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# Methods of Parameter Addition to a Family of Multivariate Exponential and Weibull Distributions

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***Abstract***

***Methods of introducing additional parameter to a family of multivariate exponential and Weibull distributions are presented. One of them is used to give a new two-parameter extension of the multivariate exponential distribution which may appear to be easier to deal with than those such commonly used two-parameter family of multivariate life distributions as the Weibull, gamma and lognormal distributions. Another general method that allows additional new three-parameter to a family of multivariate Weibull distribution is also introduced and studied. All the families of distributions expanded by either or both of these methods have the property that the minimum of a geometric number of independent random variables with common distribution in the family has a distribution also in the family.***

**Keywords:** Geometric extreme stability; Multivariate geometric distribution; Life distribution; Parametric family.

## 1. Introduction:

Exponential and Weibull distributions play important roles in analysis of survival or life time data. Ali, Mikhail and Haq [1] discussed this in detail in their paper. These two distributions play such role simply because of their constant hazard rates, convenient statistical theory as well as their important property of lacking of memory. Cox and Oakas [5] stated that whenever the one-parameter family of univariate or bivariate exponential distribution is found to be insufficient, a number of wider families such as Gamma, Weibull and Gompartz-Makeham distributions are mostly used, instead. Also, Cox and Oakas [5] discussed the usefulness of these distributions in detail. Johnson et al [10] explained these families of distributions in broader way. Genest et al [8] presented the usefulness and important properties of these distributions in detail.

There are many methods that can be used to introduce new parameters in order to expand and simplify families of distributions for either adding flexibility or to construct either covariate or correlation models. This is stated clearly in Marshall and Olkin [14]. According to [14], whenever a scale parameter is added to a family of distributions, it accelerate life model and taking powers of the bivariate survival function introduces a parameter that give rises to the proportional hazards rate model. According to [17] as well as [7], the family of Weibull distributions contains the exponential distributions and it is constructed by taking powers of exponentially distributed random variables. Similarly, the family of gamma distributions contains the exponential distributions and in this case constructed by taking powers of the lap lace transform of the exponentially distributed random vectors. Arnold [2] as well as[14] presented and studied the method of adding parameter to a family of univariate exponential distributions in order to expand and make it more flexible distributions. According to [9] and more recently [14], the families of Weibull and gamma distributions were expanded and became more flexible whenever new parameter is introduced into it. Marshall and Olkin [13] also studied the properties of the new families of these distributions formed by addition of the new parameter. However, more detail about this can be seen in [14].

In this write up, an attempt has been made to present and discuss a general method of adding new parameter to the families of multivariate exponential and Weibull distributions. In particular, starting with a multivariate survival function $\overline{F}\left(x_1, x_2, ..., x_n\right)$, the one-parameter family of multivariate survival function

$$\overline{G}\left(x_1, x_2, \ldots, x_n; \alpha\right) = \frac{\alpha \overline{F}\left(x_1, x_2, \ldots, x_n\right)}{1 - \overline{\alpha}\,\overline{F}\left(x_1, x_2, \ldots, x_n\right)}$$

$$= \frac{\alpha \overline{F}\left(x_1, x_2, \ldots, x_n\right)}{F\left(x_1, x_2, \ldots, x_n\right) + \alpha \overline{F}\left(x_1, x_2, \ldots, x_n\right)}, \quad -\infty < x_1, \ldots, x_n < \infty, \quad 0 < \alpha < \infty \qquad (1.1)$$

with $\overline{\alpha} = 1 - \alpha,$ is introduced and discussed in section 2 of this paper. As in univariate and bivariate distributions cases, it also worth noting here that $\overline{G} = \overline{F}$ whenever $\alpha = 1.$

The particular case that $\overline{F}\left(x_1, x_2, ..., x_n\right)$ is an exponential distribution gives a new two-parameter family of distributions

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that may sometimes be used in place of usual multivariate Weibull and gamma families of distributions. This extended families of exponential and Weibull distributions are discussed in detail in section 3 of this paper. Section 4 of this paper provides the method used to derive a three-parameter version of the multivariate Weibull family of distributions.

All the methods used of introducing an additional parameter have a stability property. That is, if the method is applied twice, nothing new is obtained the second time around. Therefore, a power of an exponential random vectors have a multivariate Weibull distribution, but the power of a Weibull random vectors is nothing but another Weibull random vectors. Similarly, if in (1.1) above, a multivariate survival function of the form $\overline{G}$ is introduced for $\overline{F}$, then the equation (1.1) gives nothing new. This stability property and the derivation of equation (1.1) is introduced and studied in section 5 of this paper. General conclusion of this paper is given in section 6 of this write up.

