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A Three Step Explicit Method for Direct Solution of Third Order Differential Equations

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Abstract

This study produces a three step discrete Linear Multistep Method for Direct solution of third order initial value problems of ordinary differential equations of the form $y''' = f(x,y,y',y'')$. Taylor series expansion technique was adopted in the development of the method. The differential system from the basis polynomial function to the problem is expanded by Taylor series expansion approach. The method is consistent and zero-stable. This is tested on a number of problems to show the accuracy and efficiency. A predictor of y_{n+k} where K>3 in the main method is also proposed.

Keywords: Linear multistep method, basis function, Taylor series expansion, Predictor, discrete method, Zero stable.

1.0 Introduction:

The solution of third order differential equation of initial value problem

$$y''' = f(x,y,y',y'') \qquad (1)$$

is considered in this paper. Literature has revealed or showed that the commonest method for solving a 3rd order ordinary differential equation is by reduction of the problem into a system of first order ordinary differential equation of the form

$$y' = f(x,y),\ y(x_0)=y_0 \qquad (2)$$

There are several numerical methods that can cater for the problem in equation (2). They are [1, 3, 4, 5, 6] proposed a continuous method for (1). The results obtained showed an order 5 scheme for k = 3.The conventional method which involves the reduction of (1) into a system of first order ordinary differential equation suffers some major setbacks. The setbacks include computational burden and cost implication.

Furthermore, a more serious drawback to such technique arises when the given system of equations to be solved cannot be solved explicitly for the derivatives of the highest order. [2]. The method then becomes inefficient and uneconomical for a general purpose application.

In this paper a three step explicit Linear multistep method of the form

$$y_{n+k} \sum_{i=0}^{k-1} \alpha_j y_{n+j} \quad h^3 \sum_{i=0}^{k-1} \beta_j y^m{}_{n+j} \qquad k \geq 3 \qquad (3)$$

is to be developed for direct solution of third order initial value problems of ordinary differential equations of the form

$$y''' = f(x,y.y',y''); \ \ y(x_0) = y_0, \ \ y'(x_0) = y'(0), y''(x_0) = y'' \qquad (4)$$

The Method

The method discussed in this paper is motivated by the Linear multistep method

$$\sum_{i=0}^{k-1} \alpha_j y_{n+j} h \sum_{i=0}^{k-1} \qquad (5)$$

Proposed by Adams (1883) for numerical solution of Ist order ordinary differential equation (2)
In this paper, equation (5) is redefined for the 3rd order ordinary differential equation
$y''' = f(x,y.y',y'')$ as

$$y_{n+k} = \sum_{i=0}^{k-1} \alpha_j y_{n+j} + h^3 \sum_{i=0}^{k-1} \beta_j y^m{}_{n+j} \qquad (6)$$

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Where the parameters α_j and β_j are to be determined to ensure that the resultants methods are Consistent, Zero-stable and Convergent, where $\alpha_k = 1$ and α_0 and β_0 are not both zero. Assuming that a truncation error T_{n+k} defined as

$$T_{n+k} = \sum_{i=0}^{k-1} \alpha_j y_{n+j} + h^3 \sum_{i=0}^{k-1} \beta_j y'''_{n+j} \qquad (7)$$

is made at each step of application of the method for solution of (1). Then the parameter α_j's and β_j's, j = 0(1)k can be determined as to ensure that T_{n+k} is minimum. The procedures are as follows:-

