Journal of the Nigerian Association of Mathematical Physics
Volume **18 (May, 2011), pp 345 - 348**
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On The Left Tail-End Probabilities and the Probability Generating Function

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Abstract

In this paper, another tail-end probability function is proposed using the left tail-end probabilities, $p(x \le i) = \pi_i$. The resulting function, $\Pi_x(t)$, is continuous and converges uniformly within the unit circle, $| t | < 1$. A clear functional link is established between $\Pi_x(t)$ and two other well known versions of the probability generating function. When known, $\Pi_x(t)$ uniquely generates the components of the probability mass function of the discrete random variable, and indirectly generates moments.

Keywords: Probability Generating Function, Tail – end Probabilities, Convergence, Moments.

Introduction:

We recall that for a non-negative integer-valued random variable X with probability mass function,

$$P(X = i) = p_i, \quad \ldots \quad \ldots \tag{1}$$

The probability generating function is defined as

$$P_x(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + \ldots \quad = \sum_{i=1}^{\infty} p_i t^i \tag{2}$$

which is equivalent to $P_x(t) = E(t^x)$, for X = 0, 1, 2,…

The basic properties of this function as presented in [3, 7, 8] are as follows:

i) $P_x(t)$ converges absolutely and uniformly within and on the unit circle, $| t | < 1$

ii) $P_x(t)$ is analytic, regular and infinitely differentiable for $| t | < 1$

iii) For every discrete probability distribution $\{p_i\}$, there is a unique probability generating function, $P_x(t)$; and conversely, every probability generating function, $P_x(t)$ corresponding to $\{p_i\}$ (where $p_i \ge 0$ and $\sum_i p_i = 1$) determines a unique probability mass function $\{p_i\}$.

iv) The nth component of $\{p_i\}$ can be obtained from $P_x(t)$ by the relation

$$p_n = \frac{P_x^{(n)}(0)}{n!} \tag{3}$$

which means that $P_x(t)$ is a transform of the probability mass function.

v) Moments of the random variable X may be computed from $P_x(t)$ provided the appropriate derivatives exist at t = 1[5]. In particular

$$P_x'(1) = \sum_{i=1}^{\infty} i\, p_i \; = E(X) \tag{4}$$

$$P_x''(1) = \sum i(i-1)p_i = E[X(X-1)] = E[X_{(1)}] \tag{5}$$

$$P_x^{(k+1)}(1) = E[X(X-1)(X-2)\ldots(X-k)] = E[X_{(k)}] \tag{6}$$

where $E[X_{(k)}]$ is the k^{th} factorial moment of the random variable X.[6]

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Tail-end Probabilities and the Probability Generating Function

The well known tail-end probability generating function is defined in [1, 2, 3] as

$$Q_x(t) = \sum_{i=0}^{\infty} q_i t^i \tag{7}$$

where $q_i = P(X > i),\ i = 0, 1, 2, \ldots$ (8)

We shall refer to this as the right tail-end probability, and to $Q_x(t)$ as the associated right tail probability function. That is,

$$q_i = p_{i+1} + p_{i+2} + p_{i+3} + \ldots \tag{9}$$

Between $P_x(t)$ and $Q_x(t)$, there exists the following fundamental relationship

$$1 - P_x(t) = (1 - t)\, Q_x(t), \text{ for } |t| < 1 \tag{10}$$

Therefore, just like $P_x(t)$, $Q_x(t)$ generates both probabilities and moments of a discrete random variable [4]. Specifically, the following expressions hold:

$$E[X] = P_x'(1) = Q_x(1) \tag{11}$$

$$Var[X] = P''(1) + P'(1) - [P'(1)]^2$$

$$= 2Q_x'(1)) + Q_x(1) - [Q_x(1)]^2 \tag{12}$$

i.e. $$E[X_{(r)}] = P_x^{(r+1)}(1) = (r+1)\, Q_x^{(r)}(1) \tag{13}$$

The tail-end probabilities are obtained using

$$q_n = \frac{1}{n!} Q_x^{(n)}(0), \quad n = 0, 1, 2, \ldots \tag{14}$$

By substitution, the probability mass function is generated using

$$P_n = q_{n-1} - q_n, \quad n = 1, 2, 3, \ldots \tag{15}$$

and $$p_0 = 1 - q_0 \tag{16}$$

Definition: (The Left Tail-end Probability Function)

Let $\pi_i = P(X \le i),\ i = 0,1,2, \ldots , n$ (17)

be the left tail-end probabilities of the discrete random variable X.

Suppose $P(X=n) \neq 0$ and $\sum_{i=0}^{n} p_i = 1$, (18)

then the power series

$$\Pi_x(t) = \sum_{i=1}^{n} \pi_i t^i \tag{19}$$

is the left tail-end probability generating function of the discrete random variable X

Theorem :

Within the unit circle $|t| < 1$, $\Pi_x(t)$ generates probabilities as well as moments and satisfies the following relations:

i) $P_x(t) = (1 - t)\, \Pi_x(t)$

ii) $\Pi_x(t) + Q_x(t) = \frac{1-t^n}{1-t}$

Proof

Now from (19) $\Pi_x(t) = \pi_0 + \pi_1 t + \pi_2 t^2 + \pi_3 t^3 + \ldots + \pi_n t^n$

Observe that $0 \le \pi_i \le 1$ for all i , hence $\{\pi_i\}$ is bounded. Therefore, within the unit circle $|t| < 1$, (19) will converge absolutely and uniformly.