## 2. Multivariate density and Hazard rate of the new family of distributions

As far as the multivariate function $\overline{F}$ has a multivariate density function, then the multivariate survival function $\overline{G}$ stated in (1.1), have easily-computed multivariate densities. In Particular, whenever $\overline{F}$ has a multivariate density $f(x_1,x_2,...,x_n)$ and rate of hazard $r_{\overline{F}}$, then the multivariate survival function $\overline{G}$ has the multivariate density function $g(x_1,x_2,...,x_n)$ which is given by:

$$\begin{aligned}\overline{G}(x_1,x_2,...,x_n;\alpha) &= \frac{\alpha f(x_1,x_2,...,x_n)}{\left\{1-\overline{\alpha}\overline{F}(x_1,x_2,...,x_n)\right\}^2}\\ &= \frac{\alpha f(x_1,x_2,...,x_n)}{\left\{1-\overline{F}(x_1,x_2,...,x_n)+\alpha\overline{F}(x_1,x_2,...,x_n)\right\}^2}\\ &= \frac{\alpha f(x_1,x_2,...,x_n)}{\left\{F(x_1,x_2,...,x_n)+\alpha\overline{F}(x_1,x_2,...,x_n)\right\}^2} \qquad (2.1)\end{aligned}$$

and the corresponding hazard rate is given by:

$$\begin{aligned}r(x_1,x_2,...,x_n;\alpha) &= \frac{1}{\left\{1-\overline{\alpha}\overline{F}(x_1,x_2,...,x_n)\right\}} r_F(x_1,x_2,...,x_n)\\ &= \frac{1}{1-(1-\alpha)\overline{F}(x_1,x_2,...,x_n)} r_F(x_1,x_2,...,x_n)\\ &= \frac{1}{F(x_1,x_2,...,x_n)+\alpha\overline{F}(x_1,x_2,...,x_n)} r_F(x_1,x_2,...,x_n) \qquad (2.2)\end{aligned}$$

Hence, $\lim\limits_{x_1\to\infty}\lim\limits_{x_2\to\infty}...\lim\limits_{x_n\to\infty} r(x_1,x_2,...,x_n;\alpha) = \lim\limits_{x_1\to\infty}\lim\limits_{x_2\to\infty}...\lim\limits_{x_n\to\infty}\frac{r_F(x_1,x_2,...,x_n)}{\alpha}$.

Similarly, as in bivariate case, it is also true here that

$$\lim_{x_1\to\infty}\lim_{x_2\to\infty}...\lim_{x_n\to\infty} r(x_1,x_2,...,x_n;\alpha) = \lim_{x_1\to\infty}\lim_{x_2\to\infty}...\lim_{x_n\to\infty} r_F(x_1,x_2,...,x_n).$$

From the result obtained (2.2) and what is stated in [8], we can establish the following:

$$\frac{r_F(x_1,x_2,...,x_n)}{\alpha} \le r(x_1,x_2,...,x_n;\alpha) \le r_F(x_1,x_2,...,x_n), \quad -\infty < x_1,x_2,...,x_n < \infty, \ \ \alpha \ge 1. \qquad (2.3)$$

$$r_F(x_1,x_2,...,x_n) \le r(x_1,x_2,...,x_n;\alpha) \le \frac{r_F(x_1,x_2,...,x_n)}{\alpha}, \quad -\infty < x_1,x_2,...,x_n < \infty, \ \ \alpha \le 1. \qquad (2.4)$$

Similarly,

$$\overline{F}(x_1,x_2,...,x_n) \le \overline{G}(x_1,x_2,...,x_n;\alpha) \le \overline{F}^{1/\alpha}(x_1,x_2,...,x_n), \quad -\infty < x_1,x_2,...,x_n < \infty, \ \ \alpha \ge 1. \qquad (2.5)$$

$$\overline{F}^{1/\alpha}(x_1,x_2,...,x_n) \le \overline{G}(x_1,x_2,...,x_n;\alpha) \le \overline{F}(x_1,x_2,...,x_n), \quad -\infty < x_1,x_2,...,x_n < \infty, \ \ \alpha \le 1. \qquad (2.6)$$

Using the same equation (2.2) above, we can establish that $\frac{r(x_1,x_2,...,x_n;\alpha)}{r_F(x_1,x_2,...,x_n)}$ is an increasing function in $x_i$, $i=1,2,...,n$ for $\alpha \ge 1$ and it is a decreasing function in $x_i$, $i=1,2,...,n$ for $0<\alpha\le 1$.

