Journal of the Nigerian Association of Mathematical Physics
Volume **19 (November, 2011), pp 21 – 32**

Coefficient Inequalities for Certain New Classes of Analytic and Univalent Functions

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Abstract

In this paper, the Authors introduce and study the classes $\Phi(\omega,\beta,b,m,\lambda,l)$ ***and*** $\Phi(\omega,\beta,b,m,\lambda,l)$ ***which provides an interesting movement from starlikeness to convexity using Aouf et al derivative operator. The consequences of the parametrics are also discussed.***
Keywords: Analytic functions, univalent functions, ω***-starlike,*** ω***-convex, coefficient inequality, Aouf et al derivative operator.***

1. Introduction:

Let Γ denote the class of functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \tag{1.1}$$

which are analytic in the unit disc $U=\{z:|z|<1\}$ and normalized with $f(0)=0$ and $f'(0)-1=0$. Also $S\subset\Gamma$ denote the class of analytic and univalent in Γ.

The Authors wish to give the following well known definitions of classes of starlike functions of order α and convex function of order β respectively as spread out in many existing literatures.

$$S^*=\left\{f(z)\in S:\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\beta,\ z\in \mathrm{E}\right\}$$

$$K=\left\{f(z)\in S:\mathrm{Re}\left\{1+\frac{zf'(z)}{f'(z)}\right\}>\beta,\ z\in \mathrm{E}\right\}$$

Recently, Kanas and Ronning [1] introduced a new concept of analytic functions by introducing ω is an arbitrary fixed point in U and this class of functions is denoted by $\Gamma(\omega)$ and is of the form

$$f(z)=(z-\omega)+\sum_{k=2}^{\infty}a_k(z-\omega)^k,\ z\in \mathrm{E} \tag{1.2}$$

which are analytic in the unit disk $U=\{z:|z|<1\}$ and normalized with $f(\omega)=0$ and $f'(\omega)-1=0$ and ω is a fixed point in U.

Kanas and Ronning [1] also define the following classes

$$ST(\omega)=S^*(\omega)=\left\{f\in S(\omega):\mathrm{Re}\left(\frac{(z-\omega)f'(z)}{f(z)}\right)>0,\ z\in U\right\}$$

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$$CV(\omega) = S^c(\omega) = \left\{ f \in S(\omega) : 1 + \operatorname{Re}\left(\frac{(z-\omega) f''(z)}{f'(z)} \right) > 0,\ z \in U \right\}$$

The two classes are respectively the classes of ω–starlike and ω–convex functions.

Several other authors, the likes of Acu and Owa [2], Oladipo [3], Aouf et al [4], Oladipo et al [5] have studied various aspect of these classes of functions and they obtained many interesting results:

Let $P(\omega) \subset P$ (class of caratheodory functions) denote the class of all functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} B_k (z-\omega)^k,\ z \in U \tag{1.3}$$

where

$$|B_k| \le \frac{2}{(1+d)(1-d)^k},\ k \ge 1,\ d = |\omega|,\ [6].$$

The authors here wish to give the following definitions

Definition 1.1

A function $f \in \Gamma(\omega)$ is said to be in the class $\Phi(\omega, \beta, b., m, \lambda, l)$ if and only if

$$\operatorname{Re}\left\{ 1 - \frac{2}{b} + \frac{2}{b} \frac{I_\omega^{m+1}(\lambda,l) f(z)}{I_\omega^{m}(\lambda,l) f(z)} \right\} > \beta \tag{1.4}$$

where $b \ne 0, m > -1,\ 0 \le \beta < 1, \omega$ is an arbitrary fixed point in U and $I_\omega^m(\lambda, l)$ is the Aouf et al derivative operator such that

$$I_\omega^m(\lambda,l) : \Gamma(\omega) \to \Gamma(\omega).$$

and

$$I_\omega^0(\lambda,l) f(z) = f(z)$$

$$I_\omega^1(\lambda,l) f(z) = I_\omega(\lambda,l) f(z) = I_\omega^0(\lambda,l) f(z) \frac{1-\lambda+l}{1+l} + \left((I_\omega^0(\lambda,l) f(z))' \frac{\lambda(z-\omega)}{1+l} \right)$$

$$= (z-\omega) + \sum_{k=2}^{\infty} \left(\frac{1+\lambda(k-1)+l}{1+l} \right) a^k (z-\omega)^k$$

$$I_\omega^m(\lambda,l) f(z) = I_\omega(\lambda,l)\left(I_\omega^{m-1}(\lambda,l) f(z)\right) = (z-\omega) + \sum_{k=2}^{\infty} \left(\frac{1+\lambda(k-1)+l}{1+l} \right)^m a^k (z-\omega)^k$$

for $\lambda \ge 0,\ l \ge 0, m \in N_0 \text{ and } z \in U$ [4,7].

Definition 1.2

A function $f(z) \in \Gamma(\omega)$ is said to be in the class $\Phi(\omega, \alpha, \beta, b, m, \lambda, l)$ if and only if

$$\operatorname{Re}\left\{ 1 - \frac{2}{b} + \frac{2}{b} \frac{I_\omega^{m+1}(\lambda,l) f(z)}{I_\omega^{m}(\lambda,l) f(z)} \right\} > \alpha \left| \frac{2 I_\omega^{m+1}(\lambda,l) f(z)}{b I_\omega^{m}(\lambda,l) f(z)} - 1 \right| + \beta \tag{1.5}$$

where $b \ne 0,\ \alpha \ge 0,\ 0 \le \beta < 1,\ m > -1,\quad \lambda \ge 0, l \ge 0$ and ω is an arbitrary fixed point in U.

