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On the Densities of the Scale-Invariant Statistics of the Multiple and Partial Correlation Coefficients

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Abstract

This work examines both the elliptically contoured Wishart density and the resulting density of the total correlation coefficient, and reaffirms the invariance property of the squared sampled multiple correlation coefficient. This invariance property is then exploited to show that the densities of the multiple correlation coefficients for the standard normal model and for the elliptically contoured model are identical. The same is shown to also hold for the density of the partial correlation coefficients.

Keywords: Scale invariance, elliptically contoured model, multiple and partial correlation coefficients.

1. Introduction:

Let A be an m x m positive definite matrix, having the elliptically contoured Wishart density [4, 5, 8] given by

$$f(A) = kg\,(\text{tr}\,(PA))\ \left|A_{22}\right|^{(n-m-1)/2} \tag{1}$$

where g is an absolutely continuous function, and k the normalizing constant of density functions. For (1) consider the case m = 2

i.e. $$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} \ \text{and}\ P \text{ is such that } P^{-1} = \begin{pmatrix} 1 & \rho \\ \rho & 1\end{pmatrix} \tag{2}$$

where ρ is the total population correlation coefficient, and what is desired here is the density of the total sample correlation coefficient, r, defined by

$$\left(a_{11}\ a_{22}\right)^{1/2}\ r = g_{12} \tag{3}$$

[1, 5] variously found the density of r to be

$$f(r) = k(1-\rho^2)^{n/2}\ (1-r^2)^{(n-3)/2}\ \sum_{\alpha=0}^{\infty} \frac{(2\rho r)\alpha\left[\Gamma\left(\frac{n+\alpha}{2}\right)\right]^2}{\alpha!} \tag{4}$$

The squared sample multiple correlation coefficient for (1) is given as $R^2_{1.2\ldots m}$

where $$R^2_{1.2\ldots m} = A_{12}\ A_{22}^{-1}\ A_{21} \tag{5}$$

and the matrix A is partitioned as

$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}$ and such that A_{22} is m-1 by m-1

The squared population multiple correlation coefficient

$$\rho^2_{1.2\ldots m} = P_{11}^{-1}\ P_{12}\ P_{22}^{-1}\ P_{21}$$

where

$$P^{-1} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \ldots & \rho_{1m} \\ \rho_{21} & 1 & \rho_{23} \ldots & \rho_{2m} \\ \rho_{31} & \rho_{32} & 1 \ldots & \rho_{3m} \\ \ldots & \ldots & \ldots & \ldots \\ \rho_{m1} & \rho_{m2} & \rho_{m3} \ldots & 1 \end{bmatrix} = \begin{bmatrix} 1 & P'_{12} \\ P'_{21} & P'_{22}\end{bmatrix}$$

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$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1m} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2m} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{m3} \\ \cdots & \cdots & \cdots & & \\ p_{m1} & p_{m2} & p_{m3} & \cdots & p_{mm} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

and

Now for the Wishart density, $f(A) = (2\Pi)^{-mn/2} |P|^{n/2}\ 2^{-m} \prod_{i=1}^{m} c(n-m+i) \exp\left\{-\frac{1}{2} tr(PA)\right\} |A|^{(n-m-1)/2}$ (6)

with $c(n\text{-}m\text{+}i) = (2\Pi)^{(n-m+i)/2} / \left[\Gamma\left(\frac{n-m+i}{2}\right)\right]$ (7)

The density of $R^2_{1.2\ldots m}$ is known to be [5, 6]

$$f\left(R^2_{1.2\ldots m}\right) = \frac{\left(1-\rho^2_{1.2\ldots m}\right)^{n/2} \left(1-R^2_{1.2\ldots m}\right)^{(n-m-1)/2} \left(R^2_{1.2\ldots m}\right)^{\frac{m-1}{2}}}{B\left(\frac{n-m+1}{2}, \frac{m-1}{2}\right)} .Q \tag{8}$$

where $Q = F\left[\frac{n}{2}, \frac{n}{2}, \frac{m-1}{2}; R^2_{1.2\ldots m} \rho^2_{1.2\ldots m}\right]$ (9)

