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Efficient Order Seven and Eight Rational Integrators

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Abstract

In this paper we derive two new numerical integrators of orders seven and eight for the solution of initial value problems in ordinary differential systems that are stiff, singular or oscillatory. We then compare our integrator with some existing methods. Our results show good improvement over some known existing methods.

Keywords: , Rational Integrator, Initial Value Problems, Convergence, Consistency

1. Introduction:

Obtaining solutions to Initial Value Problems in Ordinary Differential equations that are Stiff, Singular or Oscillatory has been the desire of several scientific researchers. These problems easily arise from real life problems in physical situations, chemical kinetics, engineering work, population models, electrical networks, biological simulations, mechanical oscillations, process control [1 – 7, 9].

In this paper, the initial value problem (ivp),

$$\mathbf{y}^1 = \mathbf{f}(x,\mathbf{y}), \quad \mathbf{y}(x_0) = \mathbf{y}_0, \ a \le x \le b \tag{1.1}$$

is considered, where $f(x,y)$ is defined and continuous in $D \subset C[a,b]$. The ivp may be Stiff, Singular or Oscillatory.
We desire to have an order seven and order eight numerical rational integrator for the solution of initial value problems in ordinary differential equations as an extension of [2]. We then compare our results with the results obtained with the Maximum order second derivative hybrid multi – step methods of [5] and Adaptive Implicit Runge – Kutta proposed by [4]. Our results show good improvement over some known existing methods.
We shall in section 2 of this paper be concerned with the derivation of the two integrators. Section 3 we will be on Convergence and Consistency while section 4 will be devoted to numerical examples. Finally, remarks and conclusions will be given in section 5.

2. Construction of the Integrators

Let the operator: $U : R \to C^{m+2}(x)$ be defined by

$$U(x)\left[1 + q_1 x + q_2 x^2 + q_3 x^3\right] \equiv p_m(x) \tag{2.1}$$

where U(x) has at least 1st, 2nd, 3rd . . . (m + 2)th derivatives

$$U(x_{n+i}) = \begin{cases} y(x_{n+i}) & \text{for } i = 0 \\ y_{n+i} & \text{for } i = 0, 1, 2 \end{cases} \tag{2.2}$$

2.1 The Integrator of Order Seven

Here the indicator m = 4.
Comparing equations (2.1) and (2.2) we have

$$q_1 = \frac{c_7c_3^2 - c_7c_4c_2 + c_6c_5c_2 - c_5^2c_3 - c_6c_4c_3 + c_5c_4^2}{c_6c_4c_2 - c_6c_3^2 - c_5^2c_2 + 2c_5c_4c_3 - c_4^3} \tag{2.3}$$

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$$q_2 \;=\; \frac{c_6c_5c_3 - c_6^2c_2 + c_7c_5c_2 - c_7c_4c_3 - c_5^2c_4 + c_4^2c_6}{c_6c_4c_2 - c_6c_3^2 - c_5^2c_2 + 2c_5c_4c_3 - c_4^3} \tag{2.4}$$

$$q_3 \;=\; \frac{c_6^2c_3 - 2c_6c_5c_4 + c_5^3 - c_7c_5c_3 + c_7c_4^2}{c_6c_4c_2 - c_6c_3^2 - c_5^2c_2 + 2c_5c_4c_3 - c_4^3} \tag{2.5}$$

Where $$c_r = \frac{h^r y_n^{(r)}}{r!\,x_{n+1}^r} \tag{2.6}$$

By considering (2.3) – (2.6), we obtain values for our $q_1x_{n+1},\;\; q_2x_{n+1}^2 \;\; and \; q_3x_{n+1}^3$ as

$$q_1x_{n+1} = \frac{h\left[40y_n^{(7)}y_n^{(3)^2} - 30y_n^{(7)}y_n^{(4)}y_n^{(2)} + 42y_n^{(6)}y_n^{(5)}y_n^{(2)} - 84y_n^{(5)^2}y_n^{(3)} - 70y_n^{(6)}y_n^{(4)}y_n^{(3)} + 105y_n^{(5)}y_n^{(4)^2}\right]}{7\left[30y_n^{(6)}y_n^{(4)}y_n^{(2)} - 40y_n^{(6)}y_n^{(3)^2} - 36y_n^{(5)^2}y_n^{(2)} + 120y_n^{(5)}y_n^{(4)}y_n^{(3)} - 75y_n^{(4)^3}\right]} \tag{2.7}$$

