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The Derivatives of Moufang loops

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Abstract

We investigate the question over the nature of the left and right derivatives of Moufang loops and find out that the derivatives of a Moufang loop are Moufang loops that is, loops of Bol-Moufang type.

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1.0 Introduction

A groupoid is a system $(G,\cdot)$ such that G is a non-empty set and $(\cdot)$ is a binary function on G. For a groupoid $(G,\cdot)$, one can define the following two functions called left and right translation by: $\forall a \in G,\ L(a): G \to G;\ gL(a) = a \cdot g$ and $R(a): G \to G$ such that $gR(a) = g \cdot a$. If the translation maps are bijections then $(G,\cdot)$ is said to be a quasigroup. A loop is a quasigroup which has an identity element, 1, satisfying $\forall g\ (g \cdot 1 = 1 \cdot g = g)$. Quasigroups are studied not only in algebra, but also in combinatorics, where they are identified with Latin squares,and in projective geometry, where they are identified with 3-webs. For details and references to earlier literature, see [1, 2, 5, 6]. By results of [2], the following four identities:

(1): $\big(a \cdot (b \cdot c)\big) \cdot a = (a \cdot b) \cdot (c \cdot a)$ (2): $(a \cdot c) \cdot (b \cdot a) = a \cdot \big((c \cdot b) \cdot a\big)$
(3): $\big((a \cdot b) \cdot c\big) \cdot b = a \cdot \big(b \cdot (c \cdot b)\big)$ (4): $\big((b \cdot c) \cdot b\big) \cdot a = b \cdot \big(c \cdot (b \cdot a)\big)$

Definition 2.2
An isotopism of $(G,\cdot)$ into $(G,\cdot)$ is called an autotopism of $(G,\cdot)$. The concept of principal isotopy can be introduced with the following. for all $a, b, c \in G$ are equivalent in loops; by [3], they are also equivalent in quasigroups. A loop satisfying these identities is called a Moufang loop.

2.0 Preliminaries

In this section we summarize definitions, notations and known or elementary results in loops theory which will be useful to this study.

Definition 2.1

A triple (α, β, γ) of bijections from a set G into a set H is called an isotopism of a groupoid $(G,\cdot)$ into a groupoid $(H,\circ)$ provided $a\alpha \circ b\beta = (a \cdot b)\gamma$ for all $a, b \in G$. $(H,\circ)$ is then called an isotope of $(G,\cdot)$, and groupoids $(G,\cdot)$ and $(H,\circ)$ are called isotope to each other. It follows directly from this definition that a bijection $f: G \to H$ is an isomorphism from $(G,\cdot)$ to $(H,\circ)$ if and only if (f, f, f) is an isotopism of $(G,\cdot)$ into $(H,\circ)$. Consequently, isomorphism groupoids are isotopic. But isotopic groupoids need not be isomorphic. Thus, the concept of isotopy is a generalization of that of isomorphy.

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Definition 2.3

Let α and β be permutations of $(G,\cdot)$ and let ι denote the identity map on G. Then, (α,β,ι) is a principal isotopism of a groupoid $(G,\cdot)$ into a groupoid $(G,\circ)$ means that (α,β,ι) is an isotopism of $(G,\cdot)$ into $(G,\circ)$. Principal isotopy (Just as isotopy) is an equivalence relation on any non-empty set of groupoids.
Theorem 2.4 [6].
If $(G,\cdot)$ and $(H,\circ)$ are isotopic groupoids, then $(H,\circ)$ is isomorphic to some principal isotope of $(G,\cdot)$

3.0 Derivatives

If $(G,\cdot)$ is a quasigroup, we will call operation $(\cdot)$ a function F on G and write for all $a,b \in G,\ a\cdot b = F(a,b)$ and $(G,\cdot) = (G,F)$. If there are several different quasigroups on the same carrier set G, we sometimes will denote quasigroups $(G,F_1),(G,F_2),\ldots$ simply by $F_1,F_2,\ldots$

In this section, we shall use the following notation: let G be the carrier set of quasigroups $F_1,F_2,F_3,\ldots$, then

(a) if α is an isomorphism $\alpha{:}\,(G,F_1)\to(G,F_2)$, we write $F_2 = F_1\alpha$,
(b) if (α,β,γ) is an isotopism $F_1 \to F_2$, then we write $F_2 = F_1(\alpha,\beta,\gamma)$,
(c) if(U,V,W) is an autotopism of (G,F), then we write $F(\alpha,\beta,\gamma) = F$.

