Journal of the Nigerian Association of Mathematical Physics
Volume **19 (November, 2011), pp 1 – 6**

Exponential Stability of Quantum Stochastic Dynamics

Yusuf Ibrahim

Department Of Mathematics,
Nigerian Defence Academy, Kaduna
Nigeria

Abstract

We study exponential equilibrium for operators of the form $\rho_\Lambda^{\frac{1-t}{2}}.x.\rho_\Lambda^{\frac{1-t}{2}}$, $x \in \mathcal{M}$, $t \in [0,1)$ in a Trunov-type L_p-space over a quasi-local von Neumann algebras $\mathcal{M}_0$.

Key words: quasi-local von Neumann algebras, non commutative L_p-space, density matrix h_Λ, quantum stochastic dynamics.

1. Introduction:

In noncommutative analysis one of the major problems is the construction of a dissipative quantum dynamical semigroup. The description of infinite quantum spin system is far less advanced and there is no satisfactory description of quantum stochastic dynamics especially for spin systems at high temperature or for one dimensional lattice with finite interaction at arbitrary finite temperature. An attempt to use the theory of noncommutative L_p spaces for the construction and analysis of quantum stochastic dynamics was initiated by Majewski and Zegarlinski [1], [2]. Here we use the method developed by them to study the dynamics of closed operators $\rho_\Lambda^{\frac{1-t}{2}}\, x\, \rho_\Lambda^{\frac{1-t}{2}}$ in a Trunov-type L_P spaces [3], over a quasi-local von Neumann algebra $\mathcal{M}_0$. The dynamics is given by the following evolution equation,
$\frac{d}{dt}P_t^{X,\Lambda} = \mathcal{L}_{X,\Lambda}\, P_t^{X,\Lambda}$; $P_0^{X,\Lambda} = id$, where $P_t^{X,\Lambda}$ is the dynamics and $\mathcal{L}_{X,\Lambda}$ the generator, with $X \subset \Lambda$. Here Λ is a finite subset of $\mathbb{Z}^d$ a d-dimensional integer lattice. The Lindblad-type generator $\mathcal{L}_{X,\Lambda}$ considered in this paper is the operator defined by

$$\mathcal{L}_{X,\Lambda}\left(\rho_\Lambda^{\frac{1-t}{2}}\, x\, \rho_\Lambda^{\frac{1-t}{2}}\right) = E_{X,\Lambda}\left(\rho_\Lambda^{\frac{1-t}{2}}\, x\, \rho_\Lambda^{\frac{1-t}{2}}\right) - \frac{1}{2}\left\{E_{X,\Lambda}(1), \left(\rho_\Lambda^{\frac{1-t}{2}}\, x\, \rho_\Lambda^{\frac{1-t}{2}}\right)\right\}.$$

where $E_{X,\Lambda}$ is the generalized conditional expectation defined on the von Neumann algebra $\mathcal{M}$, and $\{x, y\} = xy + yx$ is the anti commutator. The general form the generator $\mathcal{L}_{X,\Lambda}$ is given in [4]. The general construction of noncommutative L_p spaces is rather involved and cumbersome for application, but in the context of quantum spin system it has been shown that we can give a direct construction [1].

We introduce the operators of the form $\rho_n^{\frac{1-t}{2}}.\rho_n^{\frac{1-t}{2}}$ as follows: Let $\rho^{1-t} \geq 0$, $t \in [0,1)$ be a strong product of locally measurable operators $\rho^{\frac{1-t}{2}}$. From the definition of locally measurable operator given in [5], there exists a sequence (e_n) of projections in the center of the von Neumann algebra $\mathcal{M}$, such that $e_n \nearrow I$ and all the operators $\rho^{\frac{1-t}{2}}e_n$ are measurable. We denote these measurable operators by $\rho_n^{\frac{1-t}{2}}$ that is, $\rho^{\frac{1-t}{2}}e_n \equiv \rho_n^{\frac{1-t}{2}}$. Thus, ρ_n^{1-t} is a strong product of locally measurable operators

$$\rho_n^{1-t} = \rho_n^{\frac{1-t}{2}}.\rho_n^{\frac{1-t}{2}} = \rho^{\frac{1-t}{2}}e_n\,.\ \rho^{\frac{1-t}{2}}e_n$$

and hence locally measurable. Now for each $x \in \mathcal{M}$ we have that $x\rho_n^{\frac{1-t}{2}}$ is a closed densely defined operator. Now since e_n is a central projection we have $\rho^{\frac{1-t}{2}}e_n = e_n\rho^{\frac{1-t}{2}}$,

thus, $$\rho_n^{\frac{1-t}{2}}.x.\rho_n^{\frac{1-t}{2}} = \rho^{\frac{1-t}{2}}e_n\,.x.\ \rho^{\frac{1-t}{2}}e_n = \rho^{\frac{1-t}{2}}e_n\,.x.e_n\,\rho^{\frac{1-t}{2}}.$$