Adopting Taylor series expansion of $y_{n+k,}$ y_{n+j} and y'''_{n+j} j = 0(1)k-1 about point (x_n,y_n) to have

$$y_{n+k} = y_n + (kh)y_n{}^1 + \frac{(kh)^2 y_n{}^2}{2!} + \frac{(kh)y_n{}^3}{3!} + \frac{(kh)^4 y_n{}^4}{4!} + \cdots \frac{(kh)^p y_n{}^p}{p!} + \frac{(kh)^{p+1} y_n{}^2}{(p+1)!} + \frac{(kh)^{p+1} y_n{}^2}{(p+2)!} + 0(h^{p+3}) \qquad (8)$$

$$y_{n+k} = y_n + (jh)y_n{}^1 + \frac{(jh)^2 y_n{}^2}{2!} + \frac{(jh)^3 y_n{}^3}{3!} + \frac{(jh)^4 y_n{}^4}{4!} + \cdots \frac{(jh)^p y_n{}^p}{p!} + \frac{(jh)^{p+2} y_n{}^{p+1}}{(p+1)!} + \frac{(jh)^{p+2} y_n{}^{p+2}}{(p+2)!} + 0(h^{p+3}) \qquad (9)$$

$$y^m{}_{n+j} = y_n{}^3 + (jh)y_n{}^4 + \frac{(jh)^2 y_n{}^{(5)}}{2!} + \frac{(jh)^3 y_n{}^{(6)}}{3!} + \ \ldots \frac{(jh)^p y_n{}^p}{p!} + \frac{(jh)\ \ y_n{}^{p+1}}{(p+1)!} + \frac{(jh)\ \ y_n{}^{p+2}}{(p+2)!} + 0(h^{p+3}) \qquad (10)$$

When the above expansions; that is (equations 8,9 and 10), are inserted into the error equation (7), and terms in equal powers of h are combined, we get,

$$T_{n+k} = \left[1 - \sum_{i=0}^{k-1} \alpha_j\right] y_n + \left[1 - \sum_{i=0}^{k-1} j\alpha_j\right] hy^1{}_n + \left[\frac{k^2}{2} \sum_{i=0}^{k-1} \alpha_j\right] h^2 y^2 + \left[\frac{k^3}{3!} \sum_{i=0}^{k-1} \frac{(j)^3}{3!} \alpha_j \sum_{i=0}^{k-1} \beta_j\right] h^3 y^3 + \left[\frac{k^4}{4!} \sum_{i=0}^{k-1} \frac{(j)^4}{4!} \alpha_j \sum_{i=0}^{k-1} \beta_j\right] h^4 y^4 + \left[\frac{k^p}{p!} - \sum_{i=0}^{k-1} \frac{(j)^p}{p!} \alpha_j - \sum_{i=0}^{k-1} \beta_j\right] h^p y^p \ 0(h^{p+3}) \qquad (11)$$

This can be compactly written as

$$T_{n+k} = C_o y_n + C_1 hy^1 n + C_2 h^2 y_n{}^2 + \ldots C_p h^p + C_{p+1} h^{p+1} + C_{p+2} h^{p+2} + O(h^{p+3}) \qquad (12)$$

Where

$$C_0 = 1 - \sum_{i=0}^{k-1} \alpha_j y_n$$
$$C_1 = K - \sum_{i=0}^{k-1} J\alpha_j hy'_n$$
$$C_2 = \frac{k^2}{2!} - \sum_{i=0}^{k-1} (J)^2 \alpha_j h^2 y^2{}_n$$
$$C_2 = \frac{k^p}{p!} - \sum_{i=0}^{k-1} \frac{(J)^2 \alpha_j}{p!} - \sum_{i=0}^{k-1} \frac{j^{p-2}\alpha_j}{(p-2)!} \beta_j \ \ h^2 y^2{}_n \ldots O(h^{p+1}) \qquad (13)$$

Solving equations 11, 12 and 13 to obtain the parameters αj's and β j's by using guassian elimination method.