Remark A.

i. If $\alpha = 0$ in Definition 1.2, Definition 1.2 gives Definition 1.1. That is, setting $\alpha = 0$ in all the results that will be obtained using definition 1.2 will hold for class of functions in Definition 1.1

ii. Also setting $b = 2, m = 0,\ \omega = 0$ we obtain

$$\operatorname{Re}\left\{ \frac{I_0^1(\lambda,l) f(z)}{I_0^0(\lambda,l) f(z)} \right\} > \alpha \left| \frac{I_0^1(\lambda,l) f(z)}{I_0^0(\lambda,l) f(z)} - 1 \right| + \beta$$

iii. If we put $\lambda = 1,\ l = 0$ in the remark A (ii), we have

$$\operatorname{Re}\left\{ \frac{I_0^1 f(z)}{I_0^0 f(z)} \right\} > \alpha \left| \frac{I_0^1 f(z)}{I_0^0 f(z)} - 1 \right| + \beta$$

which have been studied by various authors as could be seen in many existing literatures [8,9,10,11,12].

Lemma 1.1 [13]**:** A function $\mathrm{p}(z)\in B$ satisfies the following conditions:

$$R[\mathrm{p}(z)]>0 \qquad (z\in E)$$

if and only if

$$p(z)\neq\frac{\psi-1}{\psi+1} \quad (z\in E,\psi\in C,|\psi|=1)$$

Proof:

For the sake of completeness, we choose to give the proof of Lemma 1.1, even though, it is fairly obvious that the following bilinear (or Mobius) transformation:

$$h=\frac{z-1}{z+1}$$

maps the unit circle ∂u on to the imaginary axis $R(h)=0$. Indeed, for all ψ such that $|\psi|=1$ $(|\psi|=1,\psi\in C)$, we set

$$h=\frac{\psi-1}{\psi+1} \qquad (\psi\in C;|\psi|=1),$$

then

$$|\psi|=\left|\frac{1+h}{1-h}\right|=1,$$

which shows that

$$R(h)=R\left(\frac{\psi-1}{\psi+1}\right)=0 \qquad (\psi\in C;|\psi|=1)$$

Moreover, by noting that p(0)=1 for $p(z)\in B$, we know that

$$\mathrm{p}(z)\neq\frac{\psi-1}{\psi+1} \qquad (z\in E:\psi\in C;|\psi|=1)$$

This completes the proof of Lemma 1.1.

2. Coefficient inequality for the class $\Phi(\omega,\alpha,\beta,b,m,\lambda,l)$

Theorem 2.1: If $f(z)\in\Phi(\omega,\alpha,\beta,b,m,\lambda,l)$ with $0\le\alpha\le\beta$ or $\alpha>\frac{1+\beta}{2}$ **,** then $f(z)\in\Phi\left(\omega,\frac{\beta-\alpha}{1-\alpha},b,m,\lambda,l\right)$, and ω is an arbitrary fixed point in U .

Proof: Since $\mathrm{Re}(\gamma)\le|\gamma|$ for any complex number γ. $f(z)\in\Phi(\omega,\alpha,\beta,b,,m,\lambda,l)$ implies that

$$\mathrm{Re}\left\{1-\frac{2}{b}+\frac{2}{b}\frac{I_\omega^{m+1}(\lambda,l)f(z)}{I_\omega^{m}(\lambda,l)f(z)}\right\}>\alpha\left|\frac{2I_\omega^{m+1}(\lambda,l)f(z)}{bI_\omega^{m}(\lambda,l)f(z)}-\frac{2}{b}\right|+\beta$$

Or equivalently

$$\mathrm{Re}\left\{1-\frac{2}{b}+\frac{2}{b}\frac{I_{\omega}^{m+1}(\lambda,l)f(z)}{I_{\omega}^{m}(\lambda,l)f(z)}\right\}>\frac{\beta-\alpha}{1-\alpha},(z\in U)$$

If $0\le\alpha\le\beta$, we have $0\le\frac{\beta-\alpha}{1-\alpha}<1$ and if $\alpha>\frac{1+\beta}{2}$ $\Rightarrow -1<\frac{\alpha-\beta}{\alpha-1}\le 0$

Corollaries

(i) For $\omega=0$, we have $f(z)\in VD(\alpha,\beta,b,m,\lambda,l)$ with $0\le\alpha\le\beta$ or $\alpha>\frac{1+\beta}{2}$ then $f(z)\in V\left(\frac{\beta-\alpha}{1-\alpha},b,m,\lambda,l\right)$ i.e $\Phi(0,\alpha,\beta,b,m,\lambda,l)\equiv VD(\alpha,\beta,b,m,\lambda,l)$

that is

$$\mathrm{Re}\left\{1-\frac{2}{b}+\frac{2}{b}\frac{I_{0}^{m+1}(\lambda,l)f(z)}{I_{0}^{m}(\lambda,l)f(z)}\right\}>\alpha\left|\frac{2}{b}\frac{I_{0}^{m+1}(\lambda,l)f(z)}{I_{0}^{m}(\lambda,l)f(z)}-\frac{2}{b}\right|+\beta$$

(ii) With various choices of the parameters involved both new and existing results could be obtained.

We shall derive the following Lemma for the purpose of our next result.

Lemma 2.1: A function $f(z)\in\Gamma(\omega)$ is in the class $\Phi(\omega,\alpha,\beta,b,m,\lambda,l)$ if and only if

$$1+\sum_{k=2}^{\infty}A_k(z-\omega)^{k-1}\neq 0 \qquad (2.1)$$

where

$$A_k=\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m\left[\frac{\lambda(1-\alpha)(k-1)+b(1-\beta)(1+l)+\lambda(1-\alpha)(k-1)\psi}{b(1-\beta)(1+l)}\right]a_k \qquad (2.2)$$

Proof: Upon setting

$$p(z)=\frac{1-\alpha\left[1-\frac{2}{b}+\frac{2I_{\omega}^{m+1}(\lambda,l)f(z)}{bI_{\omega}^{m}(\lambda,l)f(z)}\right]-(\beta-\alpha)}{1-\beta},\quad \left(f(z)\in\Phi(\omega,\mathrm{m},\alpha,\beta,\lambda,\mathrm{l},\mathrm{b})\right)$$

we find that $p(z)\in B$ *and* $\mathrm{Re}[p(z)]>0\quad(z\in U)$.