If (G,F) is a non-associative quasigroup, then a fixed element $a\in G$ determines a new operation $(\circ)$ on G such that

$$(a\cdot x)\cdot y = a\cdot(x\circ y) \qquad (1)$$

for all $x,y\in G$. The operation $(\circ)$ depends entirely on our choice of $a\in G$.
We shall denote $(\circ)$ by F^a and call F^a the left derivatives of F with respect to a. From (1), we have $xL(a)\cdot y = (x\circ y)L(a)$. Thus, we have the isotopism $(L(a),\iota,L(a){:}\,F^a\to F$. We now can write

$$F(L(a)^{-1},\iota,L(a)^{-1}) = F^a. \qquad (2)$$

Similarly, a fixed element $a\in G$ determines another operation $(*)$ on G such that

$$x\cdot(y\cdot a) = (x*y)\cdot a. \qquad (3)$$

We denote $(*)$ by F_a and call F_a the right derivatives of F with respect to a. Rewriting (3) as $x\cdot yR(a) = (x*y)R(a)$, we get the isotopism $\big(\iota,R(a),R(a)\big){:}\,F_a\to F$ or

$$F(\iota,R(a)^{-1},R(a)^{-1}) = F_a. \qquad (4)$$

We summarize this in the

Definition 3.1
The isotopes $F(L(a)^{-1},\iota,L(a)^{-1}) = F^a$ and $F(\iota,R(a)^{-1},R(a)^{-1}) = F_a$ are called left and right derivatives of F with respect to a fixed element $a\in G$.

4.0 Main results:
Theorem.

The left and right derivatives of Moufang loops are Moufang loops.
To prove this theorem, we need some lemmas.

Lemma 1

If $(M,\cdot)$ is a Moufang loop then any loop-isotope of $(M,\cdot)$ is isomorphic to a principal isotope $(M,\circ)$ such that

$$x \circ y = (x \cdot k) \cdot (k^{-1} \cdot y) \text{ for some } k \in M.$$

Proof.

Let $(M,\cdot)$ be a Moufang loop with identity element 1. We knows that every loop isotope of $(M,\cdot)$ is isomorphic to some f,g −isotope $(M,*)$ such that $x * y = (x \cdot g^{-1}) \cdot (f^{-1} \cdot y)$. The identity element of $(M,*)$ is then $e = f \cdot g$. Consider an isotope $(M,\circ)$ of $(M,*)$ such that $\big(R(e),R(e),R(e)\big)\colon (M,\circ) \to (M,*)$ where $R(e)\colon x \to x \cdot e = x \cdot (f \cdot g)$. Then we have

$$(x \circ y)R(e) = xR(e) * yR(e) = (x \cdot e) \cdot (y \cdot e), \text{ or}$$

$$x \circ y = [\big((x \cdot e) \cdot g^{-1}\big) \cdot (f^{-1} \cdot (y \cdot e)] \cdot e^{-1}.$$

Let $(x \cdot e) \cdot g^{-1} = u$ and $f^{-1} \cdot (y \cdot e) = v$.

Using Moufang identities and keeping in mind that $e = f \cdot g$, we have

$$x \circ y \quad = (u \cdot v) \cdot e^{-1} = \big[\big((u \cdot e) \cdot e^{-1}\big) \cdot v\big] \cdot e^{-1} = (u \cdot e) \cdot (e^{-1} \cdot v \cdot e^{-1})$$

$$= \big[\big((x \cdot e) \cdot g^{-1}\big) \cdot e\big] \cdot \big[e^{-1} \cdot \big(f^{-1} \cdot (y \cdot e)\big) \cdot e^{-1}\big] = (x \cdot (e \cdot g^{-1} \cdot e) \cdot [(e^{-1} \cdot f^{-1}) \cdot ((y \cdot e) \cdot e^{-1})$$

$$= (x \cdot (e \cdot g^{-1} \cdot e) \cdot ((e^{-1} \cdot f^{-1}) \cdot y).$$

Let $e \cdot g^{-1} \cdot e = k$, then $k = \big((f \cdot g) \cdot g^{-1}\big) \cdot e = f \cdot e$ and $k^{-1} = e^{-1} \cdot f^{-1}$. We now have $x \circ y = (x \cdot k) \cdot (k^{-1} \cdot y)$.

