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Asymptotic Stability Results of Solutions of Neutral Delay Systems

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Abstract

In recent time, the delay equations have been widely used in the modeling of dynamic systems, most especially the neutral delay equations. But its asymptotic stability analysis proves to be more difficult. In this paper, a retarded delay system is transformed to a class of neutral delay system using the differentiability condition of the functional on the Banach space. The Leibnez-Newton formula and symmetric properties of some chosen matrices are utilized to formulate a Lyapunov functional of the transformed system, which satisfies the Lyapunov-Krasvoskii conditions for asymptotic stability. The computation of the maximum time-lag (h_m) for the system to attain stability is approximated by the difference integral equation of the integrodifferential equation. Numerical illustration confirms the suitability of the result.

Keywords: asymptotic stability, time delay, difference integral, integrodifferential equation, positive definite matrix.

Subject classification; 34Kxx.

1.0 Introduction

The extensive applications of neutral functional equation in the modeling of dynamic (time dependent) system have attracted attention of many researchers in mathematical sciences (see;[1], [2] and [3]). This is because the incorporation of the time-lag in both the derivative and the state functions of the system equations (which account for the non-instantaneous actions and reactions of the system) make the system model more realistic.

Researchers have used various analytical concepts in the analysis of the properties of this functional equation (neutral equation), most especially the asymptotic stability of the system solution. Karsatos[4] stated that the survival of any dynamic system depends mostly on the stability properties of the system solution. Thus, the analytical approach readily employed to analyze the asymptotic stability of the neutral system are broadly base on two fronts: the delay independent results, which do not take the delay into consideration, but results are generated using matrix norm and measures ([3], [5], [6] and [7]), and the delay dependent results which take the delay into account ([3], [8], [9], [10] and [11]). Results of the delay dependent approach are mostly formulated to satisfy the Lyapunov-Krasovskii conditions for asymptotic stability. It involves the formulation of a positive definite matrix function, whose derivative is negative definite.

In this paper, the differentiability condition of the functional of a retarded system is employed to transform the system to a class of neutral delay system, and Leibnez-Newton formula is utilized to derive an integrodifferential of the transformed system. The maximum time-lag (h_m) for the system equation to attain asymptotic stability is computed by using difference integral equation of the integrodifferential. Results on the asymptotic stability of retarded systems using quadratic matrix equation and linear matrix inequality test that satisfy the Lyapunov- Krasovskii conditions for stability [12], are extended to the analysis of the asymptotic stability of the transformed system. This is achieved by the use of the derived integrodifferential equation, a symmetric positive definite matrix P and the introduction of symmetric matrices $W, N \in R^{nxn}$. Numerical computations are employed for illustration.

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1.1 Notations

P is a positive symmetric matrix, I is an n×n identity matrix.
E^n is the n-dimensional Euclidean space, with $\|\bullet\|$ as the Euclidean vector norm. $B_H\left([t-h,t],E^n\right)$ is the Banach space of continuously differentiable functions on $[t-h,t]$ such that $f:[t-h,t]\to E^n$, f is a continuous differential function on $[t-h,t]$. $\varphi(s)$ is a continuously function with norm in $B_H\left([t-h,t],E^n\right)$ defined as $\|\varphi(s)\|=\sup_{t-h\le s<t}|\varphi(s)|$. W is a symmetric matrix with diagonal elements $0\le a_{ii}\le 1$, such that each w_i defines the i[th] row of W, $0\le w_{ii}\le 1$, and $\zeta=\{x,y\in E^n; w_i x-(1-w_i)y\}\in W$ is the convex set segment of W. $L=(A+B)$ defines the approximating matrix of the integrodifferential, x_t is the delay vector function and $x(t)$ is an nx1 displacement vector solution about a position of stable equilibrium.