Using Lemma 1.1, we have3

$$p(z)=\frac{1-\alpha\left[1-\frac{2}{b}+\frac{2I_{\omega}^{m+1}(\lambda,l)f(z)}{bI_{\omega}^{m}(\lambda,l)f(z)}\right]-(\beta-\alpha)}{1-\beta}\neq\frac{\psi-1}{\psi+1} \qquad (2.3)$$

which readily gives

$$2I_{\omega}^{m+1}(\lambda,l)f(z)-2I_{\omega}^{m}(\lambda,l)f(z)+\frac{2b+2\beta b}{(\psi+1)(1-\alpha)}I_{\omega}^{m}(\lambda,l)f(z)\neq 0$$

Thus we find that

$$b(1-\beta)(z-\omega)+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k(z-\omega)^k\left[\frac{(\psi+1)(1-\alpha)(\lambda(k-1))+b(1-\beta)(1+l)}{(1+l)}\right]\neq 0$$

such that

$$1+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m\left[\frac{\lambda(1-\alpha)(k-1)+b(1-\beta)(1+l)+\lambda(1-\alpha)(k-1)\psi}{b(1+l)(1-\beta)}\right]a_k(z-\omega)^{k-1}\neq 0$$

which complete the proof of Lemma 2.1.

At $\lambda=1$, $l=0$ and $b=2$ in the Lemma above, we have

Corollary 2.1: A function $f(z) \in \Gamma(\omega)$ is in the class $\Phi(\omega, \mathrm{m}, \alpha, \beta, 1, 0, 2)$ if and only if

$$1 + \sum A_k (z-\omega)^{k-1} \neq 0$$

where

$$A_k = \frac{k^m \left[(1-\alpha)(k-1) + 2(1-\beta) + (1-\alpha)(k-1)\psi\right]}{2(1-\beta)} a_k$$

setting $\lambda = 1, l = 0, b = 2$ and m=1, we have

Corollary 2.2: A function $f(z) \in \Gamma(\omega)$ is in the class $\Phi(\omega, \alpha, \beta, 1, 1, 0, 2)$ if and only if

$$1 + \sum A_k (z-\omega)^{k-1} \neq 0$$

where

$$A_k = \frac{k \; \left[(1-\alpha)(k-1) + 2(1-\beta) + (1-\alpha)(k-1)\psi\right]}{2(1-\beta)} a_k$$

Putting $\lambda = 1, l = 0, b = 2, m = 1, \alpha = 0 \textit{ and } \beta = 0$

Corollary 2.3: A function $f(z) \in \Gamma(\omega)$ is in the class $\Phi(\omega, 1, 0, 0, 0, 1, 2)$ if and only if

$$1 + \sum A_k (z-\omega)^{k-1} \neq 0$$

where

$$A_k = \frac{k \; \left[(k-1) + 2 + (k-1)\psi\right]}{2} a_k$$

Theorem 2.2: If $f(z) \in \Gamma(\omega)$ satisfies the following condition:

$$\sum_{k=2}^{\infty} (r+d)^{k-1} \left(\left| \sum_{t=1}^{k} \left(\sum_{j=1}^{t} (-1)^{t-j} \left(\frac{1+\lambda(j-1)+l}{1+l} \right)^m (\lambda(1-\alpha)(j-1) + b(1-\beta)(1+l) \binom{\delta}{t-j} a_j \right) \binom{\gamma}{k-t} \right| \right.$$
$$\left. + \left| \sum_{t=1}^{k} \left(\sum_{j=1}^{t} (-1)^{t-j} \left(\frac{1+\lambda(j-1)+l}{1+l} \right)^m \lambda(1-\alpha)(j-1) \binom{\delta}{t-j} a_j \right) \binom{\gamma}{k-t} \right| \right) \leq b(1-\beta)(1+l) \qquad (2.4)$$

Proof: First of all, we note that

$$(1-(z-\omega))^{\delta} \neq 0 \quad \text{and} \quad (1+(z-\omega))^{\gamma} \neq 0 \qquad (z \in E, \delta \in R, \gamma \in R) \qquad 2.5$$

Hence, if the following inequality

$$\left(1 + \sum_{k=2}^{\infty} A_k (z-\omega)^{k-1}\right)(1-(z-\omega))^{\delta}(1+(z-\omega))^{\gamma} \neq 0 \qquad (z \in E, \delta \in R, \gamma \in R) \qquad (2.6)$$

holds true, then we have

$$1 + \sum_{k=2}^{\infty} A_k z^{k-1} \neq 0$$

It is easily seen that (2.6) is equivalent to

$$\left(1+\sum_{k=2}^{\infty}A_k(z-\omega)^{k-1}\right)\left(\sum_{k=0}^{\infty}(-1)^k b_k(z-\omega)^k\right)\left(\sum_{k=0}^{\infty}c_k(z-\omega)^k\right)\neq 0 \qquad (2.7)$$

where for convenience,

$$b_k=\binom{\delta}{k} \text{ and } c_k=\binom{\gamma}{k}$$

Considering the Cauchy product, the first two factors of (2.7) can be written as follows

$$\left(1+\sum_{k=2}^{\infty}B_k(z-\omega)^{k-1}\right)\left(\sum_{k=0}^{\infty}c_k(z-\omega)^k\right), \qquad (2.8)$$

where

$$B_k=\sum_{j=1}^{k}(-1)^{k-j}A_j b_{k-j}.$$

Furthermore, by applying the same method for the Cauchy product in (2.8), we find that

$$1+\sum_{k=2}^{\infty}\left(\sum_{t=1}^{k}B_t c_{k-t}\right)(z-\omega)^{k-1}\neq 0 \quad (z\in E)$$

or equivalently, that

$$1+\sum_{k=2}^{\infty}\left[\sum_{t=1}^{k}\left(\sum_{j=1}^{t}(-1)^{t-j}A_j b_{t-j}\right)c_{k-t}\right](z-\omega)^{k-1}\neq 0. \qquad (z\in E) \qquad (2.9)$$

