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Relationships Between Controllable and Observable Matrices Which Ensure Controllable System to be Observable.

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Abstract

Generally, the controllability and observability of controllable systems were treated separately. In this paper, through the duality condition, the relationships between matrices of controllable system and observable system which ensure that controllable systems are observable are derived. This new result was established as a result of comparison of controllable and observable gramian matrices.

Keywords: Controllability, Observability, Duality, Skew- Symmetric matrix, Transpose.

1 Introduction:

For a dynamical system to stand the test of time, controllability, observability and stability conditions of such systems must be known. This is one of the major problems facing control system engineers. Observability is the ability to observe or to measure the output of all the parameters or state variables in the system. Controllability is the ability to move a system from any given state to another desired state, using the input u. The controllability condition is that the existence of such an input should be assured. Stability, on its own, is often phrased as the bounded response of the system to any bounded input.

For any dynamical system to be successful, it must have these properties; that is observability, controllability and stability properties. For linear control systems such properties can be maintained with minimal conditions. For nonlinear control systems, uncertainties present a big challenge to the system engineers who work hard to maintain these properties using limited information.

For purposes of clarity, let us consider these properties one by one. Let us consider the system represented by

$$\dot{x} = Ax(t) + Bu(t) \qquad (1.1)$$
$$y(t) = Cx(t)$$

Where x, u and y are n-vector, m-vector and p-vector respectively and A,B, and C are respectively n x n, n x m, and p x n constant matrices which we here refer to as controllable matrices.

2. Controllability.

For a linear system given by (1.1), if there exists an input u which transfers the initial state $x(o) = x_0$ to the state $x(t_1) = x_1$ in a finite time t_1, the state x_0 is said to controllable. If all the initial states are controllable, the system is said to be completely controllable. If $x_1 = \bar{0}$, the zero state, the system is said to be null-controllable (Eke, 2000).

It is known (Stephen, 1975) that the unique solution of the linear state equation

$$\dot{x} = A(t)x(t) + B(t)u(t)$$
$$x(t_0) = x_0 \qquad (2.1)$$

is given by

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$$x(t) = \phi(t,t_0)[x_0 + \int_{t_0}^{t} \phi(t,s)B(s)u(t)ds] \tag{2.2}$$

where $\phi(t,s)$ is the transition matrix satisfying

$$\dot{\phi}(t,t_0) = A(t)\phi(t,t_0) \quad \text{for} \quad t \geq t_0 .$$

and $\phi(0,0) = I,$ the identity matrix. The solution of (2.1) by Katsuhisa et al (1988), is given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}B(s)u(s)ds \tag{2.3}$$

If the system is null-controllable, there exists an input $x(t_1) = x_1 = 0$ at a finite time $t = t_1$ such that after multiplying (2.3) by e^{-At_1}, we get

$$-x_0 = \int_0^{t_1} e^{-As}B(s)u(s)ds \tag{2.4}$$

So any controllable system will satisfy (2.4) and for a completely controllable systems, every state x_0 in $\Re^n$ satisfies (2.4) with $t_1 > 0$. From (2.4) it is found that complete controllability of a system depends on controllable matrices A and B and is independent of the output matrix C.

Definition 2.1 Dauer (1971)

A set of vector functions $\{x_1, x_2, ..., x_k\}$ is said to be linearly independent over finite interval I if for every set of non zero vectors $(a_1, a_2, ..., a_k)\ \varepsilon\ C^k$, there exists a subset J of I with positive measures such that $\sum_{i=1}^{k} a_i x_i \neq 0\ \ for\ all\ t\ \varepsilon\ J$.

The following simple theorems , due to Dauer (1971), characterize controllability; that is

Theorem 2.1 Assume $B\ \varepsilon\ L^\infty$, the system (1.1) is completely controllable in L^1 if and only if the rows of the matrix functions $\{X^{-1}(t)B(t)\}$ are linearly independent over I.

Theorem 2.2.

Let $2 \leq p < \infty$, and assume that $B\,\varepsilon\ L^P$, system (1.1) is completely controllable in L^p if and only if $W = \int X^{-1}(s)B(s)B^T(s)X^{-1^T}(s)ds$ is positive definite.

Theorem 2.3

Suppose $A\varepsilon\ L^1$ and $B\ \varepsilon\ L^p, 1 \leq p \leq \infty$ and assume that the system (2.1) is completely controllable in $L^q, \frac{1}{p} + \frac{1}{q} = 1.$ There exists $\varepsilon > 0$ such that if $|A - C\ |_p + |B - D|_p < \varepsilon$, then the system

$$\dot{y} = C(t)y + D(t)u \tag{2.5}$$

is completely controllable in L^q.