Corresponding authors: E-mail: Yusufian68@yahoo.com, Tel. +2347034972567

Let $\mathfrak{H}$ be a Hilbert space if $\xi \in \mathfrak{H}$ then the domain for $\rho_n^{\frac{1-t}{2}}.x.\rho_n^{\frac{1-t}{2}}$ is given by

$$\left\{\xi \in \mathfrak{H}:\ \xi \in D\left(x\rho_n^{\frac{1-t}{2}}\right), x\rho_n^{\frac{1-t}{2}}\xi \in D\left(\rho_n^{\frac{1-t}{2}}\right)\right\},$$

where $x\rho_n^{\frac{1-t}{2}}$ is a sequence of closed densely defined operators and $\rho_n^{\frac{1-t}{2}}$ is a sequence of bounded locally measurable operators. For any two operators $\left(\rho_n^{\frac{1-t}{2}}.x.\rho_n^{\frac{1-t}{2}}\right)$, $\left(\rho_n^{\frac{1-t}{2}}.y.\rho_n^{\frac{1-t}{2}}\right)$ addition is defined as follows $\left(\rho_n^{\frac{1-t}{2}}.x.\rho_n^{\frac{1-t}{2}}\right) + \left(\rho_n^{\frac{1-t}{2}}.y.\rho_n^{\frac{1-t}{2}}\right) = \rho_n^{\frac{1-t}{2}}(x+y)\rho_n^{\frac{1-t}{2}}$. We have the inverse operator $\left(\rho_n^{\frac{-(1-t)}{2}}.x^{-1}.\rho_n^{\frac{-(1-t)}{2}}\right)$, the identity operator $\left(\rho_n^{\frac{(1-t)}{2}}.I.\rho_n^{\frac{-(1-t)}{2}}\right)$ and the adjoint operator $\left(\rho_n^{\frac{1-t}{2}}.x.\rho_n^{\frac{1-t}{2}}\right)^* = \rho_n^{\frac{1-t}{2}}.x^*.\rho_n^{\frac{1-t}{2}}$.

Remark:
The indexing set to be used in our work is the finite subset Λ of the integer lattice space $\mathbb{Z}^d$ with $d \geq 1$.

2 Quasilocal Algebras

A quantum spin system consists of a set of particles confined to a lattice and interacting at a distance [6]. Let $\mathbb{Z}^d, d \geq 1$ be the dimensional integer lattice space, whose sites are occupied by spin-$\frac{1}{2}$ particles. The self adjoint operators at site $j \in \mathbb{Z}^d$ are elements of the algebra $\mathcal{M}_{\{j\}}$ and is isomorphic to $\mathcal{M}_2(\mathbb{C})$, where $\mathcal{M}_2(\mathbb{C})$ is a matrix algebra. The algebra of self adjoint operators localized to a finite region $X \in \mathbb{Z}^d$ is defined by $\mathcal{M}_X = \bigotimes_{j\in X}\mathcal{M}_{\{j\}}$, and is the full matrix algebra $\mathcal{M}_{2^{|X|}}(\mathbb{C})$. Let $\mathcal{F}$ be the class of all the finite subsets of $\mathbb{Z}^d$ ordered by inclusion. Let $X_1, X_2 \in \mathcal{F}$ be disjoint finite regions, $X_1 \cap X_2 = \emptyset$, and let us write $\mathcal{M}_{X_1 \cup X_2} = \mathcal{M}_{X_1} \otimes \mathcal{M}_{X_2}$ for the matrix algebra. On the other hand if $X_1 \subseteq X_2$ and I_{X_2} denote the identity of $\mathcal{M}_{X_2}$, then the algebra $\mathcal{M}_{X_1}$ is identified with the subalgebra $\mathcal{M}_{X_1} \otimes I_{X_2}$, that is, $\mathcal{M}_{X_1} \equiv \mathcal{M}_{X_1} \otimes I_{X_2}$. With this identification the algebra $\left(\mathcal{M}_{X_i}\right)_{X_i \in \mathcal{F}}$ form an increasing sequence of matrix algebras, with the property, that if, $X_1 \subseteq X_2$, then $\mathcal{M}_{X_1} \subseteq \mathcal{M}_{X_2}$. The union $\bigcup_{X\in\mathcal{F}}\mathcal{M}_X = \mathcal{M}_\Lambda$ is a normed algebra of all local self adjoint operators defined by,

