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Regular algebraic monoids

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Abstract

The purpose of this paper is to provide a proper identification of normal irreducible, regular algebraic monoids. The results from the work of Renner [3,4] suggest that we should be able to find a classification of these monoids in terms of their unit groups, and related toroidal data. That is what we accomplish here.

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1.0 Introduction

Assume that M is a normal, regular, algebraic monoid with unit group G. All our algebraic monoids are defined over an algebraically closed field of arbitrary characteristic. Let $e \in M$ be a minimal idempotent, and define

$$G_e = \{g \in G \mid ge = eg - e\}^0$$

Assume, for simplicity, that G_e is a Levi factor of G. Thus

$$G \cong G_e \alpha R_u(G) \text{ (semidirect product)}$$

where $U = R_u(G) \Delta G$ is the unipotent radical of G.

***Theorem* 1.1**

(a) *Let $T \subseteq G$ be a maximal torus and let $\overline{T} \subseteq M$ be Zariski closure of T in M. So $T \subseteq \overline{T}$ induces $X(\overline{T}) \subseteq X(T)$. Let $\Phi_U \subseteq X(T)$ be the weights of the action $Ad :\to Aut(L(U))$ on the Lie algebra of U. Then $\Phi_U \subseteq X(\overline{T}) \cup -X(\overline{T})$.*

(*b*) *Conversely, suppose we are given an algebraic group $G = G_0 \alpha R_u(G)$ (where $G_0 \subseteq G$ is a Levi factor) along with a normal torus embedding $T \subseteq \overline{T}$ of the maximal torus $T \subseteq G_0$. Let M_0 be the normal, reductive monoid with 0 and unit group G and maximal D-monoid $\overline{T}$* [3]. *Consider the action $Ad :\to Aut(L(U))$ and assume that $\Phi_U \subseteq X(\overline{T}) \cup -X(\overline{T})$. Then there exists a unique, normal, algebra monoid M with unit group G and maximal D-monoid $\overline{T} \subseteq M$.*

(c) *Any monoid M, as in (b), has the following structure: Let $e = e^2 \in M$ be the zero element of M_0. Define $U_+ = \{u \in U \mid eu = e\}, U_0 = \{u \in U \mid eu = ue\}$ and $U_- = \{u \in U \mid eu = e\}$. Then $M \cong U_+ \times M_e \times U_-$ and the monoid multiplication of M can be defined explicitly with these coordinates.*

The above theorem is an organized summary of Corollary 2.3, Proposition 2.6 and Theorem 3.3.

We should note that Theorem 1.1 classifies only those normal regular monoids with unit group G of a particular type (that is, G is related to the monoid in a particular way). The general case is explained in Section 4. It is a relatively minor modification of the above theorem. For convenience we describe it here.

So, let M be ***any*** normal, irreducible, regular, algebraic monoid with unit group G, and let $e \in E(M)$ be a minimal idempotent. Let $N = \overline{G_e R_u(G)}$ (Zariski closure), and set $H = G_e R_u(G)$. The following theorem is an organized summary

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Theorem 1,2

(a) *N is a regular monoid of the type considered in Theorem 1.1. Furthermore* $gNg^{-1} = N$ *for* $g \in G$.

(b) Define $N \times H_G =: \{[x,g] \mid x \in N, g \in G\}$ *where* $[x,g] = [y,h]$ *if there exists* $k \in H$ *such that* $y = xk^{-1}$ *and* $h = k\dot{g}$. *Then* $N \times H_G$ *is a regular monoid with multiplication* $[x,g][y,h] = [xgyg^{-1}, gh]$. *Furthermore,*

$$\phi : N \times H_G \to M$$

$$\phi([x,g]) = xg$$

is an isomorphism of algebraic monoids.

2.0 Disintegration of regular monoid

A monoid is regular for any $x \in M$ such that $xax = x$. Let M be a normal, regular, irreducible, algebraic monoid with unit group G, and let $e \in E(M) = \{e \in M\}e = e^2\}$ be a minimal idempotent. By Theorem 7.4 of [1] $G_e = \{g \in G \mid ge = eg = e\}^0$ is a reductive subgroup of G.

Assumption 2.1

$G_e \subseteq G$ is a Levi factor, so that $G = G_e \times R_u(G)$, where $R_u(G) \lhd G$ is the unipotent radical.

As pointed out in the introduction, the general case can easily be derived from this one. We adhere strictly to Assumption 2.1 except in Section 4.