1.2 Definitions

Consider a general delay system,

$$f\,(t,x(t),\ x(t-nh)^n,x^{(n)}(t))=0,\quad for\ n=1,2,.... \tag{1.0}$$

For any given initial condition $x(t_0)=\varphi(s)$, for $s\in[t-h,t]$, Han[8] stated the following definitions for stability of the system;

(i) the solution $x(t)=0$ of the system is Lyapunov stable if for any $\varepsilon>0$, there exists $\delta=\delta(t_0,\ \varepsilon)>0$ such that if $\|\varphi(s)\|<\delta$ then $\|x(t,t_0,\varphi(s))\|<\varepsilon,$.

(ii) the solution $x(t)=0$ of the system is asymptotically stable if it is Lyapunov stable, and there exists a $\delta_1=\delta_1(t_0)$ satisfying $\|\varphi(s)\|<\delta_1$ such that $\|x(t,t_0,\varphi(s))\|\to 0$ as $t\to\infty$.

2.0 Main Result

The aim of this section is to transformed retarded system (2.0) to a class of neutral system with delay (h). Results on the asymptotic stability conditions of the transformed model and theorem on the computational approach of the delay $(h>0)$ for the system solution to attain asymptotic stability are presented.

Consider a first order linear retarded system of (1.0) in the form,

$$\begin{aligned}\dot{y}(t)&=Ax(t)+Bx(t-h)\\ x(t)&=y(t)\quad+Cx(t-h).\end{aligned} \tag{2.0}$$

By system (2.0) $y(t)$ is differentiable, hence $x(t)-Cx(t-h)$ is also differentiable. Therefore, the following transformation holds for (2.0);

$$\dot{x}(t)-C\dot{x}(t-h)=Ax(t)+Bx(t-h). \tag{2.1}$$

System (2.1) is the desired neutral delay system with delay $h>0$. Utilizing the Leibniz-Newton formula stated as

$$Bx(t)-Bx(t-h)=B\int_{t-h}^{t}\dot{x}(\tau)d\tau, \tag{2.2}$$

The integrodifferential equation of (2.1) is defined as

$$\frac{d}{dt}\left[x(t)-Cx(t-h)+B\int_{t-h}^{t}x(\tau)d\tau\right]=(A+B)x(t). \tag{2.3}$$

Han[8] stated that the asymptotic stability analysis of the difference integral equation $x(t)-Cx(t-h)+B\int_{t-h}^{t}x(\tau)d\tau$ of (2.3) implies that of system (2.1).

Theorem 1

Let $f:(x_t,[t-h,t]) \to E^n$ be a continuous, differentiable linear neutral functional system, for $x_t \in B_H([t-h,t],E^n)$, and consider the integrodifferential equation (2.3) written in the form $\frac{d}{dt}f(t,x_t)-Lx(t)$ to be singular such that the approximating matrix $L=(A+B)$ is a stable. Then the difference integral equation $x(t)-Cx(t-h)+B\int_{t-h}^{t}x(\tau)d\tau$ at $t=0$ satisfies

$$\left|\frac{1}{\lambda}(B+\lambda I)-(B+\lambda C)e^{-\lambda h}\right| = 0 \cdot$$

Proof

Assume the system is stable. Then by the hypothesis the following equation holds

$$\frac{d}{dt}f(t,x_t)-Lx(t)=0. \quad (2.4)$$

Also since L is a stable matrix, there exists $\lambda: -\delta \le \lambda < 0$ for $\delta > 0$ being a real value such that the vector solution of (2.4) has the form

$$x(t)=e^{\lambda t}. \quad (2.5)$$

Then, writing (2.4) in terms of (2.5) yield $\lambda x(t)-Lx(t)=0$, so that

$$(\lambda I - L)x(t)=0. \quad (2.6)$$

Since $x(t)\neq 0,$ then $|\lambda I - L|=0$ and λ is the desired eigenvalue which must be negative for L to be a stable matrix.

