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Permutative semigroup whose congruences form a chain

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Abstract

Semigroups whose congruences form a chain are often termed Δ-semigroups. The commutative Δ-semigroups were determined by Schein and Tamura. A natural generalization of commutativity is permutativity: a semigroup is permutative if it satisfies a non-identity permutational identity. We completely determine the permutative Δ-semigroups. It turns out that there are only six noncommutative examples, each of which has at most three elements.

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1.0 Introduction

A semigroup is called **permutative** if it satisfies an identity $x_1, x_2, \dots, x_n = x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}$, for some non-identity permutation of $\{1, 2, \dots, n\}$. A Δ-semigroups is one whose congruences form a chain. The commutative Δ-semigroups were completely determined by Schein [1], [2] and Tamura [3]. In conjunction with their result, stated below as Result 1.1, our main theorem completely determines the permutative Δ-semigroups.

Theorem 1.1.

A semigroup S is a permutative Δ-semigroup if and only if it satisfies one of the following conditions.

(*i*) S is a commutative Δ-semigroup.
(*ii*) S is isomorphic to either R or R^0, where R is a two-element right zero semigroup.
(*iii*) S is isomorphic to the semigroup $Z = \{0, e, a\}$, obtained by adjoining to a null semigroup $\{0, a\}$ an idempotent element e that is both a right identity and a left annihilator for Z.
(*iv*) S is isomorphic to the dual of a semigroup of type (*ii*) or (*iii*).

Let R^+ denote the semigroup of positive real numbers under addition and let Q denote the Rees quotient semigroup by the ideal $I = [1, \infty)$. Similarly, let R denote the Rees quotient semigroup by the ideal $I = (1, \infty)$. A subsemigroup G of Q or R is *0-unitary* if $x, x + y \in G, x + y \notin I$ together imply $y \in G$.

Result 1.1 [1], [2], [3]

A semigroup S is a commutative Δ-semigroup if and only if it satisfies one of the following conditions:

(*i*) S is isomorphic to subgroup of a quasicyclic p-group (p is a prime).
(*ii*) S is a cyclic nilpotent semigroup.
(*iii*) S is an infinite 0-unitary subsemigroup of either Q or R.
(*iv*) S is obtained from a group of type (*i*) by adjoining a zero element.
(*v*) S is obtained from a semigroup of type (*ii*) or (*iii*) by adjoining an identity element.

As may also be easily verified directly, it follows from this result that a semilattice S is a Δ-semigroup if and only if $|S| \leq 2$. Several authors have considered Δ-semigroups satisfying various generalizations of commutativity, for instance in [4], [5], [6], [7], [8]. The outline of the proof of Theorem 1.1 is as follows.

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A key role is played by the *Archimedean semigroups:* those semigroups S with the property that, for arbitrary elements $a, b \in S$, there are positive integers i and j such that $a^i \in SbS$ and $b^j \in SaS$. In [9], it is proved that every permutative semigroup is a semilattice of Archimedean semigroups, that is, a *Putcha* semigroup [10]. In conjunction with the observation above, on semilattices, it follows that a permutative Δ-semigroup is either Archimedean or is a chain of two Archimedean semigroups. In the description of the commutative Δ-semigroups, those of types (i) - (iii) fall in the former category, (*iv*) and (*v*) in [10].

A semigroup S is *nil* if it has a zero element and for each $a \in S, a^n = 0$ for some positive integer n; in particular, S is *nilpotent* if $S^n = \{0\}$ for some positive integer n. Clearly, every nil semigroup is Archimedean.

A second key role is played by the *medial semigroups*: those that satisfy the permutational identity $axyb = ayxb$. This is evident from the following.

Result 1.2 (Theorem 1 of [1])

For any permutative semigroup S, there is a positive integer k such that, for all $u, v \in S^k$ *and all* $a, b \in S$, we have $uabv = ubav$. *In particular,* S^k is medial.

A semigroup S is called an *idempotent semigroup* if it satisfies the condition $S^2 = S$. From Result 1.2, it is obvious that every permutative idempotent semigroup is medial.

In Section 2, a detailed study of the permutative Archimedean case reveals that any such Δ-semigroup is medial. An important step is a proof that every permutative, Archimedean semigroup without idempotent element has a non-trivial group homomorphic image. It is then shown that *every* permutative Δ-semigroup is medial.