$$q_2x_{n+1}^2 = \frac{h^2\left[28y_n^{(6)}y_n^{(5)}y_n^{(3)} - 14y_n^{(6)^2}y_n^{(2)} + 12y_n^{(7)}y_n^{(5)}y_n^{(2)} - 20y_n^{(7)}y_n^{(4)}y_n^{(3)} - 42y_n^{(5)^2}y_n^{(4)} + 35y_n^{(4)^2}y_n^{(6)}\right]}{14\left[30y_n^{(6)}y_n^{(4)}y_n^{(2)} - 40y_n^{(6)}y_n^{(3)^2} - 36y_n^{(5)^2}y_n^{(2)} + 120y_n^{(5)}y_n^{(4)}y_n^{(3)} - 75y_n^{(4)^3}\right]} \tag{2.8}$$

$$q_3x_{n+1}^3 = \frac{h^3\left[70y_n^{(6)^2}y_n^{(3)} - 210y_n^{(6)}y_n^{(5)}y_n^{(4)} + 126y_n^{(5)^3} - 60y_n^{(7)}y_n^{(5)}y_n^{(3)} + 75y_n^{(7)}y_n^{(4)^2}\right]}{210\left[30y_n^{(6)}y_n^{(4)}y_n^{(2)} - 40y_n^{(6)}y_n^{(3)^2} - 36y_n^{(5)^2}y_n^{(2)} + 120y_n^{(5)}y_n^{(4)}y_n^{(3)} - 75y_n^{(4)^3}\right]} \tag{2.9}$$

Our integrator from (2.1) for m = 4 is given as,

$$y_{n+1} = \frac{p_0 + p_1x_{n+1} + p_2x_{n+1}^2 + p_3x_{n+1}^3 + p_4x_{n+1}^4}{1 + q_1x_{n+1} + q_2x_{n+1}^2 + q_3x_{n+1}^3} \tag{2.10}$$

where

$$\left.\begin{aligned}
p_0 &= y_n\\
p_1 &= c_0q_1 + c_1 = y_nq_1 + \frac{hy_n^{(1)}}{x_{n+1}} \Rightarrow p_1x_{n+1} = y_nq_1x_{n+1} + hy_n^{(1)}\\
p_2 &= c_0q_2 + c_1q_1 + c_2 = y_nq_2 + \frac{hy_n^{(1)}}{x_{n+1}}q_1 + \frac{h^2y_n^{(2)}}{2!x_{n+1}^2} \Rightarrow p_2x_{n+1}^2 = y_nq_2x_{n+1}^2 + hy_n^{(1)}q_1x_{n+1} + \frac{h^2y_n^{(2)}}{2!}\\
p_3 &= c_0q_3 + c_1q_2 + c_2q_1 + c_3 = y_nq_3 + \frac{hy_n^{(1)}}{x_{n+1}}q_2 + \frac{h^2y_n^{(2)}}{2!x_{n+1}^2}q_1 + \frac{h^3y_n^{(3)}}{3!x_{n+1}^3}\\
&\Rightarrow p_3x_{n+1}^3 = y_nq_3x_{n+1}^3 + hy_n^{(1)}q_2x_{n+1}^2 + \frac{h^2y_n^{(2)}}{2!}q_1x_{n+1} + \frac{h^3y_n^{(3)}}{3!}\\
p_4 &= c_1q_3 + c_2q_2 + c_3q_1 + c_4 = \frac{hy_n^{(1)}}{x_{n+1}}q_3 + \frac{h^2y_n^{(2)}}{2!x_{n+1}^2}q_2 + \frac{h^3y_n^{(3)}}{3!x_{n+1}^3}q_1 + \frac{h^4y_n^{(4)}}{4!x_{n+1}^4}\\
&\Rightarrow p_4x_{n+1}^4 = hy_n^{(1)}q_3x_{n+1}^3 + \frac{h^2y_n^{(2)}q_2x_{n+1}^2}{2!} + \frac{h^3y_n^{(3)}q_1x_{n+1}}{3!} + \frac{h^4y_n^{(4)}}{4!}
\end{aligned}\right\} \tag{2.11}$$

Substituting and rearranging we have our order seven integrator as

$$y_{n+1} = \frac{\displaystyle\sum_{r=0}^{4}\frac{h^ry_n^{(r)}}{r!} + A\sum_{r=0}^{3}\frac{h^ry_n^{(r)}}{r!} + B\sum_{r=0}^{2}\frac{h^ry_n^{(r)}}{r!} + C\sum_{r=0}^{1}\frac{h^ry_n^{(r)}}{r!}}{1 + A + B + C} \tag{2.12}$$

where,

A = q_1x_{n+1} as in equation (2.7)

B = $q_2x_{n+1}^2$ as in equation (2.8)