Lemma 2.1

Dauer (1971) Suppose W is positive definite n x n constant matrix. There exists $\varepsilon > 0$ such that if the constant n x n matrix V satisfies $|W - V| < \varepsilon,$ then V is positive definite.

If we assume the conditions of Theorem 2.3. then, for every r, $p \leq r < \infty,$ there exists $\varepsilon > 0$ such that if $|A-C|_r + |B-D|_r < \varepsilon,$ then (2.5) is completely controllable in L^q.

Apart from the above three rather simple theorems due to Dauer (1971), we have the following crucial fact in complete controllability, that is

Theorem 2.4 Katsuhisa et al (1988)

The necessary and sufficient condition for the system (1.1) to be completely controllable is one of the following conditions;

1 $W(0,t_1) = \int_0^{t_1} e^{-At} BB^T e^{-A^T t} dt$ is nonsingular.

[$W(0,t_1)$ is called controllability gramian and $(.)^T$ denotes matrix transpose]

2 The controllability matrix $\xi = [B, AB, A^2B, ..., A^{n-1}B]$ has rank n.

Proof:

Condition 1. Sufficiency:

If $W(0,t_1)$ given above is singular, the following input can be applied to the system

$$u(t) = -B^T e^{-A^T t} W^{-1}(0,t_1) x_0 \tag{2.6}$$

For the input (2.6), the state of the system (2.3) is given by

$$x(t_1) = e^{At_1} x_0 - e^{At_1} \{ \int_0^{t_1} e^{-As} BB^T e^{-A^T s} ds \} W^{-1}(0,t_1) x_0 = 0$$

for any initial state x_0. Therefore, the system (A , B) is controllable.

Necessity:

Assume that although $W(0,t_1)$ is singular for any $t_1 > 0$, there exists a non zero n-vector α such that

$$\alpha^T W(0,t_1)\alpha = \int_0^{t_1} \alpha^T e^{-As} BB^t e^{-A^T s} \alpha ds = 0 \tag{2.7}$$

which yields for any t

$$\alpha^T e^{-At} = 0^T, t \geq 0, \alpha \neq 0 \tag{2.8}$$

From the assumption of controllability, there exists an input u satisfying (2.4). Therefore from (2.4) and (2.8)

$$-\alpha^T x_0 = \alpha^T \int_0^{t_1} e^{-As} B(s)u(s)ds = 0 \tag{2.9}$$

holds for any initial state x_0. By choosing $x_0 = \alpha,$ (2.9) gives $\alpha = 0,$ which contradicts the non-zero property of α.Therefore the non-singularity of $W(0,t_1)$ is proved.

Condition 2. Sufficiency;

Let us first assume that if the rank of $\xi = n$, still the system is not controllable. We show that this is a contradiction leading to the conclusion that the system is controllable. By the above assumption, that the system is not controllable and that the rank of $\xi = n, W(0,t_1)$ is singular. Therefore (2.8), that is $\alpha^T e^{At} B = 0^T, T \geq 0, \alpha \neq 0,$ holds. Derivatives of equation (2.8) at t = 0 yields

$$\alpha^T A^k B = 0, \quad k = 0,1,2,...,n-1. \tag{2.10}$$

which is equivalent to

$$\alpha^T [B, AB, A^2B,..., A^{n-1}B] = \alpha^T \xi = 0^T \tag{2.11}$$

This contradicts the assumption that the rank $\xi = n,$ so the system is completely controllable.

Necessity:

Let us assume that the system is completely controllable, but rank $\xi < n.$ From this assumption, there exists a non-zero α satisfying (2.11).From the Cayley-Hamilton theorem (Chi-Tsong, 1984), A^{n+1} can be expressed as a linear combination of $I, A, ..., A^{n+1}$, which yields

$$\alpha^T e^{-As} B = 0^T, t \geq 0, \alpha \neq 0.$$

So,

$$0 = \int_0^{t_1} \alpha^T e^{-As} BB^T e^{-A^T s} \alpha ds = \alpha^T W(0, t_1)\alpha \qquad (2.12)$$

Since the system is completely controllable $W(0, t_1)$ is non singular from condition 1 above. Then α in (2.12) is zero, which contradicts the assumption that α is non zero. So rank $\xi = n,$ completing the proof of the theorem.