$$\mathcal{M}_\Lambda = \left\{\rho_\Lambda^{\frac{1-t}{2}}.x.\rho_\Lambda^{\frac{1-t}{2}}:\ x \in \mathcal{M}, t \in [0,1)\right\}.$$

The closure $\overline{\bigcup_{X\in\mathcal{F}}\mathcal{M}_\Lambda} = \mathcal{M}_0$ is the quasilocal (spin) von Neumann algebra. Henceforth we will denote the operators of the form $\rho_\Lambda^{\frac{1-t}{2}}.x.\rho_\Lambda^{\frac{1-t}{2}}$ by x_Λ. The mutual influence of spins, confined in finite region Λ, is represented by an interaction, i.e a family $\Phi \equiv \{\Phi_\Lambda\}$ self adjoint operators such that $\Phi_\Lambda \in \mathcal{M}_\Lambda$. The interaction Φ is said to have finite range if there exist a $d_\Phi \geq 1$ such that $\Phi(X) = 0$ whenever $Diam\ of\ X = \underset{i,j}{SUP}\ d(i,j) > d_\Phi$ where $X \subseteq \Lambda$ is finite set. The interactions between particles yield an evolution in which the spin orientation are constantly changing. We define a Hamiltonian H_Λ by setting

$H_\Lambda(\Phi) \equiv \sum_{X\subset\Lambda}\Phi_X$. The density matrix given by $h_\Lambda \equiv \frac{e^{-\beta H_\Lambda}}{Tre^{-\beta H_\Lambda}}$ with $\beta \in (0,\infty)$. A finite volume Gibbs state φ_Λ is defined as follows

$\varphi_\Lambda(f) \equiv Tr(h_\Lambda f)$.

Non Commutative L_p –Spaces : Trunov - type L_p spaces

The $L_p(\varphi,\Lambda)$ spaces with $p \in [1,\infty]$, for operators of the form $\rho_\Lambda^{\frac{1-t}{2}}.x.\rho_\Lambda^{\frac{1-t}{2}}$, $t \in [0,1)$ on a quasilocal von Neumann algebra $\mathcal{M}_0$ with a faithful locally normal state φ_Λ is defined as follows,

$L_p(\varphi,\Lambda) = \{x_\Lambda \in \mathcal{M}_\Lambda:\ \|x_\Lambda\|_p < \infty\}$

with the norm given by,

$$\|x_\Lambda\|_{L_p(\varphi,\Lambda,)} = \left(Tr\left|h_\Lambda^{\frac{1}{2p}}\left(\rho_\Lambda^{\frac{1-t}{2}}x\,\rho_\Lambda^{\frac{1-t}{2}}\right)h_\Lambda^{\frac{1}{2p}}\right|^p\right)^{\frac{1}{p}} \qquad t \in [0,1)$$

where we have assumed that $h_\Lambda \in L_1(\varphi,\Lambda)$ is a self adjoint nonsingular operator called the density matrix with Tr $h_\Lambda =$ 1. We set $L_\infty(\varphi,\Lambda) = \mathcal{M}$.