Thus, if $f(z)\in A$ satisfies the following inequality

$$\sum_{k=2}^{\infty}\left|\sum_{t=1}^{k}\left(\sum_{j=1}^{t}(-1)^{t-j}A_j b_{t-j}\right)c_{k-t}\right|(r+d)^{k-1}\leq 1$$

$$\sum_{k=2}^{\infty}\left|\sum_{t=1}^{k}\left(\sum_{j=1}^{t}(-1)^{t-j}\left(\frac{1+\lambda(j-1)+l}{1+l}\right)^m\left[\frac{\lambda(1-\alpha)(j-1)+b(1-\beta)(1+l)+\lambda(1-\alpha)(j-1)\psi}{b(1+l)(1-\beta)}\right]a_j b_{t-j}\right)c_{k-t}\right|(r+d)^{k-1}$$ That is, if ≤ 1.

$$\frac{1}{b(1-\beta)(1+l)}\sum_{k=2}^{\infty}\left|\sum_{t=1}^{k}\left(\sum_{j=1}^{t}(-1)^{t-j}\left(\frac{1+\lambda(j-1)+l}{1+l}\right)^m(\lambda(1-\alpha)(j-1)+b(1-\beta)(1+l)+\lambda(1-\alpha)(j-1)\psi)a_j b_{t-j}\right)c_{k-t}\right|(r+d)^{k-1}$$

$$\leq\frac{1}{b(1-\beta)(1+l)}\sum_{k=2}^{\infty}\left(\left|\sum_{t=1}^{k}\left(\sum_{j=1}^{t}(-1)^{t-j}\left(\frac{1+\lambda(j-1)+l}{1+l}\right)^m(\lambda(1-\alpha)(j-1)+b(1-\beta)(1+l))a_j b_{t-j}\right)c_{k-t}\right|(r+d)^{k-1}\right.$$

$$\left.+\left|\sum_{t=1}^{k}\left(\sum_{j=1}^{t}(-1)^{t-j}\left(\frac{1+\lambda(j-1)+l}{1+l}\right)^m\lambda(1-\alpha)(j-1)a_j b_{t-j}\right)c_{k-t}\right|(r+d)^{k-1}\right) \qquad (2.10)$$

This completes the proof:

Setting $\lambda=1, l=0$ in Theorem 2.1 we have the following

Corollary 2.4: if $f(z)\in\Gamma(\omega)$ satisfies the inequality

$$\sum_{k=2}^{\infty}(r+d)^{k-1}\left(\left|\sum_{t=1}^{k}\left(\sum_{j=1}^{t}(-1)^{t-j}j^{m}(1-\alpha)(j-1)+b(1-\beta)\binom{\delta}{t-j}a_j\right)\binom{\gamma}{k-t}\right|+\left|\sum_{t=1}^{k}\left(\sum_{j=1}^{t}(-1)^{t-j}(1-\alpha)(j-1)\binom{\delta}{t-j}a_j\right)\binom{\gamma}{k-t}\right|\right)\le b(1-\beta)$$

Then $f(z)\in\Phi(\omega,\alpha,\beta,b,m,1,0)$

$At\ \ \lambda=1, l=0, b=2\ \ \ we\ have$

Corollary 2.5: if $f(z)\in\Gamma(\omega)$ satisfies the inequality

$$\sum_{k=2}^{\infty}(r+d)^{k-1}\left(\left|\sum_{t=1}^{k}\left(\sum_{j=1}^{t}(-1)^{t-j}j^{m}(1-\alpha)(j-1)+2(1-\beta)\binom{\delta}{t-j}a_j\right)\binom{\gamma}{k-t}\right|+\left|\sum_{t=1}^{k}\left(\sum_{j=1}^{t}(-1)^{t-j}(1-\alpha)(j-1)\binom{\delta}{t-j}a_j\right)\binom{\gamma}{k-t}\right|\right)\le 2(1-\beta)$$

then $f(z)\in\Phi(\omega,\alpha,\beta,2,m,1,0)$

On putting $\lambda=1,\ l=0, \beta=0, b=2\ \ \ and\ \ m=1$ we have

Corollary 2.6: if $f(z)\in\Gamma(\omega)$ satisfies the inequality

$$\sum_{k=2}^{\infty}(r+d)^{k-1}\left(\left|\sum_{t=1}^{k}\left(\sum_{j=1}^{t}(-1)^{t-j}j^{m}(1-\alpha)(j-1)+2\binom{\delta}{t-j}a_j\right)\binom{\gamma}{k-t}\right|+\left|\sum_{t=1}^{k}\left(\sum_{j=1}^{t}(-1)^{t-j}(1-\alpha)(j-1)\binom{\delta}{t-j}a_j\right)\binom{\gamma}{k-t}\right|\right)\le 2$$

Then $f(z)\in\Phi(\omega,\alpha,0,2,1,1,0)$,

3.Coefficient Bounds

Theorem 3.1: If $f(z)\in\Phi(\omega,\alpha,\beta,b,m,\lambda,l)$, then

$$|a_2|\le\left(\frac{1+l}{1+\lambda+l}\right)^m\frac{b(1-\beta)(1+l)}{\lambda|1-\alpha|(1-d^2)},\ d=|\omega| \tag{3.1}$$

and

$$|a_3|\le\left(\frac{1+l}{1+2\lambda+l}\right)^m\left[\frac{b(1-\beta)(1+l)}{2\lambda|1-\alpha|(1-d^2)(1-d)}+\frac{b^2(1-\beta)^2(1+l)^2}{2\lambda^2|1-\alpha|^2(1-d^2)^2}\right] \tag{3.2}$$

$$|a_4| \le \left(\frac{1+l}{1+3\lambda+l}\right)^m \left[\frac{b(1-\beta)(1+l)}{3\lambda|1-\alpha|(1-d^2)(1-d)^2} + \frac{b^2(1-\beta)^2(1+l)^2}{2\lambda^2|1-\alpha|^2(1-d^2)^2(1-d)} + \frac{b^3(1-\beta)^3(1+l)^3}{6\lambda^3|1-\alpha|^3(1-d^2)^3}\right] \quad (3.3)$$