C = $q_3x_{n+1}^3$ as in equation (2.9)

2.2 The Integrator of order Eight

Our indicator in this case is m = 5.
Comparing equations (2.1) and (2.2) we have

$$q_1 = \frac{c_8c_4^2 - c_8c_5c_3 + c_7c_6c_3 - c_6^2c_4 - c_7c_5c_4 + c_6c_5^2}{c_7c_5c_3 - c_7c_4^2 - c_6^2c_3 + 2c_6c_5c_4 - c_5^3} \tag{2.13}$$

$$q_2 = \frac{c_7c_6c_4 - c_7^2c_3 + c_8c_6c_3 - c_8c_5c_4 - c_6^2c_5 + c_5^2c_7}{c_7c_5c_3 - c_7c_4^2 - c_6^2c_3 + 2c_6c_5c_4 - c_5^3} \tag{2.14}$$

$$q_3 = \frac{c_7^2c_4 - 2c_7c_6c_5 + c_6^3 - c_8c_6c_4 + c_8c_5^2}{c_7c_5c_3 - c_7c_4^2 - c_6^2c_3 + 2c_6c_5c_4 - c_5^3} \tag{2.15}$$

where the $c_r{}'s$ are specified by (2.6).

If we examine equations (2.13) – (2.15) with (2.6) we obtain values for our q_1x_{n+1}, $q_2x_{n+1}^2$ *and* $q_3x_{n+1}^3$ as

$$q_1x_{n+1} = \frac{h\left[75y_n^{(8)}y_n^{(4)^2} - 60y_n^{(8)}y_n^{(5)}y_n^{(3)} + 80y_n^{(7)}y_n^{(6)}y_n^{(3)} - 140y_n^{(6)^2}y_n^{(4)} - 120y_n^{(7)}y_n^{(5)}y_n^{(4)} + 168y_n^{(6)}y_n^{(5)}y_n^{(5)}\right]}{8\left[60y_n^{(7)}y_n^{(5)}y_n^{(3)} - 75y_n^{(7)}y_n^{(4)^2} - 70y_n^{(6)^2}y_n^{(3)} + 210y_n^{(6)}y_n^{(5)}y_n^{(4)} - 126y_n^{(5)^3}\right]} \tag{2.16}$$

$$q_2x_{n+1}^2 = \frac{h^2\left[140y_n^{(7)}y_n^{(6)}y_n^{(4)} - 80y_n^{(7)^2}y_n^{(3)} + 70y_n^{(8)}y_n^{(6)}y_n^{(3)} - 105y_n^{(8)}y_n^{(5)}y_n^{(4)} - 196y_n^{(6)^2}y_n^{(5)} + 168y_n^{(5)^2}y_n^{(7)}\right]}{56\left[60y_n^{(7)}y_n^{(5)}y_n^{(3)} - 75y_n^{(7)}y_n^{(4)^2} - 70y_n^{(6)^2}y_n^{(3)} + 210y_n^{(6)}y_n^{(5)}y_n^{(4)} - 126y_n^{(5)^3}\right]} \tag{2.17}$$

$$q_3x_{n+1}^3 = \frac{h^3\left[120y_n^{(7)^2}y_n^{(4)} - 336y_n^{(7)}y_n^{(6)}y_n^{(5)} + 196y_n^{(6)^3} - 105y_n^{(8)}y_n^{(6)}y_n^{(4)} + 126y_n^{(8)}y_n^{(5)^2}\right]}{336\left[60y_n^{(7)}y_n^{(5)}y_n^{(3)} - 75y_n^{(7)}y_n^{(4)^2} - 70y_n^{(6)^2}y_n^{(3)} + 210y_n^{(6)}y_n^{(5)}y_n^{(4)} - 126y_n^{(5)^3}\right]} \tag{2.18}$$

Our integrator from equations (2.1) at m = 5, is then given as,

$$y_{n+1} = \frac{\sum_{r=0}^{5} p_r x_{n+1}^r}{1 + q_1x_{n+1} + q_2x_{n+1}^2 + q_3x_{n+1}^3} \tag{2.19}$$

Values for $p_r x_{n+1}^r$ r = 0(1)4 is as given in equation (2.11).

$$p_5 = c_2q_3 + c_3q_2 + c_4q_1 + c_5 = \frac{h^2y_n^{(2)}}{2!x_{n+1}^2}q_3 + \frac{h^3y_n^{(3)}}{3!x_{n+1}^3}q_2 + \frac{h^4y_n^{(4)}}{4!x_{n+1}^4}q_1 + \frac{h^5y_n^{(5)}}{5!x_{n+1}^5}$$

$$\Rightarrow p_5x_{n+1}^5 = \frac{h^2y_n^{(2)}q_3x_{n+1}^3}{2!} + \frac{h^3y_n^{(3)}q_2x_{n+1}^2}{3!} + \frac{h^4y_n^{(4)}q_1x_{n+1}}{4!} + \frac{h^5y_n^{(5)}}{5!} \tag{2.20}$$