Remark: We give basic properties for L_p spaces.

i. $L_\infty(\varphi,\Lambda) \subset L_q(\varphi,\Lambda) \subset L_p(\varphi,\Lambda) \subset L_1(\varphi,\Lambda)$ and $\|x_\Lambda\|_p \leq \|x_\Lambda\|_q$ for $x_\Lambda \in L_q(\varphi,\Lambda,t)$, and $q > p > 1$.

ii. For $p = 2$, the corresponding norm induced by the state $\varphi_\Lambda(x_\Lambda) = Tr(h_\Lambda x_\Lambda)$ is given by the following scalar product

$$\langle x_\Lambda, y_\Lambda\rangle_\varphi \equiv Tr\left(h_\Lambda^{\frac{1}{2}} y_\Lambda{}^* h_\Lambda^{\frac{1}{2}} x_\Lambda\right) = Tr\left(\left(h_\Lambda^{\frac{1}{4}} y_\Lambda{}^* h_\Lambda^{\frac{1}{4}}\right)\left(h_\Lambda^{\frac{1}{4}} x_\Lambda h_\Lambda^{\frac{1}{4}}\right)\right)$$

iii. If $x_\Lambda \in L_q(\varphi,\Lambda)$, $y_\Lambda \in L_p(\varphi,\Lambda)$ and $p \in [1,\infty]$ then $Tr_X(x_\Lambda y_\Lambda) = Tr_X(y_\Lambda x_\Lambda)$.

Theorem

For $p, q \in [1,\infty)$ such that $p^{-1} + q^{-1} = 1$ we have ,

(a) For any $c \in \mathbb{C}$

$0 \leq \|c x_\Lambda\|_p = |c|.\|x_\Lambda\|_p$

(b) Holder inequality

For $1 \leq p \leq \infty$, satisfying $\frac{1}{p} + \frac{1}{q} = 1$, and $x_\Lambda, y_\Lambda \in \mathcal{M}_0$ then $|\langle x_\Lambda y_\Lambda\rangle| \leq \| x_\Lambda\|_q \| y_\Lambda\|_p$.

(c) Minkowski Inequality:

$For\ p, q\ with$ $1 \leq p \leq \infty$, satisfying $1/p + 1/q = 1$, $x_\Lambda, y_\Lambda \in \mathcal{M}_0$ then $\| x_\Lambda + y_\Lambda\|_p \leq \| x_\Lambda\|_p + \| y_\Lambda\|_p$

3: Stochastic Dynamics

To get a nontrivial analogue of classical stochastic dynamics, we need to consider a quantum generalization of conditional expectation as in [1], [2]. We begin with a definition of the generalized conditional expectation $E_{X,\Lambda}$ on the localized closed operators of the form $\rho_\Lambda^{\frac{1-t}{2}} x\, \rho_\Lambda^{\frac{1-t}{2}}$ for a finite set $X \subseteq \Lambda$. Let $E_{X,\Lambda}: \mathcal{M}_0 \longrightarrow \mathcal{M}_0$ be a map defined as follows,

$$E_{X,\Lambda}\left(\rho_\Lambda^{\frac{1-t}{2}} x\, \rho_\Lambda^{\frac{1-t}{2}}\right) = Tr_X\left(\gamma_{X,\Lambda}^*\left(\rho_\Lambda^{\frac{1-t}{2}} x\, \rho_\Lambda^{\frac{1-t}{2}}\right)\gamma_{X,\Lambda}\right), \quad x \in \mathcal{M}.$$

where $\gamma_{X,\Lambda} = h_\Lambda^{\frac{1}{2}}(Tr_X h_\Lambda)^{-\frac{1}{2}}$, $\gamma_{X,\Lambda}^* = (Tr_X h_\Lambda)^{-\frac{1}{2}} h_\Lambda^{\frac{1}{2}}$ and $\gamma_{X,\Lambda} \in \mathcal{M}_0$

The generator of a quantum dynamical semi-group was discussed in Lindblad [4]. In that paper, Lindblad gave the explicit form of the generator as

$$L(x) = \psi(x) - \tfrac{1}{2}\{\psi(I), x\} + i[H, x]$$

where ψ is a completely positive map defined by $\psi(x) = \sum_j v_j^* x v_j$ .To have a dynamics that describe irreversible processes like dissipation, we will need a generator of the form

$$L(x) = \psi(x) - \tfrac{1}{2}\{\psi(I), x\}$$

and since we are interested in a nontrivial analogue of classical stochastic dynamics, we replace the completely positive map ψ with a generalized conditional expectation $E_{X,\Lambda}$. Then the operator $\mathcal{L}_{X,\Lambda}$ for a finite set $X \subseteq \Lambda$ is defined to be the map $\mathcal{L}_{X,\Lambda}: \mathcal{M}_0 \longrightarrow \mathcal{M}_0$ defined by $\mathcal{L}_{X,\Lambda}\left(\rho_\Lambda^{\frac{1-t}{2}} f\, \rho_\Lambda^{\frac{1-t}{2}}\right) = E_{X,\Lambda}\left(\rho_\Lambda^{\frac{1-t}{2}} f\, \rho_\Lambda^{\frac{1-t}{2}}\right) - \frac{1}{2}\left\{E_{X,\Lambda}(1), \left(\rho_\Lambda^{\frac{1-t}{2}} f\, \rho_\Lambda^{\frac{1-t}{2}}\right)\right\}$