Setting k=3 in equation (6) we have a 3-step scheme of the form

$$y_{n+3} = \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} + h^3[\beta_0 y'''_n + \beta_1 y'''_{n+1} + \beta_2 y'''_{n+2}] \qquad (14)$$

With local truncation error

$T_{n+3} = y_{n+3}$ (- α1 - 2α2)$hy^1{}_n$ + (9/2 - $\alpha_{1/2}$ - 2α_2) $h^2y^2{}_n$ + (27/6 - α1/6 - 8α2/6 - $\beta_0 - \beta_1 - \beta_2$) $h^3y^3{}_n$ + (81/24 - α1/24 – 16/24α2 - β_1 – 2 β_2) $h^4y^4{}_n$ + (243/120 - α1/120 – 32/120α2 - $\beta_{1/}2$ – 2 β_2) $h^5y^5{}_n$ + (729/720- α1/720– 64/120α2 - $\beta_{1/}6$– 8β_2) $h^6y^6{}_n$ + (2187/5040- α1/5040– 128 α2/5040 - $\beta_{1/}24$–16/24 β_2) $h^7y^7{}_n$ ...$O(h)^8$ (15)

Solving this by Gaussian elimination technique on the system of liner equation; the values of $\alpha_{j'}$s and $\beta_{j'}$s are obtained as $\alpha_0 = 1$, $\alpha_1 = -3$, $\alpha_2 = 3$, $\beta_0 = 0$, $\beta_1 = 1/2$, $\beta_3 = 0$ (16)

Substituting the values of these parameters into the expansion in (14) the general 3-step linear multistep method is obtained as

$$y_{n+3} = 3y_{n+2} - 3y_{n+1} + y_n + \frac{h^3}{2} [f_{n+2} + f_{n+1}] \qquad (17)$$

Order P =5, and error constant C_7 =1/240 or 0.0041667 and interval of periodicity $x(\theta) = (0,\infty)$

Predictors for the methods:

To implement the scheme(method) (17) for the solution of third order initial value problem of type (4) there is need to develop the predictors for the evaluation of y_{n+2}, y_{n+1} and its derivatives y'_{n+2}, y'_{n+1} in order to evaluate $f_{n+j}=f(x_{n+j}, y_{n+j}, y'_{n+j})$ j = 1,2. These predictors must have the same order of accuracy as the main method in order to avoid initial data error in the computation.

$$y_{n+1} = y_n + \frac{(h)^2 y_n{}^2}{2!} + \frac{(h)^3 y_n{}^3}{3!} + \frac{(2h)^4 y_n{}^4}{4!} + \frac{(2h)^5 y_n{}^5}{5!} + \ \ldots O(h)^6 \qquad (18)$$

$$y_{n+2} = y_n + 2(h)y'_n + \frac{(2h)^2 y_n{}^2}{2!} + \frac{(2h)^3 y_n{}^3}{3!} + \frac{(2h)^4 y_n{}^4}{4!} + \frac{(2h)^5 y_n{}^5}{5!} + \ldots O(h)^6 \quad (19)$$

Differentiating (18) and (19) once yields

$$y_{n+1} = y_n{}^{(1)} + \frac{(h)y_n{}^2}{2!} + \frac{(h)^2 y_n{}^3}{3!} + \frac{(h)^3 y_n{}^4}{4!} + \frac{(h)^4 y_n{}^5}{5!} + \ldots O(h)^6 \quad (20)$$

$$y_{n+2} = y_n{}^{(1)} + \frac{2(h)y_n{}^2}{2!} + \frac{(2h)^2 y_n{}^3}{3!} + \frac{(2h)^3 y_n{}^4}{4!} + \frac{(2h)^4 y_n{}^5}{5!} + \ldots O(h)^6 \quad (21)$$

Equations (18, 19, 20) are used for the evaluation of y_{n+2} and y'_{n+2} respectively in the function $f_{n+1} = f(x_{n+2}, y_{n+2}, y'_{n+1})$ and $f_{n+2} = f(x_{n+2}, y_{n+2}, y'_{n+2})$ step numbers will be adopting the lower step schemes, For example, the 4-step method will adopt the 3-step scheme as predictor for y_{n+3} appearing in its equation.

Numerical Experiment

The method developed or derived is applied to solve some sample problems of third order ordinary differential equations in order to test for its accuracy.