When $F(0,0,...,0)=0$, the corresponding hazard rate $r(0,0,...,0;\alpha)$ at the origin of multivariate function behaves quite differently then it does for the Weibull or gamma distributions; for both these families, the distribution can be an exponential distribution, or $r(0,0,...,0)=0$, or $r(0,0,...,0)=\infty$, so that $r(0,0,...,0)$ is discontinuous in the shape parameter. This is not the case with the multivariate family having hazard rates as stated in equation (2.2). Therefore, the multivariate family may be useful to make the multivariate function $F(x_1,x_2,...,x_n)$ easier to understand. However, in spite of what are already stated in both equations (2.3) as well as (2.4) above, it need not be that multivariable function $F(x_1,x_2,...,x_n)$ and its corresponding multivariate survival function $G(x_1,x_2,...,x_n)$ are at all similar to each other.

## 3. A new family of two-parameter multivariate Exponential Distributions

Given the multivariate function $\overline{F}(x_1,x_2,...,x_n)=\exp\left(-\sum_{i=1}^{n}\beta x_i\right)$, the two-parameter family of multivariate survival function

$$\overline{G}(x_1,x_2,...,x_n;\alpha,\beta)=\frac{1}{(\alpha-1)+e^{\left(\sum_{i=1}^{n}\beta x_i\right)}},\qquad (x_i>0\quad for\ all\ i,\ and\ \alpha>0,\beta>0)\qquad (3.1)$$

can be derived from equation (1.1). The multivariate exponential distribution can be obtained as a special case of (3.1) when $\alpha=\beta=1$. When $\alpha=\beta\ge 1$, this multivariate distribution is the conditional multivariate distribution, given $Z>0$, of a random variable Z with the multivariate logistic survival function

$$P(Z>z)=\frac{\alpha}{\left(1-(1-\alpha)e^{\sum_{i=1}^{n}\beta x_i}\right)},\qquad for\quad -\infty<z<\infty.$$

Regarding equation (3.1) above as a special case of equations (2.1) and (2.2), it can be seen that the multivariate survival $G(x_1,x_2,...,x_n)$ has the multivariate density function $g$ which can be defined as:

$$g(x_1,x_2,...,x_n;\alpha,\beta)=\frac{\alpha\beta e^{-\sum_{i=1}^{n}\beta x_i}}{\left[1-(1-\alpha)e^{-\sum_{i=1}^{n}\beta x_i}\right]^2}=\frac{\alpha\beta e^{\sum_{i=1}^{n}\beta x_i}}{\left[e^{\sum_{i=1}^{n}\beta x_i}-(1-\alpha)\right]^2},\qquad (x_i>0,\ \forall i;\ \alpha>0,\beta>0),$$

and the corresponding hazard rate of this multivariate density function is given as:

$$r(x_1,x_2,...,x_n;\alpha,\lambda)=\frac{\lambda}{1-(1-\alpha)e^{-\sum_{i=1}^{n}\lambda x_i}}=\frac{\lambda e^{\sum_{i=1}^{n}\lambda x_i}}{e^{\sum_{i=1}^{n}\lambda x_i}-(1-\alpha)},\qquad (x_i>0,\ \forall i;\ \alpha>0,\lambda>0).$$

At this point, it should be noted that $r(x_1,x_2,...,x_n;1,\lambda)=\lambda$, that is $r(x_1,x_2,...,x_n;\alpha,\lambda)$ is decreasing function in $x_i,\ i=1,2,...,n$ for $0<\alpha\le 1$. Similarly, $r(x_1,x_2,...,x_n;\alpha,\lambda)$ is an increasing function in $x_i,\ i=1,2,...,n$ for $\alpha\ge 1$.

Considering equations (2.3) and (2.4) above, it can seen that

$$\frac{\beta}{\alpha}\le r(x_1,x_2,...,x_n;\alpha,\beta)\le\beta,\qquad (-\infty<x_1,x_2,...,x_n<\infty,\ \ \alpha\ge 1),\qquad (3.2)$$

$$\beta\le r(x_1,x_2,...,x_n;\alpha,\beta)\le\frac{\beta}{\alpha},\qquad (-\infty<x_1,x_2,...,x_n<\infty,\ \ 0\le\alpha\le 1),\qquad (3.3)$$

$$e^{-\sum_{i=1}^{n}\beta x_i}\le\overline{G}(x_1,x_2,...,x_n;\alpha,\beta)\le e^{\frac{-\sum_{i=1}^{n}\beta x_i}{\alpha}},\qquad (-\infty<x_1,x_2,...,x_n<\infty;\quad \alpha\ge 1),\qquad (3.4)$$

$$e^{\frac{-\sum_{i=1}^{n}\beta x_i}{\alpha}}\le\overline{G}(x_1,x_2,...,x_n;\alpha,\beta)\le e^{-\sum_{i=1}^{n}\beta x_i},\qquad (-\infty<x_1,x_2,...,x_n<\infty;\quad 0\le\alpha\le 1).\qquad (3.5)$$