$$|a_5| \le \left(\frac{1+l}{1+4\lambda+l}\right)^m \left[\begin{array}{l}\frac{b(1-\beta)(1+l)}{4\lambda|1-\alpha|(1-d^2)(1-d)^3} + \frac{11b^2(1-\beta)^2(1+l)^2}{24\lambda^2|1-\alpha|^2(1-d^2)^2(1-d)^2} + \\ \frac{3b^3(1-\beta)^3(1+l)^3}{8\lambda^3|1-\alpha|^3(1-d^2)^3(1-d)} + \frac{b^4(1-\beta)^4(1+l)^4}{24\lambda^4|1-\alpha|^4(1-d^2)^4}\end{array}\right] \quad (3.4)$$

Proof :

For $f(z) \in \Phi(\omega,\alpha,\beta,b,m,\lambda,l)$

$$\left\{1-\frac{2}{b}+\frac{2}{b}\frac{I_\omega^{m+1}(\lambda,l)f(z)}{I_\omega^m(\lambda,l)f(z)}\right\} > \frac{\beta-\alpha}{1-\alpha}$$

we define the function

$$p(z) = \frac{(1-\alpha)\left[1-\frac{2}{b}+\frac{2}{b}\frac{I_\omega^{m+1}(\lambda,l)f(z)}{I_\omega^m(\lambda,l)f(z)}\right]-(\beta-\alpha)}{1-\beta}, \quad z\in U$$

and ω is an arbitrary fixed point in U . Then $p(z)$ is analytic in U with $p(\omega)=1$ and $\operatorname{Re} P(z) > 0$

Let $p(z)$ be as earlier defined in (1.3). That is,

$$p(z) = 1+\sum_{k=1}^{\infty} B_k (z-\omega)^k, \quad z\in \mathrm{E}$$

we have

$$(1-\alpha)\left[1-\frac{2}{b}+\frac{2}{b}\frac{I_\omega^{m+1}(\lambda,l)f(z)}{I_\omega^m(\lambda,l)f(z)}\right]-(\beta-\alpha) = (1-\beta)p(z) \quad (3.5)$$

$$b(1-\beta)p(z) = (1-\alpha)\left[b-2+\frac{2I_\omega^{m+1}(\lambda,l)f(z)}{I_\omega^m(\lambda,l)f(z)}\right]-b(\beta-\alpha)$$

which implies

$$2\left(I_\omega^{m+1}(\lambda,l)f(z) - I_\omega^m(\lambda,l)f(z)\right) = I_\omega^m(\lambda,l)f(z)\left[\frac{b(1-\beta)}{(1-\alpha)}\sum_{k=1}^{\infty}B_k(z-\omega)^k\right]$$

$$2\left[(z-\omega)+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^{m+1}a_k(z-\omega)^k-(z-\omega)-\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k(z-\omega)^k\right] =$$

$$\left[(z-\omega)+\sum_{k=2}^{\infty}\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m a_k(z-\omega)^k\right]\left[\frac{b(1-\beta)}{|1-\alpha|}\sum_{k=1}^{\infty}B_k(z-\omega)^k\right]$$

$$2\left[\begin{array}{l}(z-\omega)+\left(\frac{1+\lambda+l}{1+l}\right)^{m+1}a_2\ (z-\omega)^2+\left(\frac{1+2\lambda+l}{1+l}\right)^{m+1}a_3\ (z-\omega)^3+\left(\frac{1+3\lambda+l}{1+l}\right)^{m+1}a_4\ (z-\omega)^4+ \\ \left(\frac{1+5\lambda+l}{1+l}\right)^{m+1}a_5\ (z-\omega)^5+\ldots-(z-\omega)-\left(\frac{1+\lambda+l}{1+l}\right)^m a_2\ (z-\omega)^2-\left(\frac{1+2\lambda+l}{1+l}\right)^m a_3\ (z-\omega)^3- \\ \left(\frac{1+3\lambda+l}{1+l}\right)^m a_4\ (z-\omega)^4-\left(\frac{1+4\lambda+l}{1+l}\right)^{m+1}a_5\ (z-\omega)^5-\ldots\end{array}\right] =$$

$$\left[(z-\omega)+\left(\frac{1+\lambda+l}{1+l}\right)^m a_2\ (z-\omega)^2+\left(\frac{1+2\lambda+l}{1+l}\right)^m a_3\ (z-\omega)^3+\left(\frac{1+3\lambda+l}{1+l}\right)^m a_4\ (z-\omega)^4+\ldots\right]\times$$

$$\left[\frac{b(1-\beta)}{|1+\alpha|}\left(B_1(z-\omega)+B_2(z-\omega)^2+B_3(z-\omega)^3+B_4(z-\omega)^4\right)+\ldots\right]$$

$$2\left[\begin{array}{l}\left(\frac{1+\lambda+l}{1+l}\right)^m\left(\frac{1+\lambda+l}{1+l}-1\right)a_2\ (z-\omega)^2+\left(\frac{1+2\lambda+l}{1+l}\right)^m\left(\frac{1+2\lambda+l}{1+l}-1\right)a_3\ (z-\omega)^3+\\ \left(\frac{1+3\lambda+l}{1+l}\right)^m\left(\frac{1+3\lambda+l}{1+l}-1\right)a_4\ (z-\omega)^4+\left(\frac{1+4\lambda+l}{1+l}\right)^m\left(\frac{1+4\lambda+l}{1+l}-1\right)a_5\ (z-\omega)^5\end{array}\right]=$$

$$\left[(z-\omega)+\left(\frac{1+\lambda+l}{1+l}\right)^m a_2\ (z-\omega)^2+\left(\frac{1+2\lambda+l}{1+l}\right)^m a_3\ (z-\omega)^3+\left(\frac{1+3\lambda+l}{1+l}\right)^m a_4\ (z-\omega)^4+\ldots\right]\times$$