In Section 3 we first prove that every medial, *nil* Δ-semigroup is actually commutative. This completes the classification in the Archimedean case. In the non-Archimedean case, we extend some techniques and results of Trotter [17] on exponential semigroups, in order to complete the proof of Theorem 1.1. A semigroup is *exponential* if it satisfies $(xy)^n = x^n y^n$ for all positive integers n. It is easily verified that every medial semigroup is exponential.

Other papers on the topic of Δ-semigroups are by Bonzini and Cherubini [11], who determined all finite Putcha Δ-semigroups, and by Tamura [12], who described all finite inverse Δ-semigroups (and some related infinite ones).

2.0 Generalities on Δ-semigroups

We will need the following properties of Δ-semigroups. In addition, we will make use of Result 1.1, for instance its description of the Δ-semigroups that are abelian groups.

Result 2.1 [3]

Every homomorphic image of a Δ-semigroup is also a Δ-semigroup.

Since with every ideal of a semigroup there is associated its Rees congruence, it is obvious that the ideals of any Δ-semigroup are totally ordered. For nil semigroups the converse holds.

Result 2.2 (Theorem 1.56 of [13])

Let S be a nil semigroup. The following are equivalent:

(*i*) *S is a Δ-semigroup;*

(*ii*) *The ideals of S are totally ordered;*

(*iii*) *The principal ideals of S are totally ordered.*

In that case, each congruence on S is the Rees congruence corresponding to the ideal consisting of the congruence class of 0.

An ideal A of a semigroup S is said to be *dense* in S if the equality relation on S is the only congruence on S whose restriction to A is the equality relation on A. Observe that every nontrivial ideal of a Δ-semigroup S is dense, since any congruence on S whose restriction to such an ideal A is the equality relation cannot contain the Rees congruence associated with A and therefore must be contained in it instead.

Result 2.3 [4], **(**Theorem 1.61 of [13])

A non-trivial band is a Δ-semigroup if and only if it is isomorphic to either R or R^1 or R^0, where R is a two-element right zero semigroup, or L or L^1 or L^0, where L is a two-element left zero semigroup, or F, where is a two-element semilattice.

As every semigroup is a semilattice of semilattice indecomposable semi-groups, Result 2.2 and 2.3 imply that a Δ-semigroup is either semilattice indecomposable or a semilattice of two semilattice indecomposable semigroups.

Result 2.4 (Theorem 1.57 of [13])

If a Δ-semigroup S is a semilattice of a nil semigroup S_1 and an ideal S_0 of S then $|S_1| = 1$.

Result 2.5 [3]

I*f a semigroup S contains a proper ideal I and if S is a Δ-semigroup then neither S nor I has a non-trivial group homomorphic image.*

Result 2.6 (Corollary 1.3 of [13])

If a Δ-semigroup S is an ideal extension of a rectangular group K by a semigroup with zero then K is either a group or a left zero semigroup or a right zero semigroup.

We note that, in case *S* = *K*, *S* is either a group or a right zero semigroup or a left zero semigroup. If *K* is a proper ideal of *S* then (using also Result 2.5) *K* is either a right zero semigroup or a left zero semigroup.

Result 2.7 (Lemma 1.3 of [1])

No Δ-semigroup can contain an ideal that is itself an ideal extension of a non-trivial right (or left) zero semigroup by a non-trivial nil semigroup that is finite cyclic.

Proof.

(The following argument is significantly simpler than that in the cited paper). Suppose the Δ-semigroup *S* contains as an ideal an extension of the right zero semigroup *R* by the nontrivial cyclic nil semigroup *A*, generated by *a*. Then $A-R=\{a,a^2,\dots,a^{n-1}\}$, for some $n > 1$, where $a^n = z \in R$.

Let ρ denote the congruence on *S* generated by (a, a^2). Since *S* is a Δ-semigroup, ρ must contain the Rees congruence modulo the ideal *R*. Suppose $r\in R, r\neq z.$ The $(r,z)\in\rho$ and so there is a sequence of elementary transitions leading from *r* to *z* [14]. The first such transition has the form $r = sat \to sa^2t = r_1$, or $r = sa^t \to sat = r_1$, where $s,t\in S^1$ and we may assume $r_1 \neq r$, so that $a \notin R$ and *at* is therefore a power of *a*. Now since $r = r^2$, either $r=(rs)(at)$ or $r=(rsa)(at)$; in either case $r\in Ra.$ Since $z = za, z\in Ra$ also, that is, $R = Ra.$ But then, by iteration, $R = Ra^n = \{z\}$. Hence *R* cannot be non-trivial.

