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On the Existence of a Solution to the Optimal Economic Growth in an Aggregative Closed Economy

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Abstract

The problem of optimal economic growth in an aggregative closed economy in which the basic equation of economic growth is a differential equation has been investigated in various ways by different authors in the past.

This paper investigates the existence of a solution to the optimal economic growth problem by showing that a solution exists to the set of constraint equations. Essentially, the paper retains all the problem formulation stated in [1, 2] where some theoretical bases leading to the economic growth problem are laid.

Keywords: Existence of a solution, optimal economic growth, aggregative closed economy, class C of non-decreasing functions.

1.0 Introduction:

The recognition that the determinants of long-term economic growth are the central macroeconomic problem was accompanied in the late 1980s by important advances in the theory of economic growth. This period featured the development of “endogenous – growth” models with a key feature of these models- a theory of technological progress, viewed as a process whereby purposeful research and application leads over time to new and better products and methods of production.

It is now generally acknowledged that the aggregate output of an economy and hence, it’s economic development path, depend ultimately on how it earnest it’s physical, human, social and environmental capital. The accumulation of physical capital as a determinant of economic growth features prominently in the Solow-Swan model. Solow [3] and Swan [4] had a fundamental impact on “big-question” development economics tries to address. It is worth noting that Solow’s path-breaking work brought about the notion of convergence, meaning that countries with a low endowment of capital relative to labour will have a high rate of return to capital. Thus, a given addition to the capital stock will have a larger impact on per capital income. It then follows that controlling for parameters like savings rates and population growth rates, poorer nations may tend to grow faster and hence catch up or converge to the levels of well-being enjoyed by richer nations. If this view holds, development becomes largely a matter of getting some economic and demographic parameters right. Along this thought of poor nations coming close to the richer ones in the long run, Baumol [5] investigated the long-run result of productivity growth, convergence, and welfare. Bond [6] investigates the estimate of empirical growth models using GMM. Uzawa [7] proposed a human capital model to speculate about the long run consequences of an economy populated by individuals whose innovative efforts are driven not only by the desire of accumulating marketable expertise, but also by values of sharing knowledge or by intellectual curiosity. A simple calibration of the model suggests that in the long run an economy populated by individuals who accumulate human capital only because they were driven by intellectual curiosity grows faster than one in which investments in human capital are motivated by the prospect of monetary reward.

Mankiw, Romer and Weil [8] incorporated human capital into the Solow-Swan model and demonstrated that the augmented model provides an excellent explanation on the international variation in standard of living. Also, economists have turned increasingly to the use of endogenous-growth models which go beyond the Solow-Swan model to argue for externalities of the generation of human capital and technological innovations, and for profit as the driving force behind innovation.

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Tran-Nam [9] investigated an infinite- horizon aggregative closed economy in which production depends essentially on physical capital, natural capital and labour. The natural capital stock is modeled as a renewable resource in which the change in the stock of natural capital depends on its autonomous evolution, production and consumption externalities, and environment maintenance program. White and Rogers [10] review the economic context within which American families lived in the 1990s. With full employment and growing income and wealth for many Americans, problem areas existed within the society. These included persistent racial gaps in economic well-being, growing inequality, and declining wages for young men. This same trend exist in most societies and organizations today particularly in most developing economies, hence the need to address the issue of economic growth.

The neoclassical theory of economic growth and the CES production function are regarded as twins. Solow seminar contribution in 1956 not only created a new theory of macroeconomic dynamics, but also introduced a new type of aggregate production function. Over the last 40 years, not less than four different variants of the CES production function have been used. The original specification due to Solow contains not only a substitution parameter $\psi = (\sigma - 1)/\sigma$ which has become a common element of all further CES variants, but also an unrestricted term α whose economic significance is not clearly defined. Arrow et al. [11] and Brown and de Cani [12] took care of the elasticity of substitution by developing a sound theoretical and econometric foundation for the CES function. Later, the emergence of a new growth theory created interest in CES production function. Klump and Preissler [13] examine the use of CES production functions in growth models. It was found that not all variants of CES functions commonly used are consistently specified.

The link between growth and investment has formed the core of various economic growth models. The AK-type endogenous growth models suggest that investment in broadly defined capital has a positive long-run effect on growth. Dajin [14] examine the empirical validity of AK-type endogenous growth models, and the long-run relation between growth and investment. It was found that contrary to Jines [15] findings, the broadly measured rate of investment exerts a long-run positive effect on the growth rate. The result was supported by evidence from twenty-four OECD countries, 1950 – 1992 and five major industrialized countries, 1870 – 1987.