As it was in bivariate case, in multivariate also it is true that distribution with an increasing hazard rate is new better than used. Similarly, distribution with a decreasing hazard rate is new worse than used. This fact was earlier presented in [4]. From the above fact, it follows that when multivariate random variables $x_1,x_2,...,x_n$ have the multivariate distribution $G(x_1,x_2,...,x_n)$ the conditional multivariate survival function satisfies

$$P(X_1>x_1+t,...,X_n>x_n+t/X_1>x_1,...,X_n>x_n)\begin{cases}\le P(X_1>t,...,X_n>t), & (\alpha\ge 1)\\ \ge P(X_1>t,...,X_n>t), & (0<\alpha\le 1).\end{cases}$$

**Proposition 3.1:** The multivariate function $\log g(.,.,...;\alpha,\beta)$ is convex for $0<\alpha\leq 1$ and concave for $\alpha\geq 1.$

The above result can be shown simply by differentiating the multivariate function $\log g(.,.,...;\alpha,\beta)$ n-times with respect to all variables $x_1,x_2,...,x_n$. This means that, for $\alpha\leq 1$, the multivariate function $G(x_1,x_2,...,x_n)$ is a decreasing function. On the other hand, for $\alpha\geq 1$, $g(.,.,...;\alpha,\beta)$ is unimodal, with the mode of each of the n-variables given as:

$$\bmod=\begin{cases}0, & (\alpha\leq 2),\\ \beta^{-1}\log(\alpha-1), & (\alpha\geq 2).\end{cases}$$

Considering equations (3.4) and (3.5), it can be shown that the multivariate function $G(x_1,x_2,...,x_n)$ has finite moments of all positive orders. By computing directly, it can be verified that, if these n-variables have distribution function $G(x_1,x_2,...,x_n;\alpha,\beta)$, then each of the n-variables has first moment given as:

$$E[x_i]=-\frac{\alpha\log\alpha}{\beta(1-\alpha)},\quad for\quad i=1,2,...,n. \tag{3.6}$$

The above expectations are always positive quantities. In particular, for the marginal distribution of random variable $X_1$, we have

$$E\left[X_1^r\right]=r\int_0^\infty \overline{G}(x_1;\alpha,\beta)x_1^{r-1}dx_1=\frac{r\alpha}{\beta^r}\int_0^1\left\{\frac{(-\log p)^{r-1}}{1-(1-\alpha)p}\right\}dp, \tag{3.7}$$

which when r=1is substituted in it, gives equation (3.6). Similarly, for the marginal distribution of random variable $X_2$, the $r^{th}$ moment is also given as in equation (3.7) above with $X_2$ replacing $X_1$. The same argument is apply to all remaining n-2 variables.

The lap lace transform of marginal distribution $g$ of each of the n-random variables $X_1,X_2,...,X_n$ can also be obtained as follows. For the random variable $X_1$, it is given as:

$$E\left[e^{-s\beta x_1}\right]=\int_0^1\left\{\frac{\alpha p^s}{(1-(1-\alpha)p)^2}\right\}dp. \tag{3.8}$$

Similarly, that of random variable $X_2$ can be obtained in the same way as above by replacing $X_1$ with $X_2$. The same pattern is applied to all remaining (n-2) random variables.

Equations (3.7) and (3.8) can be expressed as infinite series as far as $|1-\alpha|\leq 1.$ Based on this, the integrands of (3.7) and (3.8) can be expanded in a power series and the result be integrated term by term to generate the following for the random variable $X_1$.

$$E\left[X_1^r\right]=\tfrac{r\alpha}{\beta^r}\int_0^\infty x_1^{r-1}e^{-x_1}\sum_{j=0}^\infty\alpha^{-j}e^{-jx_1}dx_1=\tfrac{r\alpha}{\beta^r}\sum_{j=0}^\infty\frac{\alpha^{-j}\Gamma(r)}{(j+1)^r}\qquad(|1-\alpha|\leq 1),$$

*and also*

$$E\left[e^{-sX_1}\right]=\alpha\int_0^1 p^s\sum_{j=0}^\infty(j+1)p^j\alpha^{-j}dp=\alpha\sum_{j=0}^\infty\alpha^{-j}\frac{j+1}{s+j+1}\qquad(|1-\alpha|\leq 1). \tag{3.9}$$

Similarly, that of marginal distribution of random variable $X_2$ can also be obtained in the same way by using the corresponding moment and lap lace transform of the random variable $X_2$. All others follow in the same way.