3.0 Every permutative Δ-semigroup is medial

We first consider Archimedean permutative semigroups in general. The Archimedean semigroups containing at least one idempotent element are characterized in [15]. Namely, a semigroup is Archimedean and contains an idempotent element if and only if it is an ideal extension of a simple semigroup containing an idempotent element by a nil semigroup. As a simple semigroup *S* satisfies $S^2 = S$, then by Result 2.2, every simple permutative semigroup is medial and thus, by [16], it is a rectangular abelian group (a direct product of a left zero semigroup, a right zero semigroup and an abelian group). Thus we have the following result.

Theorem 3.1

Every permutative Archimedean semigroup S containing at least one idempotent element is an ideal extension of a rectangular abelian group by a nil semigroup.

A subset *A* of a semigroup *S* is called a left (right) unitary subset of *S*. A subset *A* of a semigroup *S* is called a reflexive subset of *S* if $ab\in A$ implies $ba\in A$ for every $a,b\in S$.

Lemma 3.2

If a is an arbitrary element of a permutative semigroup S then

$$S_a = \{x \in S : a^i x a^j = a^h \text{ for some positive integers } i, j, k\}$$

is the smallest reflexive unitary subsemigroup of S that contains a.

Proof.

Let S be a permutative semigroup. Then there is a positive integer k such that $uabv = ubav$ for every $u, v \in S^k$ and every $a, b \in S$. Let a be an arbitrary element of S. It is clear that $a \in S_a$. To show that S_a is a subsemigroup of S, let $x, y \in S_a$ be arbitrary elements. Then $a^i x a^j = a^h$ and $a^m y a^n = a^t$ for some positive integers i, j, h, m, n, t. We can suppose that $i, n \geq k$. Then

$a^{h+t} = a^i x a^j a^m y a^n = a^i xy a^{j+m+n}$.

and so $x, y \in S_a$. To show that S_a is left unitary, assume $x, xy \in S_a$ for some $x, y \in S$. Then $a^i x a^j = a^h$ and $a^m y a^n = a^t$ for some positive integers i, j, h, m, n, t. We can suppose that $m \geq j$ and $i, n \geq k$. Then

$a^{i+t} = a^i a^m xy a^n = a^i x a^j a^m y a^n = a^i x a^j a^{(m-j)} y a^n = a^{h+m-j} y a^n$.

Hence $y \in S_a$ We can prove, in a similar way, that $y, xy \in S_a$. Thus S_a is an unitary subsemigroup of S. S_a is reflexive, because it is unitary and

$(xy)^3 = x(yx)^2 y = xy^2 x^2 y = xy(yx)xy$

holds in S. *If B* is a unitary subsemigroup of S such that $a \in B$ then, for an arbitrary element $x \in S_a$, there are positive integers i, j, k such that $a^i x a^j = a^k \in B$.. Then $x \in B$ so $S_a \subseteq B$.

The following theorem extends Lemma 11 of [3] and Theorem 9.11 of [13]. There are also analogues such as Theorem 1.2 of [17].

Theorem 3.3

Every permutative Archimedean semigroup without idempotent element has a non-trivial group homomorphic image.

Proof

Let S be a permutative Archimedean semigroup without idempotent element. Assume $S_a \neq S$ for some $a \in S$. Then the principal congruence P_{S_a} of S defined by the reflexive unitary subsemigroup S_a is a group congruence on S [3] and so the factor semigroup S / P_{S_a} is a non-trivial group homomorphic image of S. Suppose $S_a = S$ for all $a \in S$. Then, for any $a \in S$, $S_{a^2} = S$ and so $a \in S_{a^2}$. Then there are positive integers i, j, h such that we have $(a^2)^i a (a^2)^j = (a^2)^h$, that is, $a^{2i+2j+1} = a^{2h}$ contradicting the assumption that S has no idempotent element.

Next, we deal with permutative, Archimedean Δ-semigroups. First of all, we prove these lemmas that will be used in the proof of Proposition 3.7 below.

Lemma 3.4

Every nilpotent Δ-semigroup is finite cyclic. Every non-nilpotent, nil permutative Δ-semigroup is idempotent. Hence any permutative nil Δ-semigroup is medial.