(1) $y''' = x - 4y'$, $y(0) = y'(0) = 0$, $y''(0) = 1$

Theoretical solution: $y(x) = (3/16)(1-\cos 2x)+(1/8)x^2$; $h = 0.0025$

(2) $y''' = -y'$, $y(o) = 0, y'(0) = 1$, $y''(0) = 2$

Theoretical Solution: $y(x) = 2(1-\cos x)+ \sin x$ $h=0.0025$

(3) $y''' = -yy''$ $y(o) = 0, y'(0) = 0$, $y''(0) = 1$

Results

The absolute error (the comparative error analysis) obtained from the method (17) for k=3 are compared with the continuous method for k=3 in Awoyemi(2003) of the problem(1). The method is also used to solve various problems (that is 1-3). The results are as shown in the tables 1, 2 and 3 below;

Table1: Numerical solution of problem1 using the 3-step method
$y''' = x - 4y'$, $y(0) = y'(0) = 0$, $y''(0) = 1$

X	Y(x)exact	Y(n) Computed	Error
0.0025	.3128130D-05	.31249950D-05	.31880060D-08
0.0050	.12501560D-04	.12499920D-04	.16361810D-08
0.0075	.28120120D-04	.28124600D-04	.44801710D-08
0.010	.49995050D-04	.49998730D-04	.36852730D-08
0.0125	.78126340D-04	.78121920D-04	.44165060D-08
0.0150	.11249160D.03	.11249360D-03	.19936120D-08
0.0175	.15311330D-03	.15311320D-03	.72759580D-10
0.0200	.199980220D-03	.19998000D-03	.18917490D-09
0.0225	.25309220D-03	.25309300D-03	.75669960D-09
0.0250	.31244950D-03	.31245130D-03	.18044370D-08

Table 2: Numerical solution of problem 2 using the 3-step method
$y''' = -y'$ $y(o) = 0$, $y'(0) = 1$, $y''(0) = 2$

X	Y(x)exact	Y(n) Computed	Error
0.0025	.25061960D-02	.25062500D-02	.53551050D-07
0.0050	.50250130D-02	.50249790D-02	.33993270D-07
0.0075	.75561960D-02	.75561240D-02	.72177500D-07
0.010	.10099850D-01	.10099620D-01	.22724270D-06
0.0125	.12655960D-01	.12655410D-01	.54575500D-06
0.0150	.15224380D.01	.15223430D-01	.95739960D-06
0.0175	.17805350D-01	.17803610D-01	.17471610D-05
0.0200	.20398610D-01	.23000190D-01	.27287750D-05
0.0225	.23004380D-01	.23000190D-01	.27287750D-05
0.0250	.25622410D-01	.25616470D-01	.59381130D-05

Table 3: Numerical solution of problem 3(which is the Blasius equation problem)

$y''' = -yy''$ $y(o) = 0, y'(0) = 1, y''(0) = 1; h = .00025$

X	Y(n) Computed
0.0025	.31249950D-05
0.0050	.12499920D-04
0.0075	.28125260D-04
0.010	.50002240D-04
0.0125	.78132570D-04
0.0150	.11251840D-03
0.0175	.15316260D-03
0.0200	.20006810D-03
0.0225	.25323870D-03
0.0250	.31267850D-03

Table 4: Comparison of Errors for problem (1) for k = 3

X	Error in New method	Error in Awoyemi (2003)
0.0025	.31249950D-05	.3124511719D-03
0.0050	.12499920D-04	.1249218917D-02
0.0075	.28124770D-04	.2808547497D-02
0.010	.49999560D-04	.4987515640D-02
0.0125	.78124270D-04	.7782043843D-02
0.0150	.11249890D.03	.1118690437D-01
0.0175	.15312350D-03	.1519573531D-01
0.0200	.19999790D-03	.1980105449D-01
0.0225	.25312230D-03	.249942801D-01
0.0250	.31249660D-03	.3076575151D-01

Conclusion

A-three step discrete multistep method for general third order ordinary differential equations is developed. The method is of order five with principal error constant $C_{p+2} = 0.00416667$.

Numerical results of the method are compared with the fifth-order continous method developed by [1].

An examination of Table 4 clearly reveals that the new method is more accurate than that of [1] for k = 3

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