Consider a bounded Symmetric Markov elementary generator $\mathcal{L}_{X+j}$ in [1], where $X + j$ is a translate of the set X by a vector j. We define the generator of the quantum dynamical semi-group for a finite volume as a self adjoint operator $\mathcal{L}^{X,\Lambda} = \sum_{j\in\Lambda} \mathcal{L}_{X+j,\Lambda} < \infty$ on $\mathcal{M}_0$, with $X \subseteq \Lambda$ a finite set of $\mathbb{Z}^d$. Let $P_t^{X,\Lambda} = e^{t\mathcal{L}^{X,\Lambda}}$ be the corresponding finite volume dynamics, it has the following properties.

Proposition : 3.1

(i) Positivity

$P_t^{X,\Lambda}\left(\rho_\Lambda^{\frac{1-t}{2}} f \rho_\Lambda^{\frac{1-t}{2}}\right) \geq 0$, with $\left(\rho_\Lambda^{\frac{1-t}{2}} f \rho_\Lambda^{\frac{1-t}{2}}\right) \geq 0$

(ii) Unit preserving

$P_t^{X,\Lambda}(1) = 1$

(iii) L_2 − Symmetry

$\langle P_t^{X,\Lambda}\left(\rho_\Lambda^{\frac{1-t}{2}} f \rho_\Lambda^{\frac{1-t}{2}}\right), \left(\rho_\Lambda^{\frac{1-t}{2}} g \rho_\Lambda^{\frac{1-t}{2}}\right)\rangle = \langle\left(\rho_\Lambda^{\frac{1-t}{2}} f \rho_\Lambda^{\frac{1-t}{2}}\right), P_t^{X,\Lambda}\left(\rho_\Lambda^{\frac{1-t}{2}} g \rho_\Lambda^{\frac{1-t}{2}}\right)\rangle$

(iv) Contractivity

$\|P_t^{X,\Lambda}\| \leq 1$

(v) Invariance

$\varphi_\Lambda\left(P_t^{X,\Lambda}\left(\rho_\Lambda^{\frac{1-t}{2}} f \rho_\Lambda^{\frac{1-t}{2}}\right)\right) = \varphi_\Lambda\left(\rho_\Lambda^{\frac{1-t}{2}} f \rho_\Lambda^{\frac{1-t}{2}}\right)$

Definition: 3.1

The discrete gradient $\partial_j f_\Lambda$ is defined by $\partial_j f_\Lambda = f_\Lambda - Tr_j f_\Lambda$, for a vector $j \in \Lambda$, $f_\Lambda \in \mathcal{M}_\Lambda$ and Tr_j is a partial trace with $Tr_j f_\Lambda \in \mathcal{M}_{\{j\}^c}$.

Let $\mathcal{M}_1 \subseteq \mathcal{M}_0$ be the subalgebra of local operators with $\sum_{j\in\mathbb{Z}^d}\|\partial_j f_\Lambda\| \leq \infty$. We denote the semi norm $|||f_\Lambda||| \equiv \sum_{j\in\mathbb{Z}^d}\|\partial_j f_\Lambda\| \leq \infty$ by a triple bar, for $f_\Lambda \in \mathcal{M}_1$. The subalgebra $\mathcal{M}_1$ is dense in $\mathcal{M}_0$. Let $\mathcal{L}_{X+j}(f_\Lambda) = E_{X+j}(f_\Lambda) - f_\Lambda$, be a pre-markov elementary generator such that the closure defines an elementary generator $\mathcal{M}_1$, where Λ is a finite set and $X \subset \Lambda$. The conditional expectation E_{X+j} is a 2-positive unit preserving map on $\mathcal{M}_1$ such that $E_{X+j}(\mathcal{M}_\Lambda) \subseteq \mathcal{M}_{\Lambda^c+j}$. We define a finite volume generator $\mathcal{L}^{X,\Lambda}$ as follows
$\mathcal{L}^{X,\Lambda} = \sum_{j\in\Lambda}\mathcal{L}_{X+j} < \infty$. The generator $\mathcal{L}^{X,\Lambda}$ is a well defined bounded operator on all the algebra $\mathcal{M}_0$. We defined also an infinite volume generator $\mathcal{L}^X$ formally by the same formula with $\Lambda \equiv \mathbb{Z}^d$ that is, $\mathcal{L}^X = \sum_{j\in\mathbb{Z}^d}\mathcal{L}_{X+j} < \infty$. For this to be defined on a large domain, we will require that the elementary generator $\mathcal{L}_{X+j}$ satisfy the following regularity property [2].