Proof

First, suppose that S is a nonidempotent nil Δ-semigroup. Let $a, b \in S - S^2$. Since the ideals of S are totally ordered, we may assume without loss of generality that $S^1 b S^1 \subseteq S^1 a S^1$. If $b \neq a$ then $b = sat$, where either s or t is in S, contradicting $b \notin S^2$. Hence $b = a$ and $S - S^2 = \{a\}$. Let $k > 1$ be an arbitrary integer. If $c \in S^{k-1} - S^k$ then $c = c_1, c_2, \ldots, c_{k-1}$ for some $c_i \in S - S^2$. Hence $c = a^{k-1}$. If S is nilpotent, then $S^j = \{0\}$ for some least positive integer j and, by the above, $S = \{a, a^2, \ldots, a^j = 0\}$. clearly such a semigroup is medial.

If S is nonidempotent and nil, but non-nilpotent, then $S^j \neq \{0\}$ for all $j \geq 1$. Let N be any positive integer such that $a^N = 0$. Let $b \in S^{3N} - \{0\}$, $b = b_1 b_2, \ldots . b_{3N}$ say. Since $a \notin S^1 b_i S^1$ unless $a = b_i$ for each i. By the total ordering on ideals of S, for each i, there are elements $s_i, t_i \in S^1$ such that $b_i = s_i a t_i$. Now, for some index $i < N, t_i s_{i+1} \in S^m - \{0\}$ for every $m > 0$, for otherwise, the product $b = (s_1 a t_1)(s_2 a t_2) \ldots (s_N a t_N) \ldots (s_{2N} a t_{2N}) \ldots (s_{3N} a t_{3N})$ involves the power a^N. Similarly, an element $t_j s_{j+1}$ has the same property for some index $j \geq 2N$.

If S is also permutative, then there exists K such that S^K is medial. Therefore if $N \geq K$, all the terms between $t_i s_{i+1}$ and $t_j s_{j+1}$ in the product for b may be commuted, yielding a term a^N, contradicting $b \neq 0$. Thus the second statement in the lemma is proven. As noted in Section one, every idempotent, permutative semigroup is medial.

Lemma 3.5

Let S be a permutative semigroup with a dense ideal R that is a right zero semigroup. If R is nontrivial, then S/R is nilpotent.

Proof

Suppose S satisfies the identity $x_1, x_2, \ldots, x_n = x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}$, for some $n > 1$, where σ is a non-trivial permutation. Then $\sigma(n) = n$ since, otherwise, if r, s are distinct members of R, substituting $r = x_n$ and $s = x_{\sigma(n)}$ (and substituting arbitrarily for any other variables) yields $r = s$. Let i be least such that $\sigma(j) = j$ for $i \leq j \leq n$. Clearly $i > 2$. Let $r \in R$ and substitute, $x_{i-1} = r$. Then $r x_i \ldots x_n = r w x_i \ldots x_n$ for every $r \in R$, where w is a non-empty word in $\{x_1, x_2, \ldots, x_{i-2}\}$. It is easy to see that $\eta = \{(a,b) \in S \times S : (\forall r \in R) ra = rb\}$ is a congruences on S such that the restriction $\eta|_R$ of η to R equals id_R. As R is a dense ideal of S, we have $\eta = id_s$. As $(x_i, \ldots, x_n, w x_i, \ldots, x_n) \in \eta,$, we get that $x_i, \ldots, x_n = w x_i, \ldots, x_n$ is an identity satisfied in S. Now by choosing for any one of the variables in w an element of R, it follows that $x_i \ldots x_n \in R$ for $x_i, \ldots, x_n \in S$. Thus $S^{n-i+1} \in R$; equivalently, $(S/R)^{n-i+1} = \{0\}$.

Lemma 3.6

No permutative Δ-semigroup can be an ideal extension of a nontrivial right (or left) zero semigroup by a non-trivial nil semigroup.

Proof

Suppose such a semigroup S exists, with non-trivial right zero ideal R. Then, as observed in Section one, R is a dense ideal of S. By the previous Lemma, S/R is nilpotent. Since S/R is also a Δ-semigroup, it is finite cyclic. Then Result 2.7 applies.

