta^2 k - 3\eta k^2)e^{-2h} + (\eta^2 k - \eta k^2)e^{-3h}\right],$$

Therefore, the positive definite symmetric matrix P which satisfies the quadratic matrix equation (1.7) for condition (ii) to be fulfilled is

$$P = \frac{1}{p_{00}} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \tag{1.9}$$

for $p_{00}, p_{11}, p_{12}, p_{22}$ as defined above and $\eta, k > 0$ c. the choice of the symmetric matrix W that satisfied the symmetric linear matrix inequality (1.8) for condition (iii) to be fulfilled is

$$W = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where $0 \leq a_{ii} \leq 1$ and $w_i \; (for \; 0 < w_i \leq 1)$ defines the i^{th} row of matrix W.

1.6 Conclusion:

The prey/predator model equation was modified by the introduction of time-lag function which accounts for the response time of the reacting state parameters. The formulation and proof of result on uniqueness of the integral solution of the critical point of the modified model equation was achieved by the utilization of the compactness, contraction and continuity properties of the functional on the Banach space. The modified model equations show desirable asymptotic stability properties under a certain condition at the critical point (such as keeping the recruitment rate of the prey higher than the recruitment rate of the predator). This is not the case with the equivalent ordinary differential system (1.1), whose solutions have shown that the system is only stable, but not asymptotically stable at the critical point (the later being a stronger condition for sustainability of the system).

References

[1] Abderrazzak B., Hassane B. and Abdelhakim M., ' Neutral Volterra Integrodifferential Equations with Infinite Delay', *Int. Journal of Math. Analysis* 3 (2009), 187-196.

[2] Brauer F. and Castillo-Chavez C., 'Mathematical Model of the Population Biology and Epidemiology', Springer, New York (2001).

[3] Gaetano A. De and Arino O., 'Modeling of the intra venous Glucose Tolerence Test; a global study for single-distributed-delay model', *Discrete and Dynam. System Journal*, 4(2004), 407-417.

[4] Han, Q. L., 'On Delay-Dependent Stability for Neutral Delay Differential Systems', *Int. J. Appl. Comput. Sci.* 11(2001), 965-976.

[5] Hinow P., Wang S. E., Arteaga C. L. and Genn F., 'A Mathematical Model Separating Quantitatively the Cytostatic and Cytotoxic Effects of HER2 Tyrosine Kinase Inhibitor', *javascript return* 10(2007), 25-38.

[6] Igobi D. K., Egwurube M. O. and Adesola W. A., 'On the Existence and Uniqueness, and Analysis of Asymptotic Stability Properties of Retarded Systems', *International Journal of Natural and Applied Sciences (IJNAS)*, 1 (2007), 116-122.

[7] Igobi, D. K., Egwurube, M. O. and Odekunle ,M. R., 'Asymptotic Stability Results for Retarded Differential Systems', *Global Journal of Mathematical Sciences* 8(2009), 23-29.

[8] Wodarz D. and Nowark M. A, 'Mathematical models of HIV pathogenesis and treatment bioassay', *Wiley periodical, Inc.* 24(2002), 1178-1187.

[9] Volterra V., 'Variation and fluctuation of the number of individuals in animal species living together, Animal Ecology', McGraw-Hill, New York, (1928).