Proposition 2.2

Let T be a maximal torus and define $N = \overline{TR_u(G)} \subseteq M$. *Then N is regular.*

Proof

Since M is regular, G_e is reductive for any minimal idempotent e of M. so $G_e \cap R_u(G) = \{1\}$. Thus, $(TR_u(G))_e \cap R_u(G) = \{1\}$. So $(TR_u(G))_e$ has no unipotent elements other than the identity. So it must be a torus. By Theorem 7.4 of [1], N is regular.

Corollary 2.3

Let $\Phi_U \subseteq X(T)$ *be the weights of Ad:* $Ad :\to Aut(L(U))$ *on the Lie algebra L(U) of* $U = R_u(G)$. *Then* $\Phi_U \subseteq X(\bar{T}) \cup -X(\bar{T})$.

Proof

Since $\bar{T}$ has a zero, this follows from Corollary 2.4 of [4].

Proposition 2.4

Let $U = R_u(G)$ *and let*

$U_+ = \{u \in U \mid eu = e\}$,

$U_0 = \{u \in U \mid eu = ue\}$ *and*

$U_- = \{u \in U \mid ue = e\}$.

Then (*a*) $U = U_+ U_0 U_- \cong U_+ \text{ x } U_0 \text{ x } U_-$

(*b*) $G_e \subseteq N_G(U_+) \cap C_G(U_0) \cap N_G(U_-)$.

Proof

(a) Follows from Formula (3) of [4].

(b) Notice first that $G_e \subseteq C_G(e)$. So if $u \in U_+$ and $g \in G_e$, then $egug^{-1} = g(eu)g^{-1} = e$. So $gug^{-1} \in U_+$. Similarly, $G_e \subseteq N_G(U_-)$.

Now $G_e \subseteq N_G(U_0)$, by an argument similar to the above. But we can prove a little more for U_0. Indeed, let $T \subseteq G_e$ be a maximal torus and let $u \in U_0$. Then for $t \in T$, $etut^{-1} = eut^{-1} = ue = eu$. So $etut^{-1}u^{-1} = e$, which implies that $tut^{-1}u^{-1} \in U_+$. But $etut^{-1}u^{-1} \in U_0$ since $T \subseteq N_G(U_0)$. So $tut^{-1}u^{-1} \in U_0 \cap U_- = \{1\}$, so that $ut = tu$. But then $T \subseteq C_G(U_0)$ for any maximal torus $T \subseteq G_e$. On the other hand, $\bigcup_{T \subseteq G} T \subseteq G$ is Zariski dense. Thus, $G_e \subseteq C_G(U_0)$.

Proposition 2.5

Let $M_e = \overline{G}_e \subseteq M$. *Then* M_e *is normal.*

Proof

Consider

$\varphi : M_e \to M \to M // R_u(G)$

where $M // R_u(G)$ is as in Theorem 4.2 of [2]. Now $M // R_u(G)$ is normal and $o(M // R_u(G)) = o(M)^{R_u(G)}$. By Theorem 4.2 of [2], φ induces an isomorphism on $\overline{T}$, so by Corollary 4.5 of [3] φ is an isomorphism.

Proposition 2.6

$M \cong U_+ \times C_M(e)^0 \times U_-$ and $C_M(e)^0 \cong M_e \times U_0$.

Proof

Define $\phi : U_+ \times C_M(e)^0 \times U_- \to M$ by $\varphi(x,y,z) = xyz$. We define a monoid structure on $U_+ \times C_M(e)^0 \times U_-$ so that φ is a morphism, and $U_+ \times C_M(e)^0 \times U_-$ is regular. From there it follows that φ is surjective and birational. But *M* is normal, so φ is an isomorphism.

By Corollary 2.3 and the comments following Corollary 2.4 of [4], $\Phi_{U_+} \subseteq X(\overline{T})$ and $\Phi_{U_-} \subseteq -X(\overline{T})$. So we obtain $\overline{T} \to \mathrm{End}(U_+)$ extending $T \to \mathrm{Aut}(U_+), g \mapsto \mathrm{int}(g)$; and $\overline{T} \to \mathrm{End}(U_-)$ extending $T \to \mathrm{Aut}(U_-), g \mapsto \mathrm{int}(g^{-1})$. So the sought after multiplication on $U_+ \times C_M(e)^0 \times U_-$ can be defined in [4]. That is