Also consider the difference-integral of equation (2.3) written as

$$x(t)-Cx(t-h)+B\int_{t-h}^{t}x(\tau)d\tau=0. \quad (2.7)$$

Resolving (2.7) in terms of (2.5) at $t=0$ we obtained,

$$I - Ce^{-\lambda h} + B\int_{-h}^{0}e^{\lambda h}d\tau = 0,$$

and by simple integral computation the above equation yield

$$I - Ce^{-\lambda h} + B\left(\frac{I-e^{-\lambda h}}{\lambda}\right) = 0, \quad \lambda < 0 \cdot$$

Hence

$$\left|\frac{1}{\lambda}(B+\lambda I)-(B+\lambda C)e^{-\lambda h}\right| = 0$$

is satisfied.

Theorem II

Let the approximating matrix $(A+B)$ of (2.3) be a stable matrix, then for a defined difference-integral $x(t)-Cx(t-h)+B\int_{t-h}^{t}x(\tau)d\tau=0$ there exists a symmetric positive definite matrix $P\in R^{nxn}$ which is the unique solution of the quadratic matrix equation $(A+B)^T P + P(A+B) = -Q$ for $Q\in R^{nxn}$. And for any given Lyapunov functional $V(t,x(\tau));\ t\le\tau\le t-h$ which is continuous and differentiable, the resulted diagonal linear matrix inequality of $\dot{V}(t,x(\tau))$ satisfies

$$\begin{pmatrix} \sum_{11} & 0 & 0 & 0 \\ 0 & \sum_{22} & 0 & 0 \\ 0 & 0 & \sum_{33} & 0 \\ 0 & 0 & 0 & \sum_{44} \end{pmatrix} < 0,$$

where

$$\sum_{11} = \left| -Q - PB - B^T P + (A+B)^T W(I+B) + (I+B)^T W(A+B) + N \right|,$$

$$\sum_{22} = \left| (A+B)^T PC - PB - B^T PC + (A+B)^T W(C+B) \right|,$$

$$\sum_{33} = \left| C^T P(A+B) - C^T PB - B^T P + (C+B)^T W(A+B) \right|,$$

$$\sum_{44} = \left| C^T PB + B^T PC + N \right|,$$

where $W, N \in R^{nxn}$ are symmetric.

Then the solution $x(t) = 0$ of (2.1) is asymptotically stable

Proof

Consider the integrodifferential equation (2.3) in the form

$$\frac{d}{dt}\left(x(t) - Cx(t-h) \right) = (A+B)x(t) - B\int_{t-h}^{t} \dot{x}(\tau)d\tau, \tag{2.8}$$

with the approximating matrix $(A+B)$, such that there exists a symmetric positive definite matrix P which is the unique solution of $(A+B)^T P + P(A+B) = -Q$ .Then, for symmetric matrices $W, N \in R^{nxn}$, and extending the results in [11] for retarded systems, the Lyapunov functional is defined as

$$V(t, x(\tau)) = \sum_{i=1}^{3} V_i(t, x(\tau)); \quad t \le \tau \le t-h,$$

where

$$V_1 = (x(t) - Cx(t-h))^T P(x(t) - Cx(t-h)),$$

$$V_2 = \left(x(t) - Cx(t-h) + B\int_{t-h}^{t} x(\tau)d\tau \right)^T W\left(x(t) - Cx(t-h) + \int_{t-h}^{t} x(\tau)d\tau \right),$$

$$V_3 = \int x(\tau)^T Mx(\tau)d\tau.$$

And

$$\dot{V}_1(t, x(\tau)) = (x(t) - Cx(t-h))^T P \frac{d}{dt}(x(t) - Cx(t-h)) + \frac{d}{dt}(x(t) - Cx(t-h))^T P(x(t) - Cx(t-h))$$

$$= (x(t) - Cx(t-h))^T P\left((A+B)x(t) - B\int_{t-h}^{t} \dot{x}(\tau)d\tau \right) + \left((A+B)x(t) - B\int_{t-h}^{t} \dot{x}(\tau)d\tau \right)^T P(x(t) - Cx(t-h))$$