Substituting equation (2.16) – (2.18), (2.11) and (2.20) into equation (2.19), and upon simple rearrangement we obtain our order eight integrator as,

$$y_{n+1} = \frac{\sum_{r=0}^{5}\frac{h^r y_n^{(r)}}{r!} + A\sum_{r=0}^{4}\frac{h^r y_n^{(r)}}{r!} + B\sum_{r=0}^{3}\frac{h^r y_n^{(r)}}{r!} + C\sum_{r=0}^{2}\frac{h^r y_n^{(r)}}{r!}}{1 + A + B + C} \tag{2.21}$$

where,

A = q_1x_{n+1} as in equation (2.16)

B = $q_2x_{n+1}^2$ as in equation (2.17)

C = $q_3x_{n+1}^3$ as in equation (2.18)

3. Convergence and Consistency

One-step methods are normally described symbolically by

$$y_{n+1} = y_n + h\Phi(x_n, y_n; h) \tag{3.1}$$

Where

$\Phi\left(x_n, y_n; h\right)$ is called the increment function, x_n the mesh point and h the mesh size. Convergence as a property any numerical integrator must have is so vital that [7] described it as a minimal property to expect of a numerical method. [7] went further to state that there is no practical use to which we can put methods which are not convergent even though there are some convergent methods that are not suitable for practical computation. As reported in [8] by [10] a one – step method is said to be convergent, if for an arbitrary initial vector y_n at an arbitrary point $x = x_n$ in [a, b], the global truncation error $y(x_n) - y_n$ satisfies the relationship (for some norm ||.||).

$$\underset{h\to 0}{Limit}\left(\underset{n}{\mathrm{Max}}\ \left\|\mathbf{y}(x_n) - \mathbf{y}_n\right\|\right) = 0 \qquad (3.2)$$

and further that every one-step method is convergent if and only if the one step method is consistent. On consistency, a one-step method is said to be consistent if the increment function is consistent with the initial value problem, that is $\Phi\left(x_n, y_n; 0\right) = f\left(x_n, y_n\right)$. Finally, by [10], a One – step method is convergent if and only if it is consistent.

Armed with these facts we state and prove the convergence and consistency of our order seven and eight rational integrators.

3.1 The case m = 4 (Order seven)

THEOREM 1

The order seven rational integrator

$$y_{n+1} = \frac{\sum_{r=0}^{4}\frac{h^r y_n^{(r)}}{r!} + A\sum_{r=0}^{3}\frac{h^r y_n^{(r)}}{r!} + B\sum_{r=0}^{2}\frac{h^r y_n^{(r)}}{r!} + C\sum_{r=0}^{1}\frac{h^r y_n^{(r)}}{r!}}{1 + A + B + C} \qquad (2.12)$$

where the function A, B and C are specified by (2.7), (2.8) and (2.9) respectively is consistent and convergent.

Proof:

From the integrator (2.12) we have

$$y_{n+1} - y_n = \frac{\sum_{r=1}^{4}\frac{h^r y_n^{(r)}}{r!} + A\sum_{r=1}^{3}\frac{h^r y_n^{(r)}}{r!} + B\sum_{r=1}^{2}\frac{h^r y_n^{(r)}}{r!} + Chy_n^{(1)}}{1 + A + B + C} \qquad (3.3)$$

hence,

$$\frac{y_{n+1} - y_n}{h} = \frac{\sum_{r=1}^{4}\frac{h^{r-1} y_n^{(r)}}{r!} + A\sum_{r=1}^{3}\frac{h^{r-1} y_n^{(r)}}{r!} + B\sum_{r=1}^{2}\frac{h^{r-1} y_n^{(r)}}{r!} + Cy_n^{(1)}}{1 + A + B + C} \qquad (3.4)$$

$$= \frac{y_n^{(1)}\left[1 + A + B + C\right] + \sum_{r=2}^{4}\frac{h^r y_n^{(r)}}{r!} + A\sum_{r=2}^{3}\frac{h^r y_n^{(r)}}{r!} + B\frac{h^2 y_n^{(2)}}{2!}}{1 + A + B + C} \qquad (3.5)$$

Note that from (2.7), (2.8) and (2.9).