Definition: 3.2

An elementary operator $\mathcal{L}_{X+j}$ is called **regular** if and only if there are positive constants b_{jk} with $j,k \in z^d$ such that $\|\mathcal{L}_{X+j}f_\Lambda\| \leq \sum_k b_{jk}\|\partial_j f_\Lambda\|$ and $b_{jk}^X \in [0,\infty)$ such that
$sup_j \sum_k b_{jk}^X < \infty$.

Definition: 3.3

The elementary generators $\mathcal{L}_{X+j}$, $j \in \mathbb{Z}^d$ satisfy the condition

$$\|[\partial_k, \mathcal{L}_{X+j}]f_\Lambda\| \leq \sum_{l\in\mathbb{Z}^d} a_{kl}^{X+j}\|\partial_l f_\Lambda\|$$

if and only if there are positive constants a_{kl}^{X+j} $k,l \in \mathbb{Z}^d$ such that

(i) $\frac{1}{|X|}\sum_{k,l\in\mathbb{Z}^d} a_{kl}^{X+j} < \infty$

(ii) $\sum_{j:X+j\not\ni k,l\in\mathbb{Z}^d} a_{kl}^{X+j} \leq \lambda|X| < \infty$

for any $f_\Lambda \in \mathcal{M}_1$, $\lambda \in (0,1)$ and $|X|$ is the cardinality of the finite set X.

We have already shown the existence of the infinite volume quantum stochastic dynamics in [7] hence we have our main result.

Theorem: 3.1

If the condition $\sum_{j:X+j\not\ni k,l\in\mathbb{Z}^d} a_{kl}^{X+j} \leq \lambda|X| < \infty$ with $\lambda \in (0,1)$ is satisfied, then $|||P_t^X f_\Lambda||| \leq e^{-(1-\lambda)|X|t}|||f_\Lambda|||$

Proof:

Let P_t^X denote the semigroup corresponding to the generator $\mathcal{L}^X = \sum_{J\in\mathbb{Z}^d}\mathcal{L}_{X+j}$, where $\mathcal{L}_{X+j}(f_\Lambda) = E_{X+j}(f_\Lambda) - f_\Lambda$.
We note that $E_{X+j}(\mathcal{M}_\Lambda) \subseteq \mathcal{M}_{\Lambda^c+j}$, and $\partial_k\mathcal{L}_{X+j}f_\Lambda = \partial_k\big(E_{X+j}f_\Lambda - f_\Lambda\big) = -\partial_k f_\Lambda$, for $k \in X+j$.
To show the exponential decay in the triple bar we need to study the term $\|\partial_k P_s^X f_\Lambda\|$ for all $j \in \mathbb{Z}^d$.
Let $P_t^{X,k}$ be a semigroup with the corresponding generator $\mathcal{L}^{X,k} = \mathcal{L}^X - \sum_{j:X+j\ni k}\mathcal{L}_{X+j}$, $k \in X+j$. For $s \in [0,t)$,
we have $\frac{d}{ds} P_{t-s}^{X,k}(\partial_k P_s^X f_\Lambda) = P_{t-s}^{X,k}\,[\partial_k\,,\mathcal{L}^{X,k}]\,P_s^X f_\Lambda$

multiplying both sides by $e^{s|X|}$, where $|X|$ is the cardinality of the finite set $X \subset \Lambda$.