$$\left[\frac{b(1-\beta)}{|1+\alpha|}B_1(z-\omega)+\frac{b(1-\beta)}{|1+\alpha|}B_2(z-\omega)^2+\frac{b(1-\beta)}{|1+\alpha|}B_3(z-\omega)^3+\frac{b(1-\beta)}{|1+\alpha|}B_4(z-\omega)^4+\ldots\right]$$

$$2\left[\begin{array}{l}\lambda\left(\frac{1+\lambda+l}{1+l}\right)^m a_2\ (z-\omega)^2+2\lambda\left(\frac{1+2\lambda+l}{1+l}\right)^m a_3\ (z-\omega)^3+3\lambda\left(\frac{1+3\lambda+l}{1+l}\right)^m a_4\ (z-\omega)^4+\\ 4\lambda\left(\frac{1+4\lambda+l}{1+l}\right)^m a_5\ (z-\omega)^5\end{array}\right]$$

$$=\left[\begin{array}{l}\frac{b(1-\beta)}{|1+\alpha|}B_1(z-\omega)^2+\frac{b(1-\beta)}{|1+\alpha|}B_2(z-\omega)^3+\frac{b(1-\beta)}{|1+\alpha|}B_3(z-\omega)^4+\frac{b(1-\beta)}{|1+\alpha|}B_4(z-\omega)^5+\ldots+\\ \frac{b(1-\beta)}{|1+\alpha|}\left(\frac{1+\lambda+l}{1+l}\right)^m a_2B_1(z-\omega)^3+\frac{b(1-\beta)}{|1+\alpha|}\left(\frac{1+\lambda+l}{1+l}\right)^m a_2B_2(z-\omega)^4+\frac{b(1-\beta)}{|1+\alpha|}\left(\frac{1+\lambda+l}{1+l}\right)^m a_2B_3(z-\omega)^5\\ +\ldots+\frac{b(1-\beta)}{|1+\alpha|}\left(\frac{1+2\lambda+l}{1+l}\right)^m a_3B_1(z-\omega)^4+\frac{b(1-\beta)}{|1+\alpha|}\left(\frac{1+2\lambda+l}{1+l}\right)^m a_3B_2(z-\omega)^5+\ldots\\ +\frac{b(1-\beta)}{|1+\alpha|}\left(\frac{1+3\lambda+l}{1+l}\right)^m a_4B_1(z-\omega)^5+\ldots\end{array}\right] \quad (3.6)$$

Comparing the coefficient of $(z-\omega)$ in (3.6), we have

$$\frac{2\lambda}{1+l}\left(\frac{1+\lambda+l}{1+l}\right)^m a_2=\frac{b(1-\beta)}{|1-\alpha|}B_1\Rightarrow\left(\frac{1+\lambda+l}{1+l}\right)^m a_2=\frac{b(1-\beta)(1+l)}{2\lambda|1-\alpha|}B_1$$

$$a_2=\left(\frac{1+l}{1+\lambda+l}\right)^m\frac{b(1-\beta)(1+l)}{2\lambda|1-\alpha|}B_1$$

Recall

$$|B_k|=\frac{2}{(1+d)(1-d)^k}\ ,k\geq 1$$

$$|B_1|=\frac{2}{(1+d)(1-d)}=\frac{2}{(1-d^2)}$$

$$\therefore\quad |a_2|\leq\left(\frac{1+l}{1+\lambda+l}\right)^m\frac{b(1-\beta)(1+l)}{\lambda|1-\alpha|(1-d^2)}$$

$$\frac{4\lambda}{1+l}\left(\frac{1+2\lambda+l}{1+l}\right)^m a_3=\frac{b(1-\beta)}{|1-\alpha|}B_2+\frac{b(1-\beta)}{|1-\alpha|}\left(\frac{1+\lambda+l}{1+l}\right)^m a_2B_1$$

$$a_3=\left(\frac{1+l}{1+2\lambda+l}\right)^m\frac{b(1-\beta)(1+l)}{4\lambda|1-\alpha|}B_2+\frac{b(1-\beta)(1+l)}{4\lambda|1-\alpha|}\left(\frac{1+l}{1+2\lambda+l}\right)^m\left(\frac{1+\lambda+l}{1+l}\right)^m a_2B_1$$

$$=\left(\frac{1+l}{1+2\lambda+l}\right)^m\frac{b(1-\beta)(1+l)}{4\lambda|1-\alpha|}B_2+\frac{b(1-\beta)(1+l)}{4\lambda|1-\alpha|}\left(\frac{1+\lambda+l}{1+2\lambda+l}\right)^m\left[\left(\frac{1+l}{1+\lambda+l}\right)^m\frac{b(1-\beta)(1+l)}{2\lambda|1-\alpha|}\right]B_1^{\ 2}$$

$$=\left(\frac{1+l}{1+2\lambda+l}\right)^m\frac{b(1-\beta)(1+l)}{4\lambda|1-\alpha|}B_2+\frac{b(1-\beta)^2(1+l)^2}{8\lambda^2|1-\alpha|^2}\left(\frac{1+l}{1+2\lambda+l}\right)^m B_1^{\ 2}$$

$$|a_3|\le\left(\frac{1+l}{1+2\lambda+l}\right)^m\frac{b(1-\beta)(1+l)}{2\lambda|1-\alpha|(1-d^2)(1-d)}+\frac{b^2(1-\beta)^2(1+l)^2}{2\lambda^2|1-\alpha|^2(1-d^2)^2}\left(\frac{1+l}{1+2\lambda+l}\right)^m$$

$$|a_3|\le\left(\frac{1+l}{1+2\lambda+l}\right)^m\left[\frac{b(1-\beta)(1+l)}{2\lambda|1-\alpha|(1-d^2)(1-d)}+\frac{b^2(1-\beta)^2(1+l)^2}{2\lambda^2|1-\alpha|^2(1-d^2)^2}\right]$$

