Journal of the Nigerian Association of Mathematical Physics
Volume **19 (November, 2011), pp 49 – 54**

Gravitational Metric Tensor Exterior to Rotating Homogeneous Spherical Masses

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Abstract

The covariant and contravariant metric tensors exterior to a homogeneous spherical body rotating uniformly about a common ϕ axis with constant angular velocity ω is constructed. The constructed metric tensors in this gravitational field have seven non-zero distinct components.The Lagrangian for this gravitational field is constructed. It is used to derive Einstein's planetary equation of motion and photon equation of motion in the vicinity of the rotating homogeneous spherical mass.

Introduction:

In recent articles [1-5], we introduced a unique and profound method for obtaining metric tensors and hence solutions to gravitational field equations exterior and interior to all regular distributions of mass. The method yields field equations with only one unknown function satisfactorily comparable to Newton's gravitational scalar potential for the distribution under consideration. This is profound because it puts Einstein's geometrical theory of gravitation on the same footing with Newton's dynamical theory of gravitation. The determination of the unknown function paves the way for the derivation of all other parameters for the field; same as the determination of the scalar potential in Newton's theory paves the way for all other characteristics of the field such as force.

In this article, the metric tensors exterior to a massive homogeneous spherical distribution of mass rotating uniformly about a common axis with a constant angular velocity are systematically constructed. The mass distribution is placed in empty space and hence the stress tensor is zero so the field equation reduces to Einstein's tensor equal to zero.

1. Covariant and Contravariant Metric Tensors

Consider a coordinate system $\bar{X}\left(\bar{r},\bar{\theta},\bar{\phi}\right)$ fixed to a static spherical body situated in empty space. Suppose the mass distribution within the spherical body is homogeneous and rotating with uniform angular velocity about a fixed diameter. The covariant metric tensor exterior to the spherical mass distribution is found to be given [5] as;

$$g_{00} = 1 + \frac{2}{c^2} f\left(r,\theta\right) \tag{1}$$

$$g_{11} = -\left[1 + \frac{2}{c^2} f\left(r,\theta\right)\right]^{-1} \tag{2}$$

$$g_{22} = -r^2 \tag{3}$$

$$g_{33} = -r^2 \sin^2\theta \tag{4}$$

$$g_{\mu\nu} = 0; \quad otherwise \tag{5}$$

where $r > R,$ the radius of the sphere. $f\left(r,\theta\right)$ is an arbitrary function determined by the rotating mass distribution within

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the sphere and is a function of only the radial distance and polar angle. Instructively, the metric tensor satisfies Einstein's field equations and the invariance of the line element; by virtue of their construction [6]. An outstanding theoretical and astrophysical consequence of the metric tensor, equations (1) – (5) is that Einstein's field equations constructed using the metric tensor has only one unknown $f(r,\theta)$. A solution of these field equations gives an explicit expression for the function $f(r,\theta)$. In approximate gravitational fields, the arbitrary function $f(r,\theta)$ is conveniently equated to the gravitational scalar potential exterior to the homogeneous rotating spherical mass distribution [1-6].

Now, consider a second coordinate system $X(r,\theta,\phi)$ fixed to the spherical body and rotating uniformly about a common $z\overline{z}$ axis with angular velocity ω as shown in fig. 1.

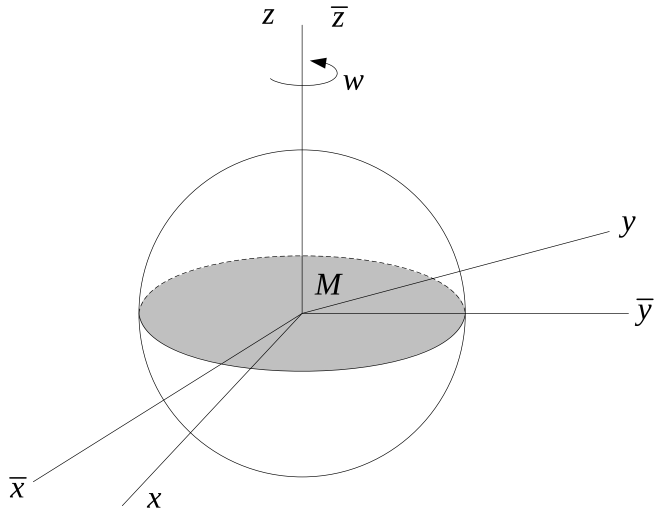

Fig.1: Uniformly Rotating Spherical Body

Thus, the spherical body as well as the homogeneous mass distribution within it is now rotating uniformly about a fixed diameter. The following transformations between the two coordinate frames can be considered

