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Modeling Hiv Epidemic Under Contact Tracing - A Remark

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Abstract

A nonlinear model on HIV epidemic under contact tracing is studied, where we assume that the rate of recruitment of HIV positives is proportional to the population. We determine the criteria for stability of the epidemic free equilibrium and the endemic equilibrium..

Keywords: Contact tracing, basic reproductive number, epidemic free, endemic.

1. Introduction:

Contact tracing has been used as a method of controlling contagious diseases [1,2]. While there is still a debate about contact tracing for the HIV infection [3,4] the resurgence of infectious tuberculosis and outbreaks of drug resistance tuberculosis secondary to HIV induced immunodepression is forcing many public health departments to re-examine this policy[5,6].

2 Mathematical Formulation

We shall consider the differential system that describes the model which is proposed by Arazazo et al [7].

The model [7] is

$$\frac{dN}{dt} = -\alpha NX - \mu N + \delta$$
$$\frac{dX}{dt} = \alpha NX - \ (k + \mu + \beta)X + v(1)$$

$$\frac{dY}{dt} = kX - (\mu + \beta)Y$$

$$\frac{dZ}{dt} = \beta X + \beta Y - \mu' Z + \rho$$

Where

N-the population of sexually active person or susceptible
X-the number of HIV positives that do not know they are infected.
Y- the number of HIV positives that do know they are infected.
Z-the number of AIDS cases
k- the rate at which the HIV positives are detected.
β- the rate at which the HIV positive develops AIDS.
μ-the death rate of the sexually active population.
μ'-the death rate due to AIDS.
δ-the recruitment into the class of the susceptible.
υ-the immigration of unknown HIV positives.
ρ-the immigration of AIDS cases.

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In this paper we assume that the rate of recruitment of HIV positive is proportional to the population.
The model becomes

$$\frac{dN}{dt} = -\alpha NX - \mu N + \delta N$$

$$\frac{dX}{dt} = \alpha NX - (k + \mu + \beta)X + vN$$

$$\frac{dY}{dt} = kX - (\mu + \beta)Y \qquad (2)$$

$$\frac{dZ}{dt} = \beta X + \beta Y - \mu' Z + \rho$$

After transformation (2) becomes

$$\frac{dN}{dt} = -\alpha NX - \mu N + \delta N$$

$$\frac{dX}{dt} = \alpha NX - (k + \mu + \beta)X + vN$$

$$\frac{dY}{dt} = kX - (\mu + \beta)Y \qquad (3)$$

$$\frac{dM}{dt} = \beta X + \beta Y - \mu' M$$

<u>Descartes' rules of sign</u>

The number of positive zeros of polynomial with real coefficients is either equals to the number of variations in sign of the polynomial or less than this by an even number [4].

3 Results

(a) Stability of the Epidemic free equilibrium

Theorem 1: If $R_0 < 1$, then the zero solution of the epidemic free equilibrium of (3) is asymptotically stable.

Proof: The Jacobian matrix of epidemic free equilibrium is

$$A = \begin{pmatrix} \delta - \mu & 0 & 0 & 0 \\ 0 & -(k + \mu + \beta) & 0 & 0 \\ 0 & k & -(\mu + \beta) & 0 \\ 0 & \beta & \beta & -\mu' \end{pmatrix}$$

The eigenvalues are

$$\lambda_1 = \delta - \mu \qquad (4)$$

$$\lambda_2 = -(k + \mu + \beta) \qquad (5)$$

$$\lambda_3 = -(\mu + \beta) \qquad (6)$$

$$\lambda_4 = -\mu' \qquad (7)$$

The zero solution epidemic free equilibrium of (3) is asymptotically stable if (4) is less than zero (i.e $\lambda_1 < 0$, if $\delta - \mu < 0$ then $\delta < \mu, \frac{\delta}{\mu} < 1, R_0 < 1$)

Theorem 2: If $R_0 > 1$, then the zero solution of the epidemic free equilibrium of (3) is unstable

Proof: The Jacobian matrix of epidemic free equilibrium is

$$A= \begin{pmatrix} \delta-\mu & 0 & 0 & 0 \\ 0 & -(k+\mu+\beta) & 0 & 0 \\ 0 & k & -(\mu+\beta) & 0 \\ 0 & \beta & \beta & -\mu' \end{pmatrix}$$

The eigenvalues are

$$(\delta-\mu-\lambda)(-(k+\mu+\beta)-\lambda)(-(v+\beta)-\lambda)(-\mu'-\lambda)=0$$

Recall that $R_0=\dfrac{\delta}{\mu}$, $R_0\,\mu=\delta$

$$-\mu(R_0-1)(k+\mu+\beta)+\lambda(k+2\mu+\beta-\delta)+\lambda^2$$

and

$$\lambda_3=-(\mu+\beta),\ \lambda_4=-\mu'$$

$$\lambda^2+\lambda(k+\beta+\mu(2-R_0))-\mu(R_0-1)(k+\mu+\beta)=0 \tag{8}$$

It is sufficient to show that at least one eigenvalue is positive. Now

$\lambda^2+\lambda(k+\beta+\mu(2-R_0))-\mu(R_0-1)(k+\mu+\beta)=0$

has one variation in sign when

$(k+\mu+\beta)>0\,, \mu>0$ *and* $R_0>1.$ Hence by Descartes' rule of sign, (8) has a positive root and then the critical point is unstable.