Comparing the coefficient of $(z-\omega)^4$, we have

$$\frac{6\lambda}{1+l}\left(\frac{1+3\lambda+l}{1+l}\right)^m a_4=\frac{b(1-\beta)}{|1-\alpha|}B_3+\frac{b(1-\beta)}{|1-\alpha|}\left(\frac{1+\lambda+l}{1+l}\right)^m a_2B_2+\frac{b(1-\beta)}{|1-\alpha|}\left(\frac{1+2\lambda+l}{1+l}\right)^m a_3B_1$$

$$a_4=\left(\frac{1+l}{1+3\lambda+l}\right)^m\frac{b(1-\beta)(1+l)}{6\lambda|1-\alpha|}B_3+\frac{b(1-\beta)(1+l)}{6\lambda|1-\alpha|}\left(\frac{1+\lambda+l}{1+3\lambda+l}\right)^m\left[\left(\frac{1+l}{1+\lambda+l}\right)^m\frac{b(1-\beta)(1+l)}{2\lambda|1-\alpha|}\right]B_1B_2+$$

$$\frac{b(1-\beta)(1+l)}{6\lambda|1-\alpha|}\left(\frac{1+2\lambda+l}{1+3\lambda+l}\right)^m\left[\left(\frac{1+l}{1+2\lambda+l}\right)^m\frac{b(1-\beta)(1+l)}{4\lambda|1-\alpha|}B_2+\frac{b(1-\beta)^2(1+l)^2}{8\lambda^2|1-\alpha|^2}\left(\frac{1+l}{1+2\lambda+l}\right)^m B_1^{\ 2}\right]B_1$$

$$=\left(\frac{1+l}{1+3\lambda+l}\right)^m\frac{b(1-\beta)(1+l)}{6\lambda|1-\alpha|}B_3+\frac{b^2(1-\beta)^2(1+l)^2}{12\lambda^2|1-\alpha|^2}\left(\frac{1+l}{1+3\lambda+l}\right)^m B_1B_2+$$

$$\frac{b^2(1-\beta)^2(1+l)^2}{24\lambda^2|1-\alpha|^2}\left(\frac{1+l}{1+3\lambda+l}\right)^m B_1B_2+\frac{b^3(1-\beta)^3(1+l)^3}{48\lambda^3|1-\alpha|^3}\left(\frac{1+l}{1+3\lambda+l}\right)^m B_1^{\ 3}$$

$$|a_4|\le\left(\frac{1+l}{1+3\lambda+l}\right)^m\frac{b(1-\beta)(1+l)}{3\lambda|1-\alpha|(1-d^2)(1-d)^2}+\frac{b^2(1-\beta)^2(1+l)^2}{3\lambda^2|1-\alpha|^2(1-d^2)^2(1-d)}\left(\frac{1+l}{1+3\lambda+l}\right)^m+$$

$$\frac{b^2(1-\beta)^2(1+l)^2}{6\lambda^2|1-\alpha|^2(1-d^2)^2(1-d)}\left(\frac{1+l}{1+3\lambda+l}\right)^m+\frac{b^3(1-\beta)^3(1+l)^3}{6\lambda^3|1-\alpha|^3(1-d^2)^3}\left(\frac{1+l}{1+3\lambda+l}\right)^m$$

$$|a_4|\le\left(\frac{1+l}{1+3\lambda+l}\right)^m\left[\frac{b(1-\beta)(1+l)}{3\lambda|1-\alpha|(1-d^2)(1-d)^2}+\frac{b^2(1-\beta)^2(1+l)^2}{2\lambda^2|1-\alpha|^2(1-d^2)^2(1-d)}+\frac{b^3(1-\beta)^3(1+l)^3}{6\lambda^3|1-\alpha|^3(1-d^2)^3}\right]$$

Comparing the coefficient of $(z-\omega)^5\ in\ which$, we have

$$\frac{8\lambda}{1+l}\left(\frac{1+4\lambda+l}{1+l}\right)^m a_5=\frac{b(1-\beta)}{|1-\alpha|}B_4+\frac{b(1-\beta)}{|1-\alpha|}\left(\frac{1+\lambda+l}{1+l}\right)^m a_2B_3+\frac{b(1-\beta)}{|1-\alpha|}\left(\frac{1+2\lambda+l}{1+l}\right)^m a_3B_2+$$

$$\frac{b(1-\beta)}{|1-\alpha|}\left(\frac{1+3\lambda+l}{1+l}\right)^m a_4B_1$$

$$a_5 = \left(\frac{1+l}{1+4\lambda+l}\right)^m \frac{b(1-\beta)(1+l)}{8\lambda|1-\alpha|} B_4 + \frac{b(1-\beta)(1+l)}{8\lambda|1-\alpha|}\left(\frac{1+\lambda+l}{1+4\lambda+l}\right)^m a_2 B_3 +$$

$$\frac{b(1-\beta)(1+l)}{8\lambda|1-\alpha|}\left(\frac{1+2\lambda+l}{1+4\lambda+l}\right)^m a_3 B_2 + \frac{b(1-\beta)(1+l)}{8\lambda|1-\alpha|}\left(\frac{1+3\lambda+l}{1+4\lambda+l}\right)^m a_4 B_1$$

$$= \left(\frac{1+l}{1+4\lambda+l}\right)^m \frac{b(1-\beta)(1+l)}{8\lambda|1-\alpha|} B_4 + \frac{b(1-\beta)(1+l)}{8\lambda|1-\alpha|}\left(\frac{1+\lambda+l}{1+4\lambda+l}\right)^m \left[\left(\frac{1+l}{1+\lambda+l}\right)^m \frac{b(1-\beta)(1+l)}{2\lambda|1-\alpha|}\right] B_1 B_3 +$$