If we use the identity

$$1-\rho^2_{1.2\ldots m} = P_{11}^{-1} |P_{22}|^{-1} |P| = 1 - P_{11}^{-1} P'_{12} P'^{-1}_{22} P'_{11} \tag{10}$$

then, (8) can be re-written as

$$F\left(R^2_{1..2\ldots m}\right) = k\left(1-P_{11}^{-1} P'_{12} P_{22}^{-1} P'_{21}\right)^{n/2} \left(1-R^2_{1.2\ldots m}\right)^{(n-m-1)/2} \left(R^2_{1.2\ldots m}\right)^{(m-3/2)} \bullet Q. \tag{11}$$

where $Q = F\left[\frac{n}{2}, \frac{n}{2}, \frac{m-1}{2}; R^2_{1..2\ldots m}\ P_{11}^{-1}\ P'_{12}\ P_{22}^{-1}\ P'_{21}\right]$ (12)

It follows from (4) that (11) is invariant under the transformation $P \rightarrow \alpha P$ for any parameter α. We shall now use this invariance property [7] of the canonical correlation matrices and of the correlation coefficients to show that the densities of the canonical correlation matrices and of the correlation coefficient are independent of any elliptically contoured modeled function g.

If y has n components, $-\infty \angle y \angle \infty$, *then* $\int_{y^T Ay} f(y^T Ay) \exp\left\lfloor (dy)^T \right\rfloor dy$

$$= k|A|^{-1/2} \quad f(u) u^{(n-2)/2} \sum_{r=0}^{\infty} \frac{u^r \left(d^T A^{-1} d\right)^r}{2^{2r} \Gamma(\frac{n}{2}+r) r!} \tag{13}$$

$$\int \exp\left\{-\frac{1}{2} tr(PA)\right\} |A|^{(n-m-1)/2} \quad \left(\delta^T A \delta\right)^r dA \cdot= 2^r k\, \Gamma\left(\frac{n}{2}+r\right)\left(\delta^T A \delta\right)^r A \tag{14}$$

For any absolutely continuous function g we have [2]

$$\left[g\left(\frac{d}{d\theta}\right) \exp(\theta x)\right]_{\theta=0} = g(x) \tag{15}$$

The Multiple Correlation Coefficient

From (6), let $P \longrightarrow \alpha P$, *and use* (11) *to get*

$$f\left(R^2_{1.2\ldots m}\right) = k \int_R \exp\left\{-\frac{1}{2}\alpha\ P_{11}\ a_{11} - P'_{12}\ A_{21} - \frac{1}{2} tr\ \left(\alpha\ P_{22}\ A_{22}\right)\right\}$$

$$\left(a_{11} - A_{12}\ A_{22}^{-1}\ A_{21}\right)^{(n-m-1)/2} |A_{22}|^{(n-m)/2}\ dA_{22} \tag{16}$$

where the region of integration R is defined by

$$a_{11} R^2_{1.2\ldots m} = A_{12}\ A_{22}^{-1}\ A_{21}, |A_{22}| > 0 \tag{17}$$

First use (13) on (16) and integrate out A_{21} to get

$$f\left(R^2_{1.2\ldots m}\right) = k \int \exp\left\{-\frac{1}{2}\alpha\, P_{11} a_{11} - \frac{1}{2} tr\left(\alpha\, P_{22}\ A_{22}\right)\right\}$$

$$|A_{22}|^{\frac{n-m}{2}} \; a_{11}^{(n+2r-2)/2}\left(1-R^2_{1.2...m}\right)^{(n-m-1)/2}\left(R^2_{1.2...m}\right)^{(m-3)/2}$$

$$\sum_{r=0}^{\infty} \frac{\left(R^2_{1.2...m}\right)^r \left(\alpha P'_{12}\, A_{22}\, P'_{21}\alpha\right)^r}{2^{2r}\,\Gamma\left[\frac{1}{2}(P-1)+r\right] r!} dP_{11}\, dA_{22} \tag{18}$$