$$\overline{t} = t \tag{6}$$

$$\overline{r} = r\cos\omega t - \theta\sin\omega t \tag{7}$$

$$\overline{\theta} = r\sin\omega t + \theta\cos\omega t \tag{8}$$

$$\overline{\phi} = \phi \tag{9}$$

The expression for the proper time in the static frame is given as

$$c^2 d\overline{\tau}^2 = c^2 g_{00} d\overline{t}^2 - g_{11} d\overline{r}^2 - g_{22} d\overline{\theta}^2 - g_{33} d\overline{\phi}^2 \tag{10}$$

and by the invariance of proper time or line element, it is related to that in the rotating frame by;

$$c^2 d\tau^2 = c^2 d\overline{\tau}^2 \tag{11}$$

Thus by the expansion of the right hand side of equation (11) using (10) and the transformations (6) to (9), the covariant metric tensor exterior to the rotating spherical mass is obtained as

$$g_{00} = 1 + \frac{2}{c^2} f(r,\theta) - r^2\omega^2\left[\theta\sin\omega t + r\cos\omega t\right]^2 - \omega^2\left[r\sin\omega t + \theta\cos\omega t\right]^2\left[1 + \frac{2}{c^2} f(r,\theta)\right]^{-1} \tag{12}$$

$$g_{01} \equiv g_{10} = r^2\omega\left[\theta\sin\omega t + r\cos\omega t\right]\sin\omega t + \omega\left[r\sin\omega t + \theta\cos\omega t\right]\left[1 + \frac{2}{c^2} f(r,\theta)\right]^{-1}\cos\omega t \tag{13}$$

$$g_{02} \equiv g_{20} = 2r^2\omega\left[\theta\sin\omega t + r\cos\omega t\right]\cos\omega t - \omega\left[r\sin\omega t + \theta\cos\omega t\right]\left[1 + \frac{2}{c^2} f(r,\theta)\right]^{-1}\sin\omega t \tag{14}$$

$$g_{11} = -r^2\sin^2\omega t - \left[1 + \frac{2}{c^2} f(r,\theta)\right]^{-1}\cos^2\omega t \tag{15}$$

$$g_{12} \equiv g_{21} = \frac{1}{2}\left\{-r^2 + \left[1+\frac{2}{c^2}f(r,\theta)\right]^{-1}\right\}\sin 2\omega t \tag{16}$$

$$g_{22} = -r^2\cos^2\omega t - \left[1+\frac{2}{c^2}f(r,\theta)\right]^{-1}\sin^2\omega t \tag{17}$$

$$g_{33} = -r^2\sin^2\theta \tag{18}$$

$$g_{\mu\nu} = 0; \quad otherwise \tag{19}$$

Remarkably, this metric tensor has seven non-zero components unlike the metric in [5]. Thus the rotation of the homogeneous mass content introduces additional off-diagonal components. Also, the leading diagonal components, equations (12), (15) and (17) are dependent on the angle of rotation ω

To construct the contravariant metric tensor for the gravitational field exterior to the rotating sphere, $g^{\mu\nu}$ we use [7]

$$g^{\mu\nu} = \frac{\text{cofactor of } g_{\mu\nu} \text{ in g}}{\text{g}} \tag{20}$$

where

$$\text{g}=\begin{vmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{vmatrix} \tag{21}$$

Thus in this exterior gravitational field, equation (21) becomes;

$$g = \begin{vmatrix} g_{00} & g_{01} & g_{02} & 0 \\ g_{10} & g_{11} & g_{12} & 0 \\ g_{20} & g_{21} & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{vmatrix} \tag{22}$$

or

$$g_{00}g_{33}\left(g_{11}g_{22} - g_{12}{}^2\right) - g_{01}g_{33}\left(g_{10}g_{22} - g_{12}g_{20}\right) - g_{02}g_{33}\left(g_{10}g_{21} - g_{11}g_{20}\right) \tag{23}$$