$$= x^T(t)\big((A+B)^T P + P(A+B)\big)x(t) - x^T(t-h)C^T P(A+B)x(t) - x^T(t)(A+B)^T PCx(t-h)$$
$$+ (x(t) - Cx(t-h))^T P(-B(x(t) - x(t-h))) + (-B(x(t) - x(t-h)))^T P(x(t) - Cx(t-h))$$

$$= x^T(t)\big((A+B)^T P + P(A+B)\big)x(t) - x^T(t-h)C^T P(A+B)x(t) - x^T(t)(A+B)^T PCx(t-h)$$
$$- x^T(t)PBx(t) + x^T(t-h)C^T PB - x^T(t)PBx(t-h) - x^T(t-h)PBx(t-h) - x^T(t-h)C^T PBx(t-h)$$
$$- x^T(t)B^T Px(t) + x^T(t-h)B^T Px(t) - x(t)B^T PCx(t-h) - x^T(t-h)B^t PCx(t-h)$$

$$
\begin{aligned}
&= x^T(t)\big((A+B)^T P + P(A+B) - PB - B^T P\big)x(t) \\
&\quad - x^T(t)\big((A+B)^T PC - PB - B^T PC\big)x(t-h) \\
&\quad - x^T(t-h)\big(C^T P(A+B) - C^T PB - B^T P\big)x(t) \\
&\quad - x^T(t-h)\big(C^T PB + B^T PC\big)x(t-h).
\end{aligned} \tag{2.9}
$$

$$
\begin{aligned}
\dot{V}_2(t,x(\tau)) &= \frac{d}{dt}\left(x(t) - Cx(t-h) + B\int_{t-h}^{t} x(\tau)d\tau \right)^T W \left(x(t) - Cx(t-h) + \int_{t-h}^{t} x(\tau)d\tau \right) \\
&\quad + \left(x(t) - Cx(t-h) + B\int_{t-h}^{t} x(\tau)d\tau \right)^T W \frac{d}{dt}\left(x(t) - Cx(t-h) + \int_{t-h}^{t} x(\tau)d\tau \right) \\
&= \big((A+B)^T x^T(t)\big)W\left(x(t) - Cx(t-h) + B\int_{t-h}^{t} x(\tau)d\tau \right) \\
&\quad + \left(x(t) - Cx(t-h) + B\int_{t-h}^{t} x(\tau)d\tau \right)^T W \big((A+B)\ \ x\ \ (t)\big) \\
&= \big((A+B)^T x^T(t)\big)W\big((I+B)x(t) - (C+B)x(t-h)\big) \\
&\quad + \big((I+B)x(t) - (C+B)x(t-h)\big)^T W\big((A+B)\ \ x\ \ (t)\big) \\
&= x(t)\big((A+B)^T W(I+B)\big)x^T(t) + x^T(t)\big((I+B)^T W(A+B)\big)x(t) \\
&\quad - x^T(t)\big((A+B)^T W(C+B)\big)x(t-h) - x^T(t-h)\big((C+B)^T W(A+B)\big)x^T(t) \\
&= x^T(t)\big((A+B)^T W(I+B) + (I+B)^T W(A+B)\big)x(t) \\
&\quad - x^T(t)\big((A+B)^T W(C+B)\big)x(t-h) \\
&\quad - x^T(t-h)\big((C+B)^T W(A+B)\big)x(t).
\end{aligned} \tag{2.10}
$$

$$
\begin{aligned}
\dot{V}_3(t,x(\tau)) &= \int_{t-h}^{t} x^T(\tau)Nx(\tau)d\tau \\
&= x^T(t)Nx(t) - x^T(t-h)\,N\,x(t-h).
\end{aligned} \tag{2.11}
$$