$e^{s|X|}\frac{d}{ds} P_{t-s}^{X,k}\,\partial_k P_s^X f_\Lambda = e^{s|X|}\, P_{t-s}^{X,k}\,[\partial_k\,,\mathcal{L}^{X,k}]\,P_s^X f_\Lambda$.

Integrating this equation from 0 to t

$$\int_0^t e^{s|X|}\frac{d}{ds} P_{t-s}^{X,k}\,\partial_k P_s^X f_\Lambda\; ds = \int_0^t ds\, e^{s|X|}\, P_{t-s}^{X,k}\,[\partial_k\,,\mathcal{L}^{X,k}]\,P_s^X f_\Lambda$$

$$e^{t|X|}\, P_0^{X,k}\partial_k P_t^X f_\Lambda - e^{0|X|}\, P_t^{X,k}\partial_k P_0^X f_\Lambda = \int_0^t ds\, e^{s|X|}\, P_{t-s}^{X,k}\,[\partial_k\,,\mathcal{L}^{X,k}]\,P_s^X f_\Lambda$$

$$e^{t|X|}\,\partial_k P_t^X f_\Lambda - P_t^{X,k}\partial_k f_\Lambda = \int_0^t ds\, e^{s|X|}\, P_{t-s}^{X,k}\,[\partial_k\,,\mathcal{L}^{X,k}]\,P_s^X f_\Lambda$$

$$e^{t|X|}\,\partial_k P_t^X f_\Lambda = P_t^{X,k}\partial_k f_\Lambda + \int_0^t ds\, e^{s|X|}\, P_{t-s}^{X,k}\,[\partial_k\,,\mathcal{L}^{X,k}]\,P_s^X f_\Lambda$$

dividing both sides by $e^{s|X|}$.

$$\partial_k P_t^X f_\Lambda = e^{-t|X|}P_t^{X,k}\partial_k f_\Lambda + \int_0^t ds\, e^{-(t-s)|X|}\, P_{t-s}^{X,k}\,[\partial_k\,,\mathcal{L}^{X,k}]\,P_s^X f_\Lambda$$

using contraction property of the Markov semigroup $P_t^{X,k}$ we have,

$$\|\partial_k P_t^X f_\Lambda\| \leq e^{-t|X|}\|\partial_k f_\Lambda\| + \left\|\int_0^t ds\, e^{-(t-s)|X|}\;[\partial_k\,,\mathcal{L}^{X,k}]\,P_s^X f_\Lambda\right\|$$

$$\|\partial_k P_t^X f_\Lambda\| \leq e^{-t|X|}\|\partial_k f_\Lambda\| + \int_0^t ds\, e^{-(t-s)|X|}\|\,[\partial_k\,,\mathcal{L}^{X,k}]\,P_s^X f_\Lambda\|$$

we note that $\mathcal{L}^{X,k} \equiv \sum_{j:X+j\not\ni k}\mathcal{L}_{X+j}$, by definition.

thus $\|\partial_k P_t^X f_\Lambda\| \leq e^{-t|X|}\|\partial_k f_\Lambda\| + \int_0^t ds\, e^{-(t-s)|X|}\sum_{j:X+j\not\ni k}\big\|\,[\partial_k\,,\;\mathcal{L}_{X+j}]\,P_s^X f_\Lambda\big\|$

since from condition (ii) we have, $\sum_{j:X+j\not\ni k}\big\|\,[\partial_k\,,\mathcal{L}_{X+j}]P_t^X f_\Lambda\big\| \leq \sum_{j:X+j\not\ni k}\;\sum_{l\in\mathbb{Z}^d} a_{kl}^{X+j}\|\partial_l P_t^X f_\Lambda\|$

where $\sup_{l\in\mathbb{Z}^d}\;\sum_k\;\sum_{j:X+j\not\ni k} a_{kl}^{X+j} \leq \lambda|X| < \infty$

$$\|\partial_k P_t^X f_\Lambda\| \leq e^{-t|X|}\|\partial_k f_\Lambda\| + \int_0^t ds\, e^{-(t-s)|X|}\sum_{j:X+j\not\ni k}\;\sum_{l\in\mathbb{Z}^d} a_{kl}^{X+j}\|\partial_l P_t^X f_\Lambda\|$$

$$\|\partial_k P_t^X f_\Lambda\| \leq e^{-t|X|}\|\partial_k f_\Lambda\| + \lambda|X|\int_0^t ds\, e^{-(t-s)|X|}\;\|\partial_l P_t^X f_\Lambda\|$$