$(u,x,v)(a,y,b) = (u\zeta_+(v,a)^x,\ x\zeta_0(u,v)y, \zeta_-(v,a)^{\bar{y}}b)$

where ζ_+, ζ_0 and ζ_- are defined by

$$\zeta_+ : U_- \text{ x } U_+ \xrightarrow[m]{} U_+U_0U_- \xrightarrow[p_1]{} U_+,$$

$$\zeta_0 : U_- \text{ x } U_+ \xrightarrow[m]{} U_+U_0U_- \xrightarrow[p_2]{} U_0 \text{ and}$$

$$\zeta_- : U_- \text{ x } U_+ \xrightarrow[m]{} U_+U_0U_- \xrightarrow[p_3]{} U_-.$$

The action of $x \in \overline{T}$ on $u \in U_+$ is denoted u^x, and $y \in \overline{T}$ on $v \in U_-$ by $v^{\bar{y}}$.

In this section we start with the pieces, and show how to construct a regular monoid.

3.0 Construction of regular monoid

Definition 3.1. Let M_0 be a normal, reductive monoid with 0, and let *U* be a connected, unipotent group with regular action $\rho : G_0 \to \mathrm{Aut}(U)$ such that $\Phi_U \subseteq X(\overline{T}) \cup -X(\overline{T})$. In the situation of 3.1 we can write $U = U_+U_0U_-$ where

$$\mathrm{L}(U_+) = \left(\bigoplus_{\alpha \in X(\overline{T})}\right) \cdots L(U)_\alpha, L(U_0) = C_{L(U)}(T) \text{ and } \mathrm{L}(U_-) = \left(\bigoplus_{\alpha \in -X(\overline{T})}\right) \cdots L(U)_\alpha.$$

Proposition 3.2

U_+, U_0 and U_- *are stabilized by* G_0 *under* ρ.

Proof

Let $\lambda : K^* \to Z(G_0) \subseteq T$ be a 1-psg (where psg is periodic subsemigroups) such that $\lim_{t \to 0} \lambda(t) = 0$. Such a λ exists because G_0 is reductive. The $\lambda^*(X(T)) \subseteq Z = X(k^*)$. One checks that $\lambda^*(X(\overline{T}) \setminus \{0\} \subseteq Z^+$ and $\lambda^*(-X(\overline{T}) \setminus \{0\} \subseteq Z^-$. Thus,

$$U_+ = \{u \in U \mid \lim_{t \to 0} \lambda((t)u\lambda(t)^{-1} = 1\}$$

$$U_- = \{u \in U \mid \lim_{t \to 0} \lambda(t)^{-1}u\lambda(t) = 1\} \text{ and}$$

$$U_0 = C_U(\lambda(k^*)).$$

But $\lambda(k^*) \subseteq G_0$ is central. Thus U_+, U_0 and U_- are stabilized by G_0 under ρ.

Theorem 3.3

Let M_0, ρ *and U be as in 3.1. Then* $U_+ \times M_0 \times U_+ \times U_-$ *has the unique structure of a regular, algebraic monoid extending the group law on* $U_+ \times G_0 \times U_+ \times U_- \cong G \times U, (u,g,v,w) \mapsto (g,uvw)$.

Proof

By Proposition 3.2, $\rho : G \to \mathrm{Aut}(U)$ stabilizes U_+, U_0 and U_-. By definition, $\rho \mid T : T \to \mathrm{Aut}(U_+)$ extends over $\overline{T}, \rho^{-1} \mid T : T \to \mathrm{Aut}(U_-)$ extend over $\overline{T}$. Thus, by Corollary 4.5 of [3] there exist unique

$\rho : G_0 \to \text{Aut}(U_+)$ and unique $\rho_- : M_0 \to \text{End}(U_-)$ extending $\rho^{-1} : G_0 \to \text{Aut}(U_-)$. Using formula (4) of [4] we can define the desired multiplication on $U_+ \times M_0 \times U_+ \times U_-$, just as we did in Proposition 2.6 above.

4.0 The general case

In this section we consider normal regular monids, but without the restrictions of Assumption 2.1. So let *M* be normal and regular. If $e \in E(M)$ is a minimal idempotent define $N = \overline{G_e R_u(G)}$.

Lemma 4.1

(a) $gNg^{-1} \subseteq N$ *for* $g \in G$.

(b) *N is a regular monoid of the type considered in* Assumption 2.1.