Dynamic optimization methods have been used in normative applications of growth theory with a view to deriving conditions for optimal economic growth but not alleging that reality could be explained by such hypotheses. Cass [16] and Arrow [17] employed dynamic optimization from the viewpoint of closed (centralized) economies in a position to control the economy completely. Since the seminar papers by Romer [18, 19] and Lucas [20] on endogenous growth were published, dynamic optimization models have become unquestioned standard in growth theory. Drastic applications of dynamic optimization models for economic growth have been carried out by various researchers. For instance, Eicher et al. [21] interpreted economic crises as being the result of the decisions of dynamically optimizing agents.

Of great interest to many researchers is the mathematical model of a highly simplified economy [1] in which the rate of output, $Y(t)$ is assumed to depend on the rates of input of capital, $K(t)$ and labor force, $L(t)$ .Specifically, the basic equation of economic growth is stated as a differential equation of the form:

$$F[K(t)] = C(t) + [\mu + \sigma(t)]K(t) + \frac{dK(t)}{dt}, \quad ,t_0 \le t \le t_1$$

Rate of output being a function of rates of input of capital and labour force implies that

$$Y = F(K, L) \tag{1.1}$$

Where, F is regarded as the production function.

Assuming that this function obeys a 'return to scale' property so that

$$F(\beta K, \beta L) = \beta F(K, L) \tag{1.2}$$

Setting $\beta = 1/L$ and defining the output per worker and capital per worker, respectively, as

$$y = Y/L \text{ and } k = K/L$$

We have

$$y = Y/L = F(K, L)/L = F(K,1) = f(k) \tag{1.3}$$

Where, $f(k)$ satisfy the conditions $f'(k) > 0$ and $f''(k) < 0$

Furthermore, since output is either consumed or invested, we have

$$Y(t) = C(t) + I(t) \tag{1.4}$$

Where, C and I are the rates of consumption and investment, respectively. Since the investment is used

To increase the capital stock and replace machinery, then

$$I = \dot{K} + \mu K \tag{1.5}$$

Where, μ is the rate of depreciation of capital and $\dot{K} = \dfrac{dK}{dt}$

Defining $c = C/L$ as the consumption rate per worker, equations (1.3) through (1.5) give

$$y = f(k) = c + \frac{1}{l}\dot{K} + \mu k \tag{1.6}$$

and since

$$\frac{d}{dt}\left[\frac{K}{L}\right] = \frac{1}{L}\dot{K} - \frac{1}{L}\dot{L}\,k$$

Which when substituted into equation (1.6) yields

$$f(k) = c + \dot{k} + \frac{\dot{L}}{L}k + \mu k \tag{1.7}$$

Assuming that labour grows exponentially with economic growth, that is $L = L_o \ell^{\sigma t}$, then

$$\dot{k} = f(k) - (\mu + \sigma)k - c \tag{1.8}$$

This last equation, due to Burghes and Graham [1], shall be regarded as the governing equation of the economic growth model.

Naturally, a planner needs to select a history of consumption per worker, c (t), on the fixed time interval $t_o \le t \le t_1$, which is subject to some constraints, such as $o \le c(t) \le f(k),\ t_o \le t \le t_1$
For instance, if it is desirable that

$$\begin{aligned} &I(t) = Y(t) - C(t) \\ &or \qquad\qquad t_o \le t \le t_1 \\ &i(t) = f[k(t)] - c(t) \end{aligned} \tag{1.9}$$

always be nonnegative, where $i(t) = I(t)/L$.

In order to make such a choice, the planner must have an economic objective, say the maximization of a standard of living as measured by W, such that

$$W = \int_{t_o}^{t_1} e^{-\delta t} U[c(t)]dt \tag{1.10}$$

where, $U(c)$ is a utility function which is a monotonic increasing function of c, δ is the discount rate for future utility, $i(t)$ is the investment per worker, $f(k)$ is the output per worker, σ is the rate of growth of the labor force and $\hat{W}$ is welfare to be maximized. It is also required that at time $t = t_1$ the remaining capital is not all exhausted so that life may continue. Hence, in addition to knowing the initial condition $k(t_o) = k_o$, it is required that $k(t_1) = k_{t_1} \ge k_1$, some minimal acceptable capital for the beginning of the second time period $t \ge t_1$.

The optimal economic growth problem then amounts to finding $c(t)$ in some class $\hat{C}$ containing only non-decreasing functions such that

$$\hat{W} = \sup.\left[\int_{t_o}^{t_1} e^{-\delta t} U[c(t)]dt\right] \tag{1.11}$$

subject to the set of constraints:

$$\begin{aligned}
&\dot{K}(t) + [\mu + \sigma(t)]k(t) + c(t) = f[k(t)], \quad t_0 \le t \le t1\\
&k(t_o) = k_o\\
&k(t_1) \ge k_1 \qquad\qquad (1.12)\\
&o \le c(t) \le \oint [k(t)]
\end{aligned}$$

Researchers have investigated the different variants of economic growth problem in various way base on individual interest [2, 22, 23]. Our interest in this paper is to establish the existence of solution to the above problem posed by Burghes and Graham [1]. The remaining path of the paper is organized as follows. Section 2 discusses the existence of solution to the economic growth problem in an aggregative closed economy by the establishment and proof of theorem 2.1. In Section 2.1 we discuss the result obtained from the proof of the theorem, while Section 3 deals with the conclusion.