Combining equations (2.9), (2.10) and (2.11) to obtain, $\dot{V}(t,x(\tau)) = \dot{V}_1(t,x(\tau)) + \dot{V}_2(t,x(\tau)) + \dot{V}_3(t,x(\tau))$.
That is

$$
\begin{aligned}
\dot{V}(t,x(\tau)) = \ & x(t)^T\big(-Q - PB - B^T P + (A+B)^T W(I+B) + (I+B)^T W(A+B) + N\big)x(t) \\
& - x(t)^T\big((A+B)^T PC - PB - B^T PC + (A+B)^T W(C+B)^T\big)x(t-h) \\
& x(t-h)^T\big(B^T P - C^T P(A+B) + C^T PB + (C+B)^T W(A+B)\big)x(t) \\
& - x(t-h)^T\big(C^T PB + B^T PC + N\big)x(t-h).
\end{aligned}
$$

$$
\dot{V}(t,x(\tau)) = \big(x(t)^T \quad x(t)^T \quad x(t-h)^T \quad x(t-h)^T\big)
\begin{pmatrix}
\sum_{11} & 0 & 0 & 0 \\
0 & \sum_{22} & 0 & 0 \\
0 & 0 & \sum_{33} & 0 \\
0 & 0 & 0 & \sum_{44}
\end{pmatrix}
\begin{pmatrix} x(t) \\ x(t-h) \\ x(t) \\ x(t-h) \end{pmatrix}. \tag{2.12}
$$

The negative definiteness of the linear matrix inequality is satisfied if one or three of the diagonal element of the matrix in equation (2.12) has/have negative value/values. By the theorem, this implies asymptotic stability of system (2.1).

3.0 Illustration

Consider the *elastic wire-mass system without friction* problem consisting of mass m that translates along a horizontal surface. The location of the mass is identified by the coordinates of its centre of mass H, which is attached to an elastic wire stretched with ends A and H. The governing modeling equation is

$$m\frac{d^2x}{dt} = -kx, \tag{3.0}$$

where x is the distance that the mass m translate, k is the tension constant which act in the opposite direction to the force. The mass m does not react instantaneously to the drag force (tension), until the later is equal to the weight of the mass. Therefore, there exists a time lag h for which the mass react to the tension kx.

Modifying system (3.0) by incorporating the time lag h yields a neutral delay system as follows

$$m\ddot{x}(t) + m\ddot{x}(t-h) = -kx(t). \tag{3.1}$$

By defining $\dot{x}_1(t) = x_2(t)$ and $\dot{x}_1(t-h) = x_2(t-h)$, a first order linear delay system of (3.1) is written as

$$\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\dot{x}_1(t-h)\\ \dot{x}_2(t-h)\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\frac{k}{m} & 0\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x_1(t-h)\\ x_2(t-h)\end{pmatrix}$$

or

$$\dot{x}(t) - C\dot{x}(t-h) = Ax(t) + Bx(t-h). \tag{3.2}$$

Resolving for the symmetric positive definite matrix P using the quadratic matrix equation

$(A+B)^T P + P(A+B) = -Q$, then

$$\begin{pmatrix}2l_{11} & 2l_{21} & 0\\ l_{12} & (l_{11}+l_{22}) & l_{21}\\ 0 & 2l_{12} & 2l_{22}\end{pmatrix}\begin{pmatrix}p_{11}\\ p_{12}\\ p_{22}\end{pmatrix} = \begin{pmatrix}-1\\ 0\\ -1\end{pmatrix}, \tag{3.3}$$

where l_{11}, l_{12}, l_{21} *and* l_{22} are the elements of the stable matrix $L = A+B$ and matrix $Q = I$. Substituting for l_{11}, l_{12}, l_{21} *and* l_{22} from the stable matrix $L = \begin{pmatrix}0 & 1\\ -\frac{k}{m} & -1\end{pmatrix}$, into system (3.3) the following equations are obtained,

$$\begin{aligned} -2\frac{k}{m}p_{12} &= -1\\ p_{11} - p_{12} - \frac{k}{m}p_{22} &= 0\\ 2p_{12} - 2p_{22} &= -1. \end{aligned} \tag{3.4}$$

Solving equation (3.4) for p_{11}, p_{12} *and* p_{22}, symmetry matrix P is obtained as

$$P = \begin{pmatrix}\frac{k^2 - km - m^2}{2km} & \frac{m}{2k}\\ \frac{m}{2k} & \frac{k-m}{2k}\end{pmatrix}.$$

This is positive definite matrix if and only if $\frac{k-m}{2k}\left(\frac{k^2 - km - m^2}{2km}\right) > \frac{m^2}{4k^2}$, and this occur at the point where $k > m$.