Hence the contravariant metric tensor has the following components;

$$g^{00} = \frac{\left(g_{11}g_{22} - g_{12}{}^2\right)}{\Xi(g_{\mu\nu})} \tag{24}$$

$$g^{01} \equiv g^{10} = \frac{\left(g_{01}g_{22} - g_{02}g_{12}\right)}{\Xi(g_{\mu\nu})} \tag{25}$$

$$g^{02} \equiv g^{20} = \frac{\left(g_{10}g_{12} - g_{11}g_{02}\right)}{\Xi(g_{\mu\nu})} \tag{26}$$

$$g^{11} = \frac{\left(g_{00}g_{22} - g_{02}{}^2\right)}{\Xi(g_{\mu\nu})} \tag{27}$$

$$g^{12} \equiv g^{21} = \frac{\left(g_{00}g_{12} - g_{01}g_{02}\right)}{\Xi(g_{\mu\nu})} \tag{28}$$

$$g^{22} = \frac{\left(g_{00}g_{11} - g_{01}{}^2\right)}{\Xi(g_{\mu\nu})} \tag{29}$$

$$g^{33} = \frac{1}{g_{33}} \tag{30}$$

$$g^{\mu\nu} = 0; \quad otherwise \tag{31}$$

where

$$\Xi(g_{\mu\nu}) = g_{00}\left(g_{11}g_{22} - g_{12}{}^2\right) - g_{01}\left(g_{10}g_{22} - g_{12}g_{20}\right) - g_{02}\left(g_{10}g_{21} - g_{11}g_{20}\right) \tag{32}$$

Thus, the metric tensor in this gravitational field has seven non-zero distinct components.

2. Orbits in the Vicinity of a Rotating Homogeneous Spherical Mass

The Lagrangian in the space time exterior to any astrophysical body is defined [8] as

$$L = \frac{1}{c}\left(-g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right)^{\frac{1}{2}} \tag{33}$$

Thus in the gravitational field of a rotating homogeneous spherical mass, the Lagrangian, equation (33) becomes

$$L = \frac{1}{c}\left(\begin{array}{l} -g_{00}\left(\frac{dt}{d\tau}\right)^2 - 2g_{01}\left(\frac{dt}{d\tau}\right)\left(\frac{dr}{d\tau}\right) - 2g_{02}\left(\frac{dt}{d\tau}\right)\left(\frac{d\theta}{d\tau}\right) - g_{11}\left(\frac{dr}{d\tau}\right)^2 \\ -2g_{12}\left(\frac{dr}{d\tau}\right)\left(\frac{d\theta}{d\tau}\right) - g_{22}\left(\frac{d\theta}{d\tau}\right)^2 - g_{33}\left(\frac{d\phi}{d\tau}\right)^2 \end{array}\right)^{\frac{1}{2}} \tag{34}$$

Considering orbits in the equatorial plane of a homogeneous spherical mass, then, $\theta \equiv \frac{\pi}{2}$ and thus the Lagrangian, equation (34) reduces to:

$$L = \frac{1}{c}\left(-g_{00}\left(\frac{dt}{d\tau}\right)^2 - 2g_{01}\left(\frac{dt}{d\tau}\right)\left(\frac{dr}{d\tau}\right) - g_{11}\left(\frac{dr}{d\tau}\right)^2 - g_{33}\left(\frac{d\phi}{d\tau}\right)^2\right)^{\frac{1}{2}} \tag{35}$$

Substituting the explicit expressions for the components of the covariant metric tensor in equation (35) yields

$$L = \frac{1}{c}\left[\begin{array}{l} -\left(W - r^2\omega^2\Phi^2 - \omega^2 R^2 W^{-1}\right)\dot{t}^2 - 2\left(r^2\omega^2\Phi\sin\omega t + \omega R W^{-1}\cos\omega t\right)^{-1}\dot{t}\dot{r} \\ +\left(r^2\sin^2\omega t - W^{-1}\cos^2\omega t\right)\dot{r}^2 + r^2\dot{\phi}^2 \end{array}\right]^{\frac{1}{2}} \tag{36}$$

where dot represents differentiation with respect to proper time and

$$W = 1 + \frac{2}{c^2}f(r,\theta), \quad \Phi = \theta\sin\omega t + r\cos\omega t, \quad R = r\sin\omega t + \theta\cos\omega t \tag{37}$$

It is well known [8] that the Lagrangian $L = \in$, with $\in = 1$ for time like orbits and $\in = 0$ for null orbits. Setting $L = \in$ in equation (36) and squaring both sides yields;

$$\begin{aligned} c^2 \in^2 = &-\left(W - r^2\omega^2\Phi^2 - \omega^2 R^2 W^{-1}\right)\dot{t}^2 - 2\left(r^2\omega^2\Phi\sin\omega t + \omega R W^{-1}\cos\omega t\right)^{-1}\dot{t}\dot{r} \\ &+\left(r^2\sin^2\omega t - W^{-1}\cos^2\omega t\right)\dot{r}^2 + r^2\dot{\phi}^2 \end{aligned} \tag{38}$$