2.0 Existence of Solution

The existence of solution to the set of constraint in (1.12) is established through theorem 2.1.

Theorem 2.1

The existence of a solution of the basic equation of economic growth corresponds to the existence of a fixed point k of F such that F(k;c) satisfies the set of constraints in expression (1.12).

Proof of theorem 2.1

Define a functional $F(x;c)$ = y from functions $x(t)$ on $t_o \le t \le t_1$ to functions $y(t)$ on $t_o \le t \le t_1$ by

$$F(x;c) = k_o + \int_t^{t_1} [f[x(s)] - c(s) - \{\mu + \sigma(s)\}x(s) \ ]ds,$$

then

$$F(k;c) = k_o + \int_t^{t_1} [f[k(s)] - c(s) - \{\mu + \sigma(s)\}k(s) \ ]ds \qquad (2.1)$$

$F(k;c) = k$ implies that:

$$k = k_o + \int_{t_o}^{t} [f\{k(s)] - c(s) - \{\mu + \sigma(s)\}k(s) \ ]ds$$

Thus,

$$k(t) = k_o + \int_{t_o}^{t} [f[k(s)] - c(s) - \{\mu + \sigma(s)\}k(s) \ ]ds$$

From where

$$k(t_o) = k_o + \int_{t_o}^{t_o} [f[k(s)] - c(s) - \{\mu + \sigma(s)\}k(s) \ ]ds$$

Since $\int_{t_o}^{t} [f[k(s)] - c(s) - \{\mu + \sigma(s)\}k(s) \ ]ds = o$ then $k(t_o) = k_o$ this satisfied the initial condition.

Now,

$$k(t_1) = k_o + \int_{t_o}^{t_1} [f[k(s)] - c(s) - \{\mu + \sigma(s)\}k(s) \ ]ds \ge k_1$$

implies that,

$$k(t_1) - k_o = \int_{t_o}^{t_1} [f[k(s)] - c(s) - \{\mu + \sigma(s)\}k(s) \ ]ds \ge k_1 - k_o$$

Imposing the condition that $k(t) \ge k_o$ for all $t \in [t_o, t_1]$, then $k(t \ \ ) - k_o \ge o$

implies that,

$$k(t)-k_o=\int_{t_o}^{t}\left[f[k(s)]-c(s)-\{\mu+\sigma(s)\}k(s)\ \right]ds\geq o$$

Making use of the fact that if g is an integrable function on $[a,b]$ and $g(x)\geq o$ for all $x\in[a,b]$, then

$$\int_a^b g(x)dx\geq o$$

We then have

$$\int_{t_o}^{t_1}\left[f[k(s)]-c(s)-\{\mu+\sigma(s)\}k(s)\ \right]ds\geq o$$

$$or$$

$$f[k(s)]-c(s)-\{\mu+\sigma(s)\}k(s)\geq 0$$

Since s is a dummy variable, we have

$$f[k(t)-c(t)-\{\mu+\sigma(t)\}k(t)\geq 0 \text{ for any t such that } t_o\leq t\leq t_1.$$

The above translates to:

$$f[k(t)-\{\mu+\sigma(t)\}k(t)=c(t) \text{ when } t=t_o$$

and

$$f[k(t)]-\{\mu+\sigma(t)\}k(t)>c(t) \text{ for all t such that } t_o<t\leq t_1.$$

If $c(t)$ is selected as above, then

$$k(t_1)=k_o+\int_{t_o}^{t_o}\left[f[k(s)]-c(s)-\{\mu+\sigma(s)\}k(s)\ \right]ds$$

Thus the final condition is satisfied and the class $\hat{C}$ containing $c(t)$ can judiciously be selected.

2.1 Discussion

The proof of theorem 2.1 shows that there exist non-decreasing functions c(t) in some class $\hat{C}$ for which welfare is maximized as stated in expression (1.11) based on boundary conditions without violating any of the constraint equation in expression (1.12). Thus, a solution exists to the set of constraint equations.

3. Conclusion

The proof of theorem 2.1 has established the fact that solution to the optimal economic growth problem in an aggregative closed economy exist. Efforts made by researchers in investigating the problem in various ways are therefore not a waste.

This paper has established a need for the development of computational procedures that will address the problem on the computer. By so doing, further investigations on the computational results can be carried out through application to various real life systems that operate close to the problem as formulated. This may lead to finding a way of optimizing the welfare of those within such organization or society.

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