Furthermore, (19) is continuous and differentiable within the same region. Hence

$$\pi_0 = \Pi_x(0) = p_0 \tag{20}$$

and $$\pi_r = \frac{1}{r!} \Pi_x^{(r)}(0) \quad r = 1, 2, 3, \ldots, n \tag{21}$$

Furthermore, observe that

$$p_r = \pi_r - \pi_{r-1} = P(X \le r) - P(X \le r-1), \quad r = 1, 2, 3, \ldots \tag{22}$$

Hence $\Pi_x(t)$ generates the components of the probability mass function uniquely

To prove (i) consider the right hand side.

$(1 - t)\ \Pi_x(t) = \Pi_x(t) - t\ \Pi_x(t)$

$= \pi_0 + (\pi_1 - \pi_0)t + (\pi_2 - \pi_1)t^2 + (\pi_3 - \pi_2)t^3 + \ldots + (\pi_n - \pi_{n-1})t^n$

$= p_0 + p_1 t + p_2 t^2 + p_3 t^3 + \ldots + p_n t^n$

$= P_x(t)$ i.e. using (20) and (22)

Hence $P_x(t) = (1 - t)\ \Pi_x(t)$ (23)

To prove (ii), consider $\Pi_x(t) + Q_x(t) = \sum_{i=1}^{\infty} (\pi_i + q_i) t^i$

$= \pi_0 + q_0 + (\pi_1 + q_1)t + (\pi_2 + q_2)t^2 + \ldots + (\pi_n + q_n)t^n$

Now $\pi_i + q_i = P(x \leq i) + P(x > i) = 1$ - - (24)

Hence using this we obtain

$$\Pi_x(t) + Q_x(t) = 1 + t + t^2 + t^3 + \ldots + t^n = \frac{1-t^n}{1-t} \quad (25)$$

$= \frac{1}{1-t}$ for $|t| < 1$ and n large.

The transformation in (23) shows that $\Pi_x(t)$ may be used indirectly to generate moments since whenever $\Pi_x(t)$ is known, $P_x(t)$ can be obtained using (20) and (22), from which moments can then be obtained.

Applications

1 Consider a random variable X which assumes values X_0, X_1, X_2, X_3 and X_4 with probability mass function $\{p_i\} = \{0.05, 0.15, 0.4, 0.3, 0.1\}$. We wish to use this to illustrate the relationships between $P_x(t)$, $Q_x(t)$ and $\Pi_x(t)$.

Now $\{q_i\} = \{0.95, 0.8, 0.4, 0.1, 0\}$

And $\{\pi_i\} = \{0.05, 0.2, 0.6, 0.9, 1.0\}$

$P_x(t) = 0.05 + 0.15t + 0.4t^2 + 0.3t^3 + 0.1t^4$

$Q_x(t) = 0.95 + 0.8t + 0.4t^2 + 0.1t^3 + 0t^4$

$\Pi_x(t) = 0.05 + 0.2t + 0.6t^2 + 0.9t^3 + t^4$

$Q_x(t) + \Pi_x(t) = 1 + t + t^2 + t^3 + t^4$

$= \frac{1-t^4}{1-t}$, which confirms theorem (ii)

Again

$(1 - t)\ \Pi_x(t) = \Pi_x(t) - t\ \Pi_x(t)$

$= 0.05 + (0.2 - 0.05)t + (0.6 - 0.2)t^2 + (0.9 - 0.6)t^3 + (1-0.9)t^4$

$= 0.05 + 0.15t + 0.4t^2 + 0.3t^3 + 0.1t^4$

$= P_x(t)$, and this confirms theorem (i)

2 Consider a discrete random variable $X \sim b(4, p)$ with

$\Pi_x(t) = 1/256\ \{1 + 13t + 67t^2 + 175t^3 + 256t^4\}$.

It is required to estimate p, E(X) and Var(X)

Solution:

Table 1: Probabilities associated with the values of X

function	x_0	x_1	x_2	x_3	x_4
π_i	$\frac{1}{256}$	$\frac{13}{256}$	$\frac{67}{256}$	$\frac{175}{256}$	1

q_i	$\frac{255}{256}$	$\frac{243}{256}$	$\frac{189}{256}$	$\frac{81}{256}$	0
p_i	$\frac{1}{256}$	$\frac{12}{256}$	$\frac{54}{256}$	$\frac{108}{256}$	$\frac{81}{256}$

From Table 1 and using (21) and (23)

$$P_0 = \frac{1}{256} \quad p_1 = \frac{3}{64} \quad p_2 = \frac{27}{128} \quad p_3 = \frac{27}{64} \quad p_4 = \frac{81}{256}$$

$$E(X) = Q_x(1) = \frac{768}{256} = 3, \qquad \text{or } E(X) = P_x'(1) = \frac{768}{256} = 3$$

But $E(X) = np$ hence, $p = 3/4$ and $q = 1/4$.

Thus, $Var(X) = npq = 3/4$

Conclusion

We have shown that the function $\Pi_x(t)$ derived from the left tail-end probabilities generates probabilities associated with the points of a discrete random variable on a finite support, and generates moments indirectly because its radius of uniform convergence is within (and not on) the unit circle (i.e. $| t | < 1$). We have also established a functional link between the new function and two other well known versions of the probability generating function, namely, $P_x(t)$ and $Q_x(t)$. With the functional links, it is generally possible to recover $P_x(t)$ and $Q_x(t)$ from $\Pi_x(t)$ and the appropriate moments are thus generated, albeit, indirectly. A left tail – end generating function has thus been proposed as a function with capacity to generate both probabilities and moments. Furthermore, this function has been shown to be analytic, uniformly convergent within the unit circle and is infinitely differentiable.

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