$$\frac{b(1-\beta)(1+l)}{8\lambda|1-\alpha|}\left(\frac{1+2\lambda+l}{1+4\lambda+l}\right)^m \left[\left(\frac{1+l}{1+2\lambda+l}\right)^m \frac{b(1-\beta)(1+l)}{4\lambda|1-\alpha|} B_2 + \frac{b(1-\beta)^2(1+l)^2}{8\lambda^2|1-\alpha|^2}\left(\frac{1+l}{1+2\lambda+l}\right)^m B_1^{\,2}\right] B_2 +$$

$$\frac{b(1-\beta)(1+l)}{8\lambda|1-\alpha|}\left(\frac{1+3\lambda+l}{1+4\lambda+l}\right)^m \left[\begin{array}{l} \left(\frac{1+l}{1+3\lambda+l}\right)^m \frac{b(1-\beta)(1+l)}{6\lambda|1-\alpha|} B_3 + \frac{b^2(1-\beta)^2(1+l)^2}{12\lambda^2|1-\alpha|^2}\left(\frac{1+l}{1+3\lambda+l}\right)^m B_1 B_2 + \\ \frac{b^2(1-\beta)^2(1+l)^2}{24\lambda^2|1-\alpha|^2}\left(\frac{1+l}{1+3\lambda+l}\right)^m B_1 B_2 + \frac{b^3(1-\beta)^3(1+l)^3}{48\lambda^3|1-\alpha|^3}\left(\frac{1+l}{1+3\lambda+l}\right)^m B_1^{\,3} \end{array}\right] B_1$$

$$= \left(\frac{1+l}{1+4\lambda+l}\right)^m \frac{b(1-\beta)(1+l)}{8\lambda|1-\alpha|} B_4 + \frac{b(1-\beta)(1+l)}{8\lambda|1-\alpha|}\left(\frac{1+l}{1+4\lambda+l}\right)^m B_1 B_3 +$$

$$\frac{b^2(1-\beta)^2(1+l)^2}{32\lambda^2|1-\alpha|^2}\left(\frac{1+l}{1+4\lambda+l}\right)^m B_2^{\,2} + \frac{b^3(1-\beta)^3(1+l)^3}{64\lambda^3|1-\alpha|^2}\left(\frac{1+l}{1+4\lambda+l}\right)^m B_1^{\,2} B_2 +$$

$$\frac{b^2(1-\beta)^2(1+l)^2}{48\lambda^2|1-\alpha|^2}\left(\frac{1+l}{1+4\lambda+l}\right)^m B_1 B_3 + \frac{b^3(1-\beta)^3(1+l)^3}{96\lambda^3|1-\alpha|^3}\left(\frac{1+l}{1+4\lambda+l}\right)^m B_1^{\,2} B_2 +$$

$$\frac{b^3(1-\beta)^3(1+l)^3}{192\lambda^3|1-\alpha|^3}\left(\frac{1+l}{1+4\lambda+l}\right)^m B_1^{\,2} B_2 + \frac{b^4(1-\beta)^4(1+l)^4}{384\lambda^4|1-\alpha|^2}\left(\frac{1+l}{1+4\lambda+l}\right)^m B_1^{\,4}$$

$$|a_5| \le \left(\frac{1+l}{1+4\lambda+l}\right)^m \left[\begin{array}{l} \frac{b(1-\beta)(1+l)}{4\lambda|1-\alpha|(1-d^2)(1-d)^3} + \frac{11b^2(1-\beta)^2(1+l)^2}{24\lambda^2|1-\alpha|^2(1-d^2)^2(1-d)^2} + \\ \frac{3b^3(1-\beta)^3(1+l)^3}{8\lambda^3|1-\alpha|^3(1-d^2)^3(1-d)} + \frac{b^4(1-\beta)^4(1+l)^4}{24\lambda^4|1-\alpha|^4(1-d^2)^4} \end{array}\right]$$

This implies that consequently we have proved that (3.1) - (3.4) holds true for $k \ge 3$

Corollaries

- The parametric value $b = 2$ and $m = 0$ yields

  If $f(z) \in \Phi(\omega, \alpha, \beta, 2, 0, \lambda, l)$ then

$$|a_2| \le \frac{b(1-\beta)(1+l)}{\lambda|1-\alpha|(1-d^2)} \;\Rightarrow\; \frac{2(1-\beta)}{\lambda|1-\alpha|(1-d^2)}$$

and

$$|a_3| \le \frac{2(1-\beta)(1+l)}{2\lambda|1-\alpha|(1-d^2)(1-d)}\left[1 + \frac{2(1-\beta)(1+l)}{|1-\alpha|(1-d^2)(1-d)^{-1}}\right]$$

- Setting $\alpha = 0$ and $m = l = 0$ we have, if $f(z) \in \Phi(\omega, 0, \beta, 2, 0, \lambda, 0)$ then

$$|a_2| \le \frac{2(1-\beta)}{\lambda(1-d^2)}$$

and

$$|a_3| \leq \frac{2(1-\beta)}{2\lambda(1-d^2)(1-d)}\left[1+\frac{2(1-\beta)}{\lambda(1-d^2)(1-d)^{-1}}\right]$$

- Setting $\beta = \alpha = 0$ and $b = 2, m = l = 0$ we have, if $f(z) \in \Phi(\omega,0,0,2,0,\lambda,0)$, then

$$|a_2| \leq \frac{2}{(1-d^2)}$$

and

$$|a_3| \leq \frac{2}{2\lambda(1-d^2)(1-d)}\left[1+\frac{2}{\lambda(1-d^2)(1-d)^{-1}}\right]$$

- Putting $d = 0$ in Theorem 3.1, we have

$$|a_2| \leq \left(\frac{1+l}{1+\lambda+l}\right)^m \frac{b(1-\beta)(1+l)}{\lambda|1-\alpha|}$$

and

$$|a_3| \leq \left(\frac{1+l}{1+2\lambda+l}\right)^m \frac{b(1-\beta)(1+l)}{2\lambda|1-\alpha|}\left[1+\frac{b(1-\beta)(+l)}{\lambda|1-\alpha|}\right]$$

- and so on for $a_{4,}\ a_5$.

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