The shapes of orbits (as a function of the azimuthal angle) are important in most applications of general relativity [8]. Hence, it is instructive to transform equation (38) into an equation in terms of the azimuthal angle ϕ. Now, consider the following transformation, with $r = r(\phi)$ and $u(\phi) = \dfrac{1}{r(\phi)}$ then

$$\dot{r} = \dot{\phi}\frac{dr}{d\phi} \qquad \text{or} \qquad \dot{r} = \frac{l}{1+r^2}\frac{dr}{d\phi} \tag{39}$$

but

$$\frac{dr}{d\phi} = \frac{dr}{du}\frac{du}{d\phi} \qquad \text{or} \qquad \frac{dr}{d\phi} = -u^{-2}\frac{du}{d\phi} \tag{40}$$

and thus

$$\dot{r} = -\frac{l}{1+u^2}\frac{du}{d\phi} \tag{41}$$

Now, imposing the transformations on equation (38) and simplifying yields

$$\begin{aligned}&\frac{1}{(1+u^2)^2}\left(\frac{du}{d\phi}\right)^2 + \frac{2\left(u^{-2}\omega^2\Phi\sin\omega t + \omega R W^{-1}\cos\omega t\right)^{-1}}{\left(u^{-2}\sin^2\omega t - W^{-1}\cos^2\omega t\right)}\dot{t}\left(\frac{1}{1+u^2}\right)\frac{du}{d\phi}\\ &-\frac{\left(W - u^{-2}\omega^2\Phi^2 - \omega^2 R^2 W^{-1}\right)}{\left(u^{-2}\sin^2\omega t - W^{-1}\cos^2\omega t\right)}\dot{t}^2 - \frac{c^2\in^2}{\left(u^{-2}\sin^2\omega t - W^{-1}\cos^2\omega t\right)} = 0\end{aligned} \tag{42}$$

For time like orbits, $\in = 1$ and hence equation (42) reduces to;

$$\begin{aligned}&\frac{1}{(1+u^2)^2}\left(\frac{du}{d\phi}\right)^2 + \frac{2\left(u^{-2}\omega^2\Phi\sin\omega t + \omega R W^{-1}\cos\omega t\right)^{-1}}{\left(u^{-2}\sin^2\omega t - W^{-1}\cos^2\omega t\right)}\dot{t}\left(\frac{1}{1+u^2}\right)\frac{du}{d\phi}\\ &-\frac{\left(W - u^{-2}\omega^2\Phi^2 - \omega^2 R^2 W^{-1}\right)}{\left(u^{-2}\sin^2\omega t - W^{-1}\cos^2\omega t\right)}\dot{t}^2 - \frac{c^2}{\left(u^{-2}\sin^2\omega t - W^{-1}\cos^2\omega t\right)} = 0\end{aligned} \tag{43}$$

This is the planetary equation of motion in the vicinity of the rotating homogeneous spherical mass. It can be solved to obtain the perihelion precision of planetary orbits in this gravitational field. Since light rays travel on null geodesics, we have $\in = 0$ and equation (42) becomes

$$\frac{1}{(1+u^2)^2}\left(\frac{du}{d\phi}\right)^2 + \frac{2\left(u^{-2}\omega^2\Phi\sin\omega t + \omega R W^{-1}\cos\omega t\right)^{-1}}{\left(u^{-2}\sin^2\omega t - W^{-1}\cos^2\omega t\right)}\dot{t}\left(\frac{1}{1+u^2}\right)\frac{du}{d\phi} - \frac{\left(W - u^{-2}\omega^2\Phi^2 - \omega^2 R^2 W^{-1}\right)}{\left(u^{-2}\sin^2\omega t - W^{-1}\cos^2\omega t\right)}\dot{t}^2 = 0 \tag{44}$$

Equation (44) is the photon equation of motion in the vicinity of a rotating massive homogeneous spherical body.

3. Conclusion

The consequences of the results in this article are:

i. With the construction of the metric tensor, gravitational field equations for this field become eminent. The field equations are constructed via the coefficients of affine connection, Riemann-Christoffel tensor and Ricci tensor. The obtained field equations have only one unknown which can be determined completely. Thus by taking into consideration physical and astrophysical properties of the massive body, the unknown function can be conveniently defined.
ii. Other gravitational phenomena can now be completely studied. These include motion of test particles, gravitational length contraction, gravitational time dilation, gravitational spectral shift of light, geodetic deviation just to name a few.

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