The identities element of $(M,\circ)$ is clearly 1, since $x \circ 1 = (x \cdot k) \cdot (k^{-1} \cdot 1) = x$ and $1 \circ x = (1 \cdot k) \cdot (k^{-1} \cdot x) = x$. The isotopism $\big(R(e),R(e),R(e)\big)\colon (M,\circ) \to (M,*)$ is an isomorphism $(M,\circ) \to (M,*)$. We now have $(M,*) \cong (M,\circ)$ which completes the proof.

In the sequel we will call $(M,\circ)$ the k −isotope of $(M,\cdot)$.

Lemma. 2

Every loop isotope to a Moufang loop is a Moufang loop.

Proof.

In view of lemma 1., we only have to consider k −isotopes. Let $(M,\circ)$ be a k −isotope of a Moufang loop $(M,\cdot)$. Then $x \circ y = (x \cdot k) \cdot (k^{-1} \cdot y)$. It is sufficient to show that $(M,\circ)$ satisfies the Moufang identity $(x \circ y) \circ (z \circ x) = x \circ [(y \circ z) \circ x]$. Let us now denote the left side of this identity by A and the right side by B. To prove the theorem, we shall show that $A = B$.

Rewriting $(\circ)$ in terms of $(\cdot)$ and using Moufang identities in $(M,\cdot)$ repeatedly, we have

$$A \;= (x \circ y) \circ (z \circ x) = \big[\big((x \cdot k) \cdot (k^{-1} \cdot y)\big) \cdot k\big] \cdot \big[k^{-1} \cdot \big((z \cdot k) \cdot (k^{-1} \cdot x)\big)\big]$$

$$= \big\{x \cdot \big[k \cdot \big(k^{-1} \cdot (y \cdot k)\big)\big]\big\} \cdot \big\{\big[\big(k^{-1} \cdot (z \cdot k)\big) \cdot k^{-1}\big] \cdot x\big\} = \big(x \cdot (y \cdot k)\big) \cdot \big((k^{-1} \cdot z) \cdot x\big),$$

$$B = x \circ [(y \circ z) \circ x] = (x \cdot k) \cdot \big\{k^{-1} \cdot \big[\big((y \cdot k) \cdot (k^{-1} \cdot z)\big) \cdot k \cdot (k^{-1} \cdot x)\big]\big\}$$

$$= (x \cdot k) \cdot \big\{k^{-1} \cdot \big[\big(y \cdot (z \cdot k)\big) \cdot (k^{-1} \cdot x)\big]\big\} = (x \cdot k) \cdot \{[k^{-1} \cdot (y \cdot (z \cdot k) \cdot k^{-1}] \cdot x\}$$

$$= x \cdot \{k \cdot [(k^{-1} \cdot y) \cdot (z \cdot k)) \cdot k^{-1}]\} \cdot x = x \cdot [(y \cdot (z \cdot k)) \cdot k^{-1}] \cdot x$$

$$= x \cdot \big\{\big[\big((y \cdot k) \cdot k^{-1}\big) \cdot (z \cdot k)\big] \cdot k^{-1}\big\} \cdot x = x \cdot \big\{(y \cdot k) \cdot \big[k^{-1} \cdot \big((z \cdot k) \cdot k^{-1}\big)\big]\big\} \cdot x = x \cdot \big[(y \cdot k) \cdot \big((k^{-1} \cdot z) \cdot (k \cdot k^{-1})\big)\big] \cdot x$$

$$= x \cdot ((y \cdot k) \cdot (k^{-1} \cdot z)) \cdot x$$

$$= \big(x \cdot (y \cdot k)\big) \cdot \big((k^{-1} \cdot z) \cdot x\big) = A.$$

In view of this lemma, it is obvious that every loop-isotope of a Moufang loop is an inverse property loop. More interesting is the fact, that the converse is also true. We can now prove our theorem.

Proof.

Mengue Mengue and Ajala [4] proved that the derivatives of a loop are loops using that result, lemma 2 and definition 3.1 for Moufang loops our proof is complete.

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