This completes the proof.

(b) Stability of the epidemic equilibrium

Theorem 3: If $r_3>$, $r_2>$,$r_1>0$ and $r_0>0$, the endemic equilibrium is asymptotically stable.

Proof: The Jacobian matrix of epidemic equilibrium (3) is

$$A=\begin{pmatrix} 0 & -\dfrac{(k+\mu+\beta)(\delta-\mu)}{(-\mu+\delta+v)} & 0 & 0 \\ v+\delta-\mu & v\dfrac{(k+\mu+\beta)}{(-\mu+\delta+v)} & 0 & 0 \\ 0 & k & -(\mu+\beta) & 0 \\ 0 & \beta & \beta & -\mu' \end{pmatrix}$$

The eigenvalues of the epidemic equilibrium is obtained by solving

$$|A-I\lambda|=0 \tag{9}$$

$$\lambda^4+\lambda^3 r_3+\lambda^2 r_2+\lambda^1 r_1+r_0=0 \tag{10}$$

Where $r_3=(\mu'+\mu+\beta)-v\dfrac{(k+\mu+\beta)}{(-\mu+\delta+v)}$

$r_2=(k+\mu+\beta)(\delta-\mu)+\mu'(\mu+\beta)-v(\mu'+\mu+\beta)\dfrac{(k+\mu+\beta)}{(-\mu+\delta+v)}$

$$r_1=(k+\mu+\beta)(\delta-\mu)(\mu'+\mu+\beta)-v\mu'(\mu+\beta)\frac{(k+\mu+\beta)}{(-\mu+\delta+v)}$$

$$r_0=(k+\mu+\beta)(\delta-\mu)\mu'(\mu+\beta)$$

(10) has zero variation is signs. Hence by Descartes' rule of signs λ's are all negative or two negative roots and two complex numbers. It they are all negative, and then the critical point is asymptotically stable. Suppose

$$\lambda_1=\alpha+i\beta,\ \lambda_2=\alpha-i\beta,\lambda_3=-a \quad and \quad \lambda_4=-b$$

We claim that $\alpha<0$. The number of complex roots will be 2 or 0. If there are two negative roots then

$$(\lambda-\alpha-i\beta)(\lambda-\alpha+i\beta)(\lambda+a)(\lambda+b)=0 \text{ and } =0 \text{ and}$$

$$\lambda^4+(a-2\alpha+b)\lambda^3+\left(\alpha^2+\beta^2+ab-2\alpha(a+b)\right)\lambda^2+\left((a+b)\left(\alpha^2+\beta^2\right)-2ab\right)\lambda+ab\left(\alpha^2+\beta^2\right)=0$$

$$r_3=a-2\alpha+b$$

$$r_2=\alpha^2+\beta^2+ab-2\alpha(a+b)$$

$$r_1=(a+b)\left(\alpha^2+\beta^2\right)-2ab$$

$$r_0=ab\left(\alpha^2+\beta^2\right)$$

But $r_3>0,r_2>0,r_1>0,r_0>0$ (given)

$$\therefore a-2\alpha+b>0,\ \alpha^2+\beta^2+ab-2\alpha(a+b)>0,\ (a+b)\left(\alpha^2+\beta^2\right)-2ab>0,\ ab\left(\alpha^2+\beta^2\right)>0$$

Since "a" and "b" can be small as "a"$\to 0$, "b" $\to 0$ $\quad r_1>0 \Rightarrow \quad \alpha<0$

Hence the equilibrium point is asymptotically stable.

This completes the proof.

Theorem 4: If $R_0>1,\mu'(\delta+v)+\delta(\mu+\beta)>\mu\left(\mu'+\beta+\mu^2\right)+vk$,

$$\begin{aligned}&(k+\mu+\beta)\left(\mu^2+\delta^2+v\delta\right)+\mu'((\mu+\beta)(\delta+v))>\\&v(\mu'+\mu+\beta)(k+\mu+\beta)+\mu(\mu'(\mu+\beta)+(k+\mu+\beta)(2\delta+v))\end{aligned} \quad \text{and}$$

$$\begin{aligned}&(\mu'+\beta+\mu)\left(\mu^2+\delta^2+v\delta\right)>\\&\mu'v(\mu+\beta)+\mu(\mu'+\mu+\beta)(2\delta+v)\end{aligned},$$

then the endemic equilibrium is asymptotically stable.