As a consequence of proposition 3.1 as well as what was earlier presented in [11], the total positivities properties yield moment inequalities that are not generally true. In particular, the coefficient of variation $\frac{\delta}{\mu}$ is less than 1 for $\alpha>1$ and is greater 1 when $\alpha<1$. $\delta^2$ is the variance while $\mu$ is the first moment of random variables $X_1,X_2,...,X_n$. It is also clear that the $k^{th}$ quartile $\tilde{x}_k$ of $\overline{G}$ can be obtained by the relation: $\tilde{x}_k=\frac{1}{\beta}\log\left(\frac{(1-k)+\alpha k}{1-k}\right)$.

Also, the median of each of the random variables $X_1,X_2,...,X_n$ is given by the formula:

Median of $\tilde{X}_i=\dfrac{\log(1+\alpha)}{\beta},\quad\forall i, i=1,2,...,n.$

From the above relations, it can be observed that median, mode and expectations of random variables $X_1, X_2, ..., X_n$ are all increasing functions in $\alpha$ and decreasing functions in the scale parameter $\beta$.

Considering the monotonic nature of $\log_e x_i, \ \forall i, \ \ i=1,2,...,n$, the fact that $\log_e x_i \le x_i - 1, \ \ \forall i, \ i=1,2,...,n$ and the values of random variables $X_1, X_2, ..., X_n$ are all positive, it can be shown that

$\mathrm{mod}\, e(X_i) \le med(X_i) \le \frac{\alpha}{\beta} \le E(X_i), \ \forall i, \ \ i=1,2,...,n$. But it should also be noted that

$\lim_{\alpha\to\infty} \frac{\mathrm{mod}\, e(X_1)}{E(X_1)} = \lim_{\alpha\to\infty} \frac{\mathrm{mod}\, e(X_2)}{E(X_2)} = ... = \lim_{\alpha\to\infty} \frac{\mathrm{mod}\, e(X_n)}{E(X_n)} = 1.$ If all $E(X_i), \ i=1,2,...,n$ are fixed constants, say equal 1, then the weak limit of $\overline{G}$, as $\alpha$ tends to infinity, is degenerate at point 1, while the limit is degenerate at point zero when $\alpha$ tends to zero. It also worth noting that $\lim_{x_1\to\infty} \lim_{x_2\to\infty} ... \lim_{x_n\to\infty} r(x_1, x_2, ..., x_n; \alpha, \beta) = \beta$ is bounded and continuous in the parameters, just like gamma distribution but not like Weibull distribution.

## 4. Extended multivariate Weibull Distributions

Consider the multivariate Weibull survival function

$$\overline{F}(x_1, x_2, ..., x_n) = e^{-\sum_{i=1}^{n} (\beta x_i)^{\lambda}}, \quad x_i \ge 0, \quad \forall i; \ \lambda > 0, \tag{4.1}$$

then using equations (1.1) and (4.1) above, we can get the new three-parameter survival function

$$\overline{G}(x_1, x_2, ..., x_n; \alpha, \beta, \lambda) = \frac{\alpha e^{-\sum_{i=1}^{n} (\beta x_i)^{\lambda}}}{1 - (1-\alpha) e^{-\sum_{i=1}^{n} (\beta x_i)^{\lambda}}}. \tag{4.2}$$

This geometric–extreme stable extension of the multivariate Weibull distribution may sometimes be a competitor to the more usual three-parameter Weibull distribution with survival function

$$\overline{F}(x_1, x_2, ..., x_n; \alpha, \beta, \delta) = Exp\left\{-\beta \sum_{i=1}^{n} (x_i - \delta)\right\}^{\lambda}, \quad x_i \ge \delta, \ \forall i; \quad \beta > 0, \lambda > 0; \quad -\infty < \delta < \infty.$$