Next using (14) we in integrate out P_{11} and A_{22} to get

$$f\left(R^2_{1.2...m}\right) = k\alpha^{-\frac{np}{2}} \left(1-R^2_{1.2....m}\right)^{\frac{1}{2}(n-p-1)} \left(R^2_{1.2...m}\right)^{\frac{1}{2}(p-3)} \bullet Q \tag{19}$$

Note that (19) is true for all $n \geq m$. Hence suppose

n = 2, m = 2, and P = I, then observe that for any conformable matrix B,

$$\int_R \exp\left\{-\frac{1}{2}\alpha\, tr(B)\right\}|B|^{-1/2}\, dB = K\alpha^{-2}\int_R \exp\left\{-\frac{1}{2}\, tr(B)\right\}|B|^{-1}\, dB$$

$$= k\alpha^{-2} \int f\left(R^2_{1.2}\right) \exp\left\{-\frac{1}{2}(b_{11}+b_{22})\right\} db_{11}\; db_{22} \tag{20}$$

$$= k \int f\left(R^2_{1.2}\right) \exp\left\{-\frac{1}{2}\alpha u\right\} u\; du$$

which is an identity in α and hence using (15) with $\theta = -\alpha/2$ *we have*

$$\int_R g\left(tr(B)\right)|B|^{-1/2}\, dB = k\; f\left(R^2_{1.2}\right)\int_0^\infty g(u)\, du \tag{21}$$

From (21) it follows that if P is given by (2), then (20) becomes

$$\int_R \exp\left\{-\frac{1}{2}\alpha\, tr\,(PB)\right\}|B|^{-1/2}\, dB = k(1-R^2_{1.2})^{-\frac{1}{2}}\left(R^2_{1.2}\right)^{-\frac{1}{2}}$$

$$\int \exp\left\{-\frac{1}{2}\alpha\left(s_1^2 - 2\rho R^2_{1.2}\, s_1 s_2 + s_2^2\right) s_1 s_2 \quad ds_1 ds_2 \tag{22}$$

where

$$S_1 = r\,Cos\theta,\; S_2 = rSin\theta,\; \left(1-\rho R^2_{1.2}\;\; Sin2\theta\right) r^2 = u \;\; and\; Sin\,2\theta = x \tag{23}$$

(22) reduces to

$$\int_R \exp\left\{-\frac{1}{2}\alpha tr\;(PB)|B|^{-1/2}\, dB = k\left(1-R^2_{1.2}\right)^{\frac{1}{2}}\left(R^2_{1.2}\right)^{-1/2}\int_0^\infty u\, \exp\left\{-\frac{1}{2}cu\right\} du \tag{24}$$

$$= \int_0^1 \frac{x}{(1-\rho R^2_{1.2}\, x)} \bullet \frac{1}{\sqrt{1-x^2}} dx$$

and hence, (21) holds, showing that the density of $R^2_{1.2}$ does not depend on g.

Finally observe that from (21) and (24)

$$\int_R g\left(tr(PB)\right)|P|^{n/2}|B|^{(n-m-1)/2}\, dB = k\; f\left(R^2_{1.2...m}\right)\int_0^\infty g(t)\, t^{n-1}\, dt \tag{25}$$

Thus we have shown that the densities of the multiple correlation coefficient (25) for the standard normal model and for the elliptically contoured model (1) are identical.