Proof

If $g \in G$ then $gG_e g^{-1} = G_{geg^{-1}}$. But from Theorem 6.30 of [1] it follows that $geg^{-1}heh^{-1}$ for some $h \in G_e R_u(G)$.

But then
$$gG_e g^{-1} = hG_e h^{-1} gG_e R_u(G) g^{-1} = gG_e g^{-1} gR_u(G) g^{-1}$$
$$= gG_e R_u(G) = hG_e h^{-1} R_u(G) = hG_e h^{-1} R_u(G) h^{-1}$$
$$= hG_e R_u(g) h^{-1} = G_e R_u(G)$$
since $h \in G_e R_u(G)$. By continuity, $gNg^{-1} \subseteq N$.

For (b), notice that G_e is reductive by Theorem 7.4 of [1]. But $(G_e R_u(G))_e = G_e$ and so, again by Theorem 7.4 of [1], *N* is regular. Furthermore, $G_e \times R_u(G) \to G$ is bijective. But we need a little more in positive characteristic. So let $k^* \subseteq Z(G_e) e \in \overline{k^*}$, as in the proof of Proposition 3.2. So $G_e \subseteq C_G(k^*) = G_e U_0 = U_0 G_e$. But also $\mathrm{L}(G) = \mathrm{L}(G)_+ \oplus \mathrm{L}(G_e U_0) \oplus \mathrm{L}(G)_-$, because global and infinitesimal centralizers correspond for torus actions. But from the proof of Proposition 3.2, $\mathrm{L}(U_+) \subseteq \mathrm{L}(G)_+$ and $\mathrm{L}(U_-) \subseteq \mathrm{L}(G)_-$. Thus $\mathrm{L}(U_+) \subseteq \mathrm{L}(G)_+$ and $\mathrm{L}(U_-) \subseteq L(G)_-$ since dim $G = \dim(U_+) + \dim(U_-) + \dim(G_e U_0)$, while $U_+ \times G_0 \times U_+ . \times U_- \cong G$. But then $G_e \cap R_u(G) = G_e \cap U_0$. But from 2.4(b), $G_e \subseteq C_G(U_0)$. So $G_e \cap U_0$ is a central, unipotent subgroup scheme of G_e. On the other hand, it is well known that $Z(G_e)$ is a disagonalizable group (possibly non reduced, in general). In any case $G_e \cap U_0 = G_e \cap R_u(G)$ must be the trivial group scheme. Thus, G_e x $R_u(G) \to G$ is separable, and therefore an isomorphism.

Let $H = G_e R_u(G)$ and define $N \text{ x }^H G = \{(x,g) \mid x \in N, g \in G\} / \sim$ where $(x,g) \sim (xh^{-1}, hg)$ if $h \in H$. Define $\varphi : N \text{ x }_H G \to M$ by $\varphi([x,g]) = xg$.

Theorem 4.2

φ is an isomorphism

Proof

From the proof of 4.1, *H* is a normal subgroup of G_-. Define a multiplication on $N \text{ x }^H G$ by $[x,g][y,h] = [xgyg^{-1}, gh]$. One checks that this is well defined. Furthermore, φ is a morphism of algebraic monoids.

Now φ is birational since $G(N \times \text{H}_G) = G = G(M)$. But also, $G\varphi(N)G = M$, since , Proposition 6.27 of [1], *N* intersects every J-class of *M. So φ is surjective and birational, while* M normal. Thus φ is an isomorphism.

5,0 Conclusion

Regular monoids, in general, are constructed from those that satisfy Assumption 2.1. Indeed, let *N* be a normal regular monoid with unit group *H*, and assume $H = H_e R_u(H)$ (as in 2.1). Assume $H \triangleleft G$ and *G/H is* reductive. Then we can define a regular monoid M with unit group *G*.

$$M = N \times H_G \text{ with multiplication}$$
$$[x,g][y,h] = [xgyg^{-1}, gh].$$

By Theorem 4.2, all normal regular algebraic monoids are obtained this way:

References

[1] Putcha, M. S., "Linear Algebraic Monoids", Cambridge University Press, Cambridge, UK, (1999).

[2] Renner, L. E., *Reductive monoids are von Neumann regular*, Jorunal of Algebra, 93 237 – 45. (1995),

[3] Renner, L. E., *Classification of semisimple algebraic monoids*, Trans. Amer. Math. Soc., 292 (93 – 223. (1995),

[4] Renner, L. E., *Completely regular algebraic monoids*, J. Pure and Applied Algebra, 59 291 – 298. (2000),