If $X_i, \ i=1,2,...,n$ have a multivariate exponential distribution with parameter $\beta = 1$ then $\frac{X_1^{1/\lambda}}{\beta}, \frac{X_2^{1/\lambda}}{\beta}, ..., \frac{X_n^{1/\lambda}}{\beta}$ have the survival function as given in equation (4.1) above. Similarly, if $X_i, \ i=1,2,...,n$ have the survival function (3.1) with parameter $\beta = 1$ then $\frac{X_1^{1/\lambda}}{\beta}, \frac{X_2^{1/\lambda}}{\beta}, ..., \frac{X_n^{1/\lambda}}{\beta}$ have the survival function as stated in (4.2). Therefore, moments of survival function given in (4.2) can be obtained from non integer moments of function (3.1). Hence, from equation (3.6), it can be seen that, whenever the random variables $X_i, \ i=1,2,...,n$ have the multivariate survival function as in equation (4.2), then

$$E\left[\prod_{i=1}^{n} X_i^{s}\right] = \prod_{i=1}^{n} \frac{x_i^{\alpha}}{\lambda} \sum_{j=1}^{k} \frac{(1-\alpha)^j}{(j+1)^{\frac{s}{\lambda}}} \Gamma\left(\frac{s}{\lambda}\right), \qquad |1-\alpha| \le 1. \tag{4.3}$$

If $|1-\alpha| > 1$, then the moments can be obtained from equation (3.4) by applying change of variable technique that was earlier applied in deriving equation (4.3). However, those moments can not be stated in closed form; therefore, even the first moment of equation (4.2) must be obtained numerically. By expressing the moments as

$$E\left[\prod_{i=1}^{n} X_i^{s}\right] = \int_0^{\infty}\int_0^{\infty}...\int_0^{\infty} s \prod_{i=1}^{n} x_i^{s-1} \overline{F}(x_1, x_2, ..., x_n)\, dx_1 ... dx_n, \qquad s > 0,$$

it can be shown that

$$\lim_{\lambda\to\infty} E\left[\prod_{i=1}^{n} X_i^{s}\right] = \beta^{-s}, \qquad s > 0 \cdot$$

Of course, these are moments of random variables that are degenerate at point $\frac{1}{\beta}$.

It should be noted that the density and hazard rate of the distribution given by the equation (4.2), can be obtained from equations (2.1) and (2.2). The hazard rate, particularly, is given by

$$r\left(x_1,x_2,...,x_n;\alpha,\beta,\lambda\right)=\frac{\beta\lambda\prod_{i=1}^{n}\left(\beta x_i\right)^{\lambda-1}}{\left[1-\left(1-\alpha\right)e^{-\sum_{i=1}^{n}\left(\beta x_i\right)^{\lambda}}\right]}.$$

In this function it can be verified, by applying calculus, that its hazard rate is increasing if $\alpha\geq1,\lambda\geq1$ and decreasing if $\alpha\leq1,\lambda\leq1$. If $\lambda>1$, then the hazard rate is initially increasing and eventually increasing, but there may be one interval where it is decreasing. On the other hand, if $\lambda<1$, then the hazard rate is initially decreasing and eventually decreasing, but there may be one interval where it is increasing. The slope changes at those intervals are subtle and hence graphical method can not be applied in this case easily.

## 5. Geometric-Extreme stability of multivariate Distribution

Let $\overline{X}_1=X_1^{(1)},X_1^{(2)},...;\ \overline{X}_2=X_2^{(1)},X_2^{(2)},...;...;\overline{X}_N=X_N^{(1)},X_N^{(2)},...;$ be the sequences of independent identically distributed multivariate random vectors with distributions as stated in the family (1.1), and if N has a geometric distribution on {1,2,3,…}, then minimum and maximum of all $\overline{X}_1,\ \overline{X}_2,...,\overline{X}_N$ also have distributions in the family. To see why this property may be of interest, recall that extreme value distributions are limiting distributions for extrema, and as such they are sometimes useful approximation. In practice, a random vector of interest may be the extreme of only a finite, possibly random, number N of random vectors. When N has a geometric distribution, the random vector has a particularly important stability property, just like that of extreme value distributions.