Proposition 3.7

Every permutative, Archimedean Δ-semigroup is either
(*a*) *simple, whence a group or a left or right zero semigroup, or*
(*b*) *nil. In any case, every such semigroup is medial.*

Proof

Let S be such a semigroup. If S is simple then S is idempotent and so is medial, thus a rectangular group [16] and so is as described, by the comments following Result 2.7.

If S is not simple then, by Theorem 3.3 and Result 2.6, S contains an idempotent element. By Theorem 3.1, Result 2.7 and the remarks that follow the latter, S is an ideal extension of a right or left zero semigroup K by a non-trivial nil semigroup. By Lemma 3.5, $[K] = 1$, that is, S is a non-trivial nil semigroup. The mediality now follows by Lemma 3.6.

Finally, we may consider the general permutative case.

Theorem 3.8

Every permutative Δ-semigroup is medial.

Proof.

Let S be such a semigroup. The Archimedean case is covered by the preceding result. We have seen that the alternative case is when S is a semilattice of two Archimedean semigroups S_1 and S_0 with $S_0 S_1 \subseteq S_0$. By Result 2.3, S_1^0 and so S_1 is an Archimedean Δ-semigroup. It is clear that S_1 is permutative. Then S_1 is either a group or a two-element right or left zero semigroup (see also Result 2.6). In all three cases $S^2 \bigcap S_0 \neq 0$ and $S_1 \subseteq S^2$. As the ideals S_0 and S^2 of S are comparable, we have $S^2 = S$. Then, by Result 2.2, s is a medial semigroup.

4.0 Medial Δ-semigroups

We shall refine the following partial description of the medial Δ-semigroups summarized by the first author, deducible from the results of Trotter's Theorems 2.7, 3.5, 3.6 of [17].

Result 4.1 (Theorem 9.20 of [8])

A medial semigroup is a Δ-semigroup if and only if it satisfies one of the following conditions.

(*i*) *S is a Δ-group (necessarily abelian), or such a group with a zero adjoined.*

(*ii*) *S is a nil Δ-semigroup.*

(*iii*) *S is isomorphic to either R or R^0, where R is a two-element right zero semigroup.*

(*iv*) *S is isomorphic to the dual of a semigroup of type* (*iii*).

(*v*) $S = N \bigcup \{e\}$, *where* $e^2 = e$, N *is a nil semigroup and* $eN, Ne \subseteq N$

Trotter [17] called any Δ-semigroup constructed in the fashion of (*v*) a T1 *semigroup*. (In our earlier notation, $N = S_0\{e\} = S_1$.)

We shall first show that every medial, nil Δ-semigroup is commutative; and then that every medial, T1 Δ-semigroup is either commutative or is isomorphic to the semigroup Z of Theorem 1.1 or its dual. In view of Result 1.1, the proof of Theorem 1.1 is then complete.

A semigroup is *left commutative* if it satisfies the identity $abx = bax$; right commutativity is defined dually. Clearly all such semigroups are medial.

Proposition 4.2

If S is a left or right commutative, nil Δ-semigroup then it is commutative.

Proof

We need only consider the identity $abx = bax$. Let $\rho = \{(a,b) \in S \times S : as = bs, \forall\, s \in S\}$. It is well known that ρ is congruence on S; from the identity it follows that S/ρ is commutative.

By Result 2.4, ρ is the Rees ideal congruence modulo, the ideal $I = 0\rho$, which is the left annihilator of S. Thus if $a \in S$, either $aS = 0$ or $a\rho = \{a\}$. now let $a, b \in S, a \neq b$. If $a, b, ab \notin I$, then since S/ρ is commutative, $ab = ba$. If $a, b \in I$ then $ab = ba = 0$.

If $a, b \notin I$ then, since the principal ideals of S are totally ordered, without loss of generality $a = xby$ for some $x, y \in S^1$. Since $a \notin I, x, y \notin I$. By the first case above, x, b, y commute. Hence $ab = ba$.