Proof: The Jacobian matrix of epidemic equilibrium (3) is

$$A=\begin{pmatrix}0 & -\dfrac{(k+\mu+\beta)(\delta-\mu)}{(-\mu+\delta+v)} & 0 & 0\\ v+\delta-\mu & v\dfrac{(k+\mu+\beta)}{(-\mu+\delta+v)} & 0 & 0\\ 0 & k & -(\mu+\beta) & 0\\ 0 & \beta & \beta & -\mu'\end{pmatrix}$$

We want to show that $r_3>0,r_2>0,r_1>0$ and $r_0>0$

$$r_3=(\mu'+\mu+\beta)-v\frac{(k+\mu+\beta)}{(-\mu+\delta+v)}$$

Note that

$$(\mu'+\mu+\beta)-v\frac{(k+\mu+\beta)}{(-\mu+\delta+v)}=\frac{(\mu'+\mu+\beta)(-\mu+\delta+v)-v(k+\mu+\beta)}{(-\mu+\delta+v)}$$

$$=\frac{-\mu'\mu-\mu^2-\mu\beta+\mu'\delta+\mu\delta+\beta\delta+\mu'v+\mu v+\beta v-v(k+\mu+\beta)}{(-\mu+\delta+v)}$$

$$\frac{-\mu'\mu-\mu^2-\mu\beta+\mu'\delta+\mu\delta+\beta\delta+\mu'v-vk}{(-\mu+\delta+v)}>0$$

since $\quad \mu'(\delta+v)+\delta(\mu+\beta)>\mu(\mu'+\mu+\beta)+vk$

this implies that $\quad r_3>0$.

Clearly $\quad r_2>0 \quad$ if $\quad \mu'(\mu+\beta)+(k+\mu+\beta)(\delta-\mu)-v(\mu'+\mu+\beta)\frac{(k+\mu+\beta)}{(-\mu+\delta+v)}>0$

$$\frac{\mu'(\mu+\beta)(-\mu+\delta+v)+(k+\mu+\beta)(\delta-\mu)(-\mu+\delta+v)-v(\mu'+\mu+\beta)(k+\mu+\beta)}{(-\mu+\delta+v)}>0$$

$$\begin{aligned}&(k+\mu+\beta)(\mu^2+\delta^2+v\delta)+\mu'(\mu+\beta)(\delta+v)-\\&\mu(\mu'(\mu+\beta)+(k+\mu+\beta)(2\delta-v))-v(\mu'+\mu+\beta)(k+\mu+\beta)>0\end{aligned}$$

$$\begin{aligned}&(k+\mu+\beta)(\mu^2+\delta^2+v\delta)+\mu'(\mu+\beta)(\delta+v)>\\&\mu(\mu'(\mu+\beta)+(k+\mu+\beta)(2\delta+v))+v(\mu'+\mu+\beta)(k+\mu+\beta)\end{aligned}$$

This implies that $r_2>0$
Here we want to show that $r_1>0$

$$r_1=(k+\mu+\beta)(\delta-\mu)(\mu'+\mu+\beta)-v\mu'(\mu+\beta)\frac{(k+\mu+\beta)}{(-\mu+\delta+v)}$$

Note that $\quad (k+\mu+\beta)(\delta-\mu)(\mu'+\mu+\beta)-v\mu'(\mu+\beta)\frac{(k+\mu+\beta)}{(-\mu+\delta+v)}$

$$=\frac{(k+\mu+\beta)(\delta-\mu)(\mu'+\mu+\beta)(-\mu+\delta+v)-v\mu'(\mu+\beta)(k+\mu+\beta)}{(-\mu+\delta+v)}$$

$$=\frac{(k+\mu+\beta)((\delta-\mu)(\mu'+\mu+\beta)(-\mu+\delta+v)-v\mu'(\mu+\beta))}{(-\mu+\delta+v)}$$

$$(k+\mu+\beta)((\delta-\mu)(\mu'+\mu+\beta)(-\mu+\delta+v)-v\mu'(\mu+\beta))>0$$

$$(k+\mu+\beta)>0 \text{ and } \quad (\delta-\mu)(\mu'+\mu+\beta)(-\mu+\delta+v)-v\mu'(\mu+\beta)>0$$

$$(k+\mu+\beta)>0 \text{ and } \quad -\mu(\mu'+\mu+\beta)(2\delta+v)-v\mu'(\mu+\beta)+(\mu'+\mu+\beta)(\mu^2+\delta^2+\delta v)>0$$

Since $\quad (\mu'+\mu+\beta)(\mu^2+\delta^2+\delta v)>\mu(\mu'+\mu+\beta)(2\delta+v)+v\mu'(\mu+\beta)$

This implies that $r_1>0$

And lastly, we want to show that $r_0>0$

$$r_0=(k+\mu+\beta)(\delta-\mu)\mu'(\mu+\beta)$$
$$for\ r_0>0\ ,\ \delta>\mu$$
$$\text{implies that } R_0>1$$

Hence $r_0>0$
By theorem (3) endemic equilibrium is locally and asymptotically stable. This completes the proof

4. Discussion of Result

From the result obtained in (4) and (8),the basic reproduction number R_o is less than 1, then the epidemic free equilibrium is globally asymptotically stable but otherwise, the epidemic free equilibrium is unstable i.e if $R_o>1$. From the result obtained in (10), the epidemic equilibrium is asymptotically stable if $R_o>1$. From what we had in (4) and (10), both the epidemic free equilibrium and endemic equilibrium are stable and their stability depends on the basic reproduction number R_o. Therefore contact tracing could be used as a method of controlling the spread of the virus.

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