2 OBSERVABILITY.

When using the output of the system (1.1) measured from time t = 0 to time $t = t_0$, if the initial state x(0) = x_0 is uniquely determined, x_0 is said to be observable. The output of the system (1.1) is given by Katsuhisa et al (1988) as

$$y(t) = Ce^{At} x_0 + C\int_0^t e^{A(t-s)} Bu(s)ds \qquad (3.1)$$

We note that the output and the input can be measured and used. So a signal η can be obtained from u and y using the formula by Katsuhisa et al (1988)

$$\eta(t) = y(t) - C\int_0^t e^{A(t-s)} Bu(s)ds \qquad (3.2)$$

$$= Ce^{At} x_0$$

Since p is usually smaller than n, x_0 can not be determined uniquely by $\eta(t)$ at a specific time t. But when the signal $\eta(t)$ is available over a time interval from 0 to t, and the system is completely observable, the initial state x_0 can be uniquely determined. If (2.14) is multiplied by $e^{A^T t} C^T$ and integrated from 0 to t_1, we get

$$\{\int_0^{t_1} e^{A^T t} C^T Ce^{At} dt\} x_0 = \int_0^{t_1} e^{A^T t} C^T \eta(t) dt \qquad (3.3)$$

Let us define an n x n matrix $M(0, t_1)$ by

$$M(0, t_1) = \int_0^{t_1} e^{A^T t} C^T Ce^{At} dt \qquad (3.4)$$

If $M(0, t_0)$ is nonsingular, x_0 is determined uniquely from (3.3) as

$$x_0 = M^{-1}(0, t_1) \int_0^{t_1} e^{At} C\eta(t) dt \qquad (3.5)$$

From (3.5), we see that the non singularity of $M(0, t_1)$ for $t_1 \geq 0$ is a sufficient condition for the system (1.1) to be completely observable. $M(0, t_1)$ is non singular is proved in Katsuhisa et al (1988) page 52. This then leads to the following theorem;

Theorem 3.1 Katsuhisa et al (1988).

A necessary and sufficient condition for the system (1.1) to be completely observable is one of the following equivalent conditions:

1. $M(0,t_1)=\int_0^{t_1} e^{A^T t}C^T C e^{At}dt$ is non singular.

2. The observability matrix defined as n x np matrix

$[C^T, A^T C^T, (A^T)^2 C^T, ..., (A^T)^{n-1}C^T]$

has rank n.

Proof:

For 1, this has been proved from the derivation of (2.7). For 2, it can be proved from 1 in a similar way as in Theorem 2.4.

We note that controllability and observability have a link between them as stated in the following duality theorem.

Theorem 3.2 (Duality) Stephen (1975)

The system

$$\begin{aligned}\dot{x} &= A(t)x(t)+B(t)u(t)\\ y &= C(t)x(t)\end{aligned} \tag{3.6}$$

is completely controllable if and only if the dual system

$$\begin{aligned}\dot{x} &= -A^T(t)x(t)+C^T(t)u(t)\\ y &= B^T(t)x(t)\end{aligned} \tag{3.7}$$

is completely observable.

MAIN RESULT.

The dual theorem leads us to the main result of this paper.

Theorem 4.1

The necessary condition for a controllable system

$$\begin{aligned}\dot{x} &= A(t)x(t)+B(t)u(t)\\ y &= C(t)x(t)\end{aligned} \tag{4.1}$$

to be observable is that the n x n matrix A be skew-symmetric and matrices B and C be transpose of each other.

Proof:

Since the system (4.1) is controllable, the controllability gramian $W(0,t_1)$ defined by

$$W(0,t_1)=\int_0^{t_1} e^{-At}BB^T e^{-A^T t}dt \tag{4.2}$$

is non singular. From duality Theorem 3.2, if (4.1) is to be observable, then the observability gramian $M(0,t_1)$ defined by

$$M(0,t_1)=\int_0^{t_1} e^{A^T t}C^T C e^{At}dt \tag{4.3}$$

must be non singular. By comparing (4.2) and (4.3), we see that if $A^T=-A \, and \, C=B^T \ (B=C^T),$ the two equations are the same. Since (4.2) is non singular, (4.3) will also be non singular, and so the observability condition of (4.1) is assured. $A^T=-A$ implies that A is skew-symmetric, and $C=B^T, B=C^T$ means that they are of transpose to each other.

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