Assume that N is independent of $\overline{X}_1,\ \overline{X}_2,...,\overline{X}_N$ with a geometric $\left(p\right)$ distribution, that is

$$P\left(N=n\right)=\left(1-p\right)^{n-1}p,\qquad n=1,2,3,...,$$

and let

$$U_1=\min\left(X_1^{(1)},X_2^{(1)},...,X_n^{(1)}\right),U_2=\min\left(X_1^{(2)},X_2^{(2)},...,X_n^{(2)}\right),...,U_N=\min\left(X_1^{(N)},X_2^{(N)},...,X_n^{(N)}\right)$$

*and*

$$V_1=\max\left(X_1^{(1)},X_2^{(1)},...,X_n^{(1)}\right),V_2=\max\left(X_1^{(2)},X_2^{(2)},...,X_n^{(2)}\right),...,V_N=\max\left(X_1^{(N)},X_2^{(N)},...,X_n^{(N)}\right)\qquad(5.1)$$

**Definition:** *If $F\in\tau$ implies that the distributions of $U_i\left(V_i\right),\ \left(i=1,2,...,n\right)$ are in $\tau$, then $\tau$ is said to be geometric-minimum stable (geometric-maximum stable). If $\tau$ is both geometric-minimum and geometric-maximum stable, then $\tau$ is said to be geometric-extreme stable.*

The term \`maximum-geometric stable\` was discussed by [15] and [14] to describe a related but more restricted concept. They apply the term not to families of distributions but to individual distributions; in their sense, a distribution is \` maximum-geometric stable\` if the location-scale parameter family generated by the distribution is geometric-maximum stable in our sense. The two ideas essentially coincide for families $\tau$ that are parameterized by location and scale. Most of the families considered in this paper are not of that form, a notable exception being the logistic distribution. For instance the family of logistic distributions, with multivariate survival function of the form

$$\overline{F}\left(x_1,x_2,...,x_n\right)=\frac{1}{1+\theta e^{\beta x_1+\beta x_2+...+\beta x_n}},\qquad-\infty<x_1,x_2,...,x_n<\infty;\ \theta,\beta>0,$$

is a geometric-extreme stable family; indeed, distributions in this family are geometric-extreme stable even in the sense of [15]. The fact that this family is geometric-minimum stable was utilized by [3] to construct a stationary process with logistic marginal.

Considering random variables $U_i,\ \left(i=1,2,...,n\right)$ of equation (5.1),

$$\begin{aligned}\overline{G}\left(x_1,x_2,...,x_n\right)&=P\left(U_1>x_1,U_2>x_2,...,U_n>x_n\right)\\&=\sum_{n=1}^{\infty}\overline{F}^n\left(x_1,x_2,...,x_n\right)\left(1-p\right)^{n-1}p\\&=\frac{p\overline{F}\left(x_1,x_2,...,x_n\right)}{1-\left(1-p\right)\overline{F}\left(x_1,x_2,...,x_n\right)},\qquad-\infty<x_1,x_2,...,x_n<\infty.\end{aligned}\qquad(5.2)$$

As an extension of univariate and bivariate parametric family of distributions given by [14], the multivariate parametric family of distributions stated in equation (5.2), is also geometric-minimum stable.

Similarly, for random variables $V_i$, $(i=1,2,...,n)$, also given in equation (5.1), by using arguments similar to those used above, we can see that

$$G(x_1,x_2,...,x_n)=P(V_1\le x_1,V_2\le x_2,...,V_n\le x_n)$$
$$=\frac{pF(x_1,x_2,...,x_n)}{1-(1-p)F(x_1,x_2,...,x_n)},\qquad -\infty<x_1,x_2,...,x_n<\infty.$$

Hence,

$$\overline{G}(x_1,x_2,...,x_n)=\sum_{n=1}^{\infty}\overline{F}^n(x_1,x_2,...,x_n)(1-p)^{n-1}p$$
$$=\frac{p\overline{F}(x_1,x_2,...,x_n)}{1-(1-p)\overline{F}(x_1,x_2,...,x_n)},\qquad -\infty<x_1,x_2,...,x_n<\infty. \tag{5.3}$$

According to [14], the multivariate parametric family, given in equation (5.3) above, is geometric-maximum stable.
The multivariate families defined in equations (5.2) and (5.3) above, combine together to give a single parametric family $\xi=\xi(F(X_1,X_2,...,X_n))=\{G(x_1,x_2,...,x_n;\alpha),\alpha>0\}$, where $\overline{G}(x_1,x_2,...,x_n)$ is given by equation (1.1); with condition that in equation (5.2), $0<\alpha=p\le 1,$ and, in (5.3), with $\alpha=\frac{1}{p}\ge 1.$ At this point, it can be seen that $\overline{G}(x_1,x_2,...,x_n;1)=\overline{F}(x_1,x_2,...,x_n)$, hence, $F(x_1,x_2,...,x_n)\in\xi;$ furthermore, it also worth noting that $F(x_1,x_2,...,x_n)\in\xi$ is stochastically increasing function in $\alpha.$