Thus summing the inequalities over $k \in \mathbb{Z}^d$ we have ,

$$|||P_t^X f_\Lambda||| \leq e^{-|X|t}|||f_\Lambda||| + \lambda|X|\int_0^t ds\, e^{-(t-s)|X|}\;|||P_t^X f_\Lambda|||$$

solving the inequality we have,

$$|||P_t^X f_\Lambda||| - \lambda|X|\, e^{-|X|t}\int_0^t ds\, e^{|X|s}\;|||P_t^X f_\Lambda||| \;\leq\; e^{-t|X|}|||f_\Lambda|||$$

$$|||P_t^X f_\Lambda||| - \lambda|X|\, e^{-|X|t}\int_0^t ds\, e^{|X|s}\;|||P_t^X f_\Lambda||| \;\leq\; e^{-t|X|}|||f_\Lambda|||$$

multiplying by $e^{|X|t}$ and factoring $|||P_t^X f_\Lambda|||$ we have,

$$\left(e^{|X|t} - \lambda|X|\int_0^t e^{|X|s}ds\right)|||P_t^X f_\Lambda||| \;\leq\; |||f_\Lambda|||$$

hence writing $e^{|X|t} = |X|\int_0^t ds e^{s|X|} + 1$ we have,

$$\left(|X|\int_0^t ds e^{|X|s} + 1 - \lambda|X|\int_0^t ds\, e^{|X|s}\right)|||P_t^X f_\Lambda||| \;\leq\; |||f_\Lambda|||$$

collecting the terms in the bracket, we have,

$$\left((1-\lambda)|X|\int_0^t ds\, e^{|X|s} + 1\right)|||P_t^X f_\Lambda||| \;\leq\; |||f_\Lambda|||$$

we note that $e^{|X|t} = |X|\int_0^t ds e^{s|X|} + 1$ hence we have,

$(1-\lambda)\, e^{|X|t}\;|||P_t^X f_\Lambda||| \;\leq\; |||f_\Lambda|||$, with $\lambda \in (0,1)$

hence we have, $e^{(1-\lambda)|X|t}\;|||P_t^X f_\Lambda||| \;\leq\; |||f_\Lambda|||$

$$|||P_t^X f_\Lambda||| \;\leq\; e^{-(1-\lambda)|X|t}|||f_\Lambda|||$$

Conclusion*:*
In this paper we study stochastic dynamics on spin algebra using the technique and argument developed in [1].We have been able to establish that an infinite volume quantum stochastic dynamics of operators of the form $\rho_{\Lambda}^{\frac{1-t}{2}}\, x\, \rho_{\Lambda}^{\frac{1-t}{2}}$ have an exponential decay to equilibrium, hence the equilibrium of the dynamics depend on the regularity property imposed on the generator and not on the form of the operators. We have also shown that the analysis of quantum stochastic dynamics using noncommutative L_p- spaces technique is useful, especially if the underlying configuration space is infinite dimensional.

References

[1] Majewski, A.W. and Zegarlinski, B. (1996). *Quantum Stochastic Dynamics 1: Spin System on a Lattice*, MPEJ 1.

[2] Majewski, A.W. and Zegarlinski, B.(1996). *On Quantum Stochastic Dynamics and Non commutative L_p Spaces*, Letters Mathematical Physics, **(36), 337-349**.

[3] Trunov, N. V. (1979) , *On a non-Commutative analogue of the L_P Spaces*. Izvestiya Vuz, Soviet Mathematics, **23, pp. 69-77.**

[4] Lindblad, G., (1976), *On the generators of Quantum Dynamical semigroups*, Comm. Math. Phys. **48, 119-130**.

[5] Trunov, N. V. (1982) , *On the theory of normal weights on von Neumann algebras*, Izvestiya Vuz, Soviet Mathematics, **26, no8, pp. 69-77**.

[6] Bratteli, O. and Robinson, D.W. (1979), *Operator algebra and Quantum statistical Mechanics I* , Springer-verlag, Berlin.

[7] Yusuf, I. Tijjani, A.A. and Onyezulli, I. A. (2010), On the Existence of the infinite volume quantum stochastic dynamics, The J. Nigerian Association of Mathematical Physics **vol 16, pp. 85-84**.