Without loss of generality, the remaining case is where $a \in I, b \notin I$. As above, $a = xby$ for some $x, y \in S^1$. If $y \notin I$, then $xby = bxy$. Thus we may assume that either $a = bx$ or $a = xb$ for some $x \in S$. If $x \notin I$ then by the previous paragraph $bx = xb$ and so $ab = ba$. Thus we may assume $x \in I$. Now we may similarly write $x = bx_1$ or $x = x_1 b$

for some $x_1 \in S$. If $x_1 \notin I$ then, again similarly, $bx_1 = x_1 b$ and so $a = b^2 x_1$ or $a = x_1 b^2$, whence *ab* = *ba*. If $x_1 \in I$, continue this process by writing $x_1 = bx_2$ or $x_1 = x_2 b$. By induction, wither some $x_1 \notin I$ and then *ab* = *ba*, or for all *i* there exists x_i such that $a = b^{i+1} x_i$ or $a = x_i b^{i+1}$. But *S*is nil, so it follows that *a* = 0, completing the proof.

Theorem 4.3

If S is a medial, nil Δ-semigroup, then S is commutative.

Proof

Again, let ρ be the congruence $\{(a,b) \in S \times S : as = bs, \forall\ s \in S\}$. From the medial identity it is clear that S/ρ is right commutative. Since it is again a nil Δ-semigroup, it is commutative, by the previous propositions. Let $I_L = 0\rho$. Let λ be the dual congruence, so that S/λ is also commutative. Let $I_L = 0\lambda$. As in the proof of the proposition, for each $a \in S$, either $a\rho = I_L$ or $a\rho = \{a\}$, and dually.

Since the ideals of *S* are totally ordered, without loss of generality $I_L \subseteq I_R$. Let $a, b, \in S$. If $a, b \notin I_L$ then precisely as in the third and fourth paragraphs of the proof of the previous proposition, *ab* = *ba*. Otherwise, without loss of generality, $a \in I_L$, so *ab* = 0. But also $a \in I_R$, so *ba* = 0.

We now turn to T1 semigroups.

Result 4.1 (Theorem 1.58 of [13], Lemma 3.3 of [17]

Let $S = N \cup \{e\}$ be any T1 semigroup. Then every ideal of N is also an ideal of S and so N is also a Δ-semigroup.

Theorem 4.4

Let $S = N \cup \{e\}$ be a medial T1 *semigroup. Then N is a commutative Δ-semigroup and S satisfies one of the following conditions.*

(*i*) *e acts as an identity element for N and S itself is commutative.*
(*ii*) *e acts as a right identity and a left annihilator for N and S is isomorphic to the semigroup Z in Theorem 2.1 (iii).*
(*iii*) *the dual of the previous case.*

Proof

That *N* is commutative is immediate from Result 4.1 and Theorem 4.4.

Now suppose that *S* is any T1 semigroup for which *N* is commutative. We show first that for any $a \in N$, either $ea = a$ or $ea = 0$. (The dual statement obviously also holds.) Result 4.1 shows that since $N^1 a N^1$ is an ideal of *N*, it is also an ideal of *S*, whence it contains *ea*. Hence, if $ea \neq a$, then $ea = at$ for some $t \in N$. Then $ea - eat = eat^n$ for each *n* and, since $t \in N, ea = 0.$

Next suppose that *ea* = *a* for some non-zero $a \in N$. Let $b \in N$. Either *b* = *ax* or *a* = *bx*, for some $x \in S^1$. In the former case, $eb = eax = ax = b$; in the latter case, suppose *eb* = 0: then *e* = *ebx* = 0, a contradiction, so that again *eb* = *b*. Hence *e* is either a left identity for *S* or a left annihilator for *N*. Clearly the dual statement also holds.

Conclusion

In this paper, we have show that if *N* is nonzero, then *e* cannot be both a left and a right annihilator for *N*. For in that event, given $a \in N - \{0\}, S^1 a S^1 \subset S^1 e S^1$, so *a = set*, for some $s, t \in S^1$. Both *s* and *t* cannot belong to *N*, for then *se = et = 0*. But otherwise, either *a = ea* or *a = ae*, contradicting the assumption. *e* is either an identity for *S*, or is a right identity for *S* and a left annihilator for *N*, or is a left identity for *S* and a right annihilator for *N*. If $ab = (ae)b = a(eb) = 0$, then *N* is a null semigroup. But every subset of *N* that contains 0 is an ideal, so $|N| \leq 2$. When $N = \{0\}$, *e* actually acts as an identity. Otherwise, *N* = {*a*, 0}, say, where *ae = a*, *ee = e* and all other products are 0.

The concrete results obtained raise the question whether Trotter's results [17] on exponential Δ-semigroups can similarly be strengthened. In particular, is it true (c.f. Theorem 5) that every nil, exponential Δ-semigroup is commutative?

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