**Proposition 5.1:** *The parametric family $\xi$ of distributions of the form (1.1) is geometric-maximum stable.*

**Proof.** To verify this proposition, it is enough to verify closure of $\xi$ under a kind of composition, as follows. Suppose that $\overline{G}(x_1,x_2,...,x_n)=\frac{\kappa\overline{G}(x_1,x_2,...,x_n;\alpha)}{\{1-(1-\kappa)\overline{G}(x_1,x_2,...,x_n;\alpha)\}}$, where $G(x_1,x_2,...,x_n;\alpha)$ is given as stated in (5.3). Therefore,

$$\overline{H}(x_1,x_2,...,x_n)=\frac{\kappa\alpha\overline{F}(x_1,x_2,...,x_n)}{\{1-(1-\alpha\kappa)\overline{F}(x_1,x_2,...,x_n)\}}.$$

This shows that $H(x_1,x_2,...,x_n)\in\xi$, and consequently, $\xi$ has both geometric-maximum and geometric-minimum stability.

The proof of proposition (5.1) also shows that, if $F$ is replaced by any other distribution in $\xi$, then that distribution will also generate $\xi.$

Below are some properties of geometric-extreme stable families that worth noting. The same properties also hold for geometric-minimum and geometric-maximum stable families.

(a) If $P_1$ and $P_2$ are geometric-extreme stable families, then $P_1\cup P_2$ and $P_1\cap P_2$ are also geometric-extreme stable families; the empty set is vacuously such a family.

(b) For every distribution $F$ that determines a geometric-extreme stable family $P(F)$, if $G\in P(F)$ then $P(G)=P(F).$ Therefore, the minimal geometric-extreme stable families form a partition of the set of all distributions into a set of equivalence classes. In this case, a minimal geometric-extreme stable family is a family which is nonempty and has no nonempty geometric-extreme stable subfamily.

(c ) If $F$ and $G$ differ only by a scale (location) parameter, then $P(G)$ can be derived from $P(F)$ by a common scale (location) parameter change.

(d) Assume that $F\in P$ this means that $\overline{F}(0)>0,$ and also $\overline{F}_+$ is given by the formula:

$$\overline{F}_+(x_1,x_2,...,x_n)=\begin{cases}1, & x_i\le 0,\ \ \forall i,\ i=1,2,...,n,\\ \frac{\overline{F}(x_1,x_2,...,x_n)}{\overline{F}(0)}, & x_i\ge 0,\ \ \forall i,\ i=1,2,...,n.\end{cases}$$

If $F$ is geometric-extreme stable, then $\{F_+ : F\in \mathrm{T}\}$ is also geometric-extreme stable.

(e) Let $F$ be a family of distribution functions, and also suppose that

$$P_{\theta,\delta}=\{G:G(x_1,x_2,...,x_n)=F^{\theta}(x_1-\delta,x_2-\delta,...,x_n-\delta)\quad for\quad some\quad F\in P\}.$$

If $P$ is geometric-extreme stable, then $P_{\theta,\delta}$ is geometric-extreme stable for all $\theta > 0$ and all real $\delta$.

## 5.2 Application of Geometric distribution in extreme stability property

The geometric-extreme stability property of $\xi = \xi(F)$ is indeed important, and it largely depends upon the fact that a geometric sum of independent identically distributed geometric random variables has a geometric distribution. This partially explains why random-minimum stability cannot be expected if the geometric distribution is replaced by some other distribution on $\{1,2,...\}$. Therefore, if the above fact is repeated with the assumption that N-1 has a Poisson distribution, and then $\xi$ would be replaced by a family that would not be Poisson-extreme stable.

If $F$ is a distribution function and $\overline{G}(x_1, x_2, ..., x_n; \theta) = \sum_{n=1}^{\infty} \overline{F}^n (x_1, x_2, ..., x_n) t_n(\theta)$ has the stability property then the discrete distribution must satisfy the functional equation

$$\sum_{n=1}^{\infty} \left\{ \sum_{m=1}^{\infty} z^m t_m(\theta) \right\}^n t_n(\alpha) = \sum_{n=1}^{\infty} z^n t_n(\kappa), \qquad 0 \le z \le 1.$$

The only solution to this equation is the geometric distribution when some regularity conditions are applied.

### 6. Conclusion

The general method of introducing one-parameter into a family of multivariate distribution is developed and presented. The extended exponential distribution provide a new method of adding two-parameter to a family of multivariate distribution which may sometimes compete with multivariate Weibull and gamma families of distributions. New method for derivation of three-parameter type of Weibull family of distribution is introduced and discussed. It is also presented in this paper that all the methods of adding parameter to different families of different distributions commonly possessed stability properties.

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