THE PARTIAL CORRELATION COEFFICIENT

Let us partition the matrix A as

$$A = \begin{bmatrix} a_{11} & a_{12} & A_{12} \\ a_{21} & a_{22} & \\ & A_{12}{}^T & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A^T{}_{12} & A_{22} \end{bmatrix}) \tag{26}$$

and assume that P is also conformably partitioned. The squared partial correlation coefficient is defined by

$$R^2{}_{1.2...m} = d_{21}/d_{11}\, d_{22}$$

$$D_{11} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} = A_{11} - A_{12}\;\; A_{22}{}^{-1}\, A_{21} \tag{27}$$

The joint density of D_{11}, A_{12} and A_{22} is

$$f(D_{11}, A_{12}, A_{22}) = k|D_{11}|^{\frac{1}{2}(n-m-1)}$$

$$\exp\left\{-\frac{1}{2}tr\left(P_{11}\, D_{11} + P_{11}\, A_{12}\;\, A_{22}^{-1}\;\, A_{21} + 2tr\left(P_{12}\, A_{21}\right) + tr\left(P_{22}\, A_{22}\right)\;\right]\;\}\right) \tag{28}$$

Integrating out A_{12} we have

$$f\left(D_{11,}\ A_{22}\right)=k\left|D_{11}\right|^{(n-m-1)/2}\ \left|A_{22}\right|^{\frac{1}{2}(n-m-1)}\exp\left\{-\frac{1}{2}tr\left(P_{11}D_{11}\right)\right.$$

$$\left.-\frac{1}{2}tr\left(P_{22}\ A_{22}\right)+\frac{1}{2}tr\left(P_{11}^{\ -1}\ P_{12}\ \ A_{22}\ P_{22}^{\ T}\right)\ \right\} \qquad (29)$$

Hence the density of D_{11} is

$$f\left(D_{11}\right)=k\ \exp\left\{-\frac{1}{2}tr\left(P_{11}D_{11}\right)\right\}\left|D_{11}\right|^{\frac{1}{2}(n-m-1)} \qquad (30)$$

The population squared partial correlation coefficient is

$$\rho_{123...m}\ =\ p_{12}^{\ 2}\ /\ p_{11}\ p_{22} \qquad (31)$$

where

$$P_{11}\ =\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

The density of $R_{12.3...m}$ *is*

$$f\left(R^2_{12.3...m}\right)=k\left(1-\rho^2_{12.3...m}\right)^{\frac{1}{2}(n-m+2)}\left(1-R^2_{12.3...m}\right)^{\frac{1}{2}(n-m-1)}\left(R^2_{12.3...m}\right)^{-1/2}Q$$

where

$$Q=\ F\left[\frac{1}{2}(n-m+2),\frac{1}{2}(n-m+2),,\frac{1}{2};\ R^2_{12.3...m}\ \ \rho^2_{12.3...m}\right] \qquad (32)$$

This density is invariant under the transformation $P\rightarrow\alpha P$ for any n and $m\geq 2.$ Moreover, it depends only on p_{11}, p_{12} and p_{22}, and hence (28) reduces to

$\int_R\ g\left(tr\left(P_{22}\ D_{11}\right)+t\right)t^m\ \left|D_{11}\right|^{1/2(n-m-1)}\ dD_{11}$, which is of the form

$kf\left(R^2_{12.3...m}\right)\int_0^\infty g(u+t)u^q\ t^m\ \ du\ \ dt$, where t and q are some real positive numbers.

Conclusion

Following the non-central density of the sample correlation coefficient for a class of elliptically contoured models derived by [5], this work has exploited the scale invariance property of multiple and partial correlation coefficients to show that the densities of the scale invariant statistics are identical for the normal model and the elliptically contoured model. Furthermore, the density of the total sample correlation coefficient, r, is invariant under the transformation $P\ \rightarrow\alpha P$ for any parameter α. The same applies to the squared sample correlation coefficient. We also used the invariance property of the canonical correlation matrices and the correlation coefficients to show that the densities of the canonical correlation matrices and of the correlation coefficient are independent of any elliptically contoured modeled function g.

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