3.1 Computation Of The Maximum Delay

Considering the approximating matrix $L = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -1 \end{pmatrix}$, the eigen-value $\lambda < 0$ is computed as $\lambda = -1 \pm \sqrt{\frac{m-4k}{m}}$, which is a complex root with negative real part for all $k > m$. Therefore L is a stable matrix.

The value of h_m is thus computed using the hypothesis of theorem I, that is for $\lambda = -1 \pm \sqrt{\frac{m-4k}{m}}$ then,

$$\left| \frac{1}{\lambda}(B + \lambda I) - (B + \lambda C)e^{-\lambda h_{\max}} \right| = 0 \cdot$$

3.2 ASSYMPTOTIC STABILTY ANALYSIS

By theorem II, the negative definiteness of the diagonal linear matrix inequality

$$\begin{pmatrix} \sum_{11} & 0 & 0 & 0 \\ 0 & \sum_{22} & 0 & 0 \\ 0 & 0 & \sum_{33} & 0 \\ 0 & 0 & 0 & \sum_{44} \end{pmatrix} < 0,$$

implies that the system solution is asymptotically stable.

Using matrices $A, B, C, I+B$ as defined by system (3.2),

$A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I + B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, the symmetric positive definite matrix P computed above, matrix $Q = I$, and for appropriate choice of symmetric matrices $W, N \in R^{nxn}$, the following are obtained;

$$\sum_{11} = \begin{vmatrix} -1 - 2w_{12}\frac{k}{m} + n_{11} & \frac{m}{2k} + w_{11} - w_{12} + n_{12} \\ -\frac{m}{2k} + w_{11} - w_{12} + n_{12} & -1 + k - m - n_{22} \end{vmatrix},$$

$$\sum_{22} = \begin{vmatrix} \frac{1}{2} + w_{12}\frac{k}{m} & \frac{k^2 - km + m}{2k} - 2w_{22}\frac{k}{m} \\ \frac{k^2 - km + m}{2k} - w_{11} + w_{12} & \frac{1}{2} + 2(w_{12} - w_{22}) \end{vmatrix},$$

$$\sum_{33} = \begin{vmatrix} \frac{1}{2} + w_{12}\frac{k}{m} & \frac{k^2 - km + 2m^2 + 2km^2}{2km} - w_{11} + w_{12} \\ \frac{m^2 - k^2 + m}{2km} - 2w_{22}\frac{k}{m} & \frac{-k - 2m}{2k} + 2(w_{12} - w_{22}) \end{vmatrix}, \text{ and}$$

$$\sum_{44} = \begin{vmatrix} n_{11} & \frac{m}{2k} + n_{12} \\ -n_{12} & 2(\frac{k-m}{2k}) + n_{22} \end{vmatrix}.$$

Then, the value of one or three of $\sum_{11}$, $\sum_{22}$, $\sum_{33}$ and $\sum_{44}$ must be negative (for appropriate choice of symmetric matrices $W, N \in R^{nxn}$ and $k > m$) for the linear matrix inequality to be negative definite and the system solution is asymptotically stable.

4.0 Conclusion

In this paper, the difference integral equation of the integrodifferential of the transformed neutral system was used to formulate a computational approach for the maximum time-lag (h_m), at which the system equation attain asymptotic stability. The integrodifferential of the system, the symmetric positive definite matrix P which is a unique solution of quadratic matrix equation, and the symmetric matrices W, N were utilized to generate a Lyapunov functional that satisfy the Lyapunov- Krasovskii conditions for asymptotic stability. Numerical illustration employed confirms the results of the analysis.

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