$$\underset{h\to 0}{\mathrm{Limit}}\ A = 0$$

$$\underset{h\to 0}{\mathrm{Limit}}\ B = 0$$

$$\underset{h\to 0}{\mathrm{Limit}}\ C = 0$$

Hence

$$\underset{h\to 0}{\mathrm{Limit}}\left[\frac{y_{n+1} - y_n}{h}\right] = \frac{y_n^{(1)}\left[1 + 0 + 0 + 0\right] + 0 + 0 + 0}{1 + 0 + 0 + 0}$$

$$= y_n^{(1)}$$

$$\therefore \quad \underset{h\to 0}{\mathrm{Limit}}\left[\frac{y_{n+1} - y_n}{h}\right] = y_n^{(0)} = f(x_n, y_n) \qquad (3.6)$$

as required.

Thus our new rational integrator of order seven is consistent with the initial value problem, hence our integrator is convergent [10].

3.2 The case m = 5 (Order eight)

THEOREM 2.
The order eight rational integrator,

$$y_{n+1} = \frac{\sum_{r=0}^{5}\frac{h^r y_n^{(r)}}{r!} + A\sum_{r=0}^{4}\frac{h^r y_n^{(r)}}{r!} + B\sum_{r=0}^{3}\frac{h^r y_n^{(r)}}{r!} + C\sum_{r=0}^{2}\frac{h^r y_n^{(r)}}{r!}}{1 + A + B + C} \quad (2.21)$$

where the functions A, B and C are specified by (2.16), (2.17) and (2.18) respectively is consistent and convergent.
Proof:
From the integrator (2.21) we have

$$y_{n+1} - y_n = \frac{\sum_{r=1}^{5}\frac{h^r y_n^{(r)}}{r!} + A\sum_{r=1}^{4}\frac{h^r y_n^{(r)}}{r!} + B\sum_{r=1}^{3}\frac{h^r y_n^{(r)}}{r!} + C\sum_{r=1}^{2}\frac{h^r y_n^{(r)}}{r!}}{1 + A + B + C} \quad (3.7)$$

hence,

$$\frac{y_{n+1} - y_n}{h} = \frac{\sum_{r=1}^{5}\frac{h^{r-1} y_n^{(r)}}{r!} + A\sum_{r=1}^{4}\frac{h^{r-1} y_n^{(r)}}{r!} + B\sum_{r=1}^{3}\frac{h^{r-1} y_n^{(r)}}{r!} + C\sum_{r=1}^{2}\frac{h^{r-1} y_n^{(r)}}{r!}}{1 + A + B + C} \quad (3.8)$$

$$= \frac{y_n^{(1)}[1 + A + B + C] + \sum_{r=2}^{5}\frac{h^r y_n^{(r)}}{r!} + A\sum_{r=2}^{4}\frac{h^r y_n^{(r)}}{r!} + B\sum_{r=2}^{3}\frac{h^r y_n^{(r)}}{r!} + C\frac{h^2 y_n^{(2)}}{2!}}{1 + A + B + C} \quad (3.9)$$

Note that by (2.16), (2.17) and (2.18).

$$\lim_{h \to 0} A = 0$$

$$\lim_{h \to 0} B = 0$$

$$\lim_{h \to 0} C = 0$$

Hence $\displaystyle \lim_{h \to 0}\left[\frac{y_{n+1} - y_n}{h}\right] = \frac{y_n^{(1)}[1+0+0+0] + 0+0+0+0}{1+0+0+0}$

$$= y_n^{(1)}$$

$$\therefore \quad \lim_{h \to 0}\left[\frac{y_{n+1} - y_n}{h}\right] = y_n^{(0)} = f(x_n, y_n) \quad (3.10)$$

as required.
That is, the order eight numerical integrator is consistent. Hence, our integrator is convergent [10].

4. Numerical Examples

We illustrate the performance of our new integrator over the Maximum order second derivative hybrid multi – step methods of [5] and the Adaptive implicit Runge – Kutta methods of [4] in the following examples.

Example 1: [4]

$$y^{(1)} = -100y, \quad y(x_0) = 1, \quad x \in [0,1]$$

The exact solution to is $y(x) = \ell^{-100x}$ and the time constant is 0.01.
A variable step size was used starting with h = 0.10

Table 1: Errors in Numerical Integration of Example 1

H	Value of x	Analytic solution	Maximum order 2nd derivative hybrid multi – step [5]	Order Seven integrator	Order Eight integrator
1.00000D-01	1.00000D-01	4.53999D-05	2.14454D-02	-1.11022D-16	0.00000D+00
5.00000D-02	1.50000D-01	3.05902D-07	2.00536D-02	1.11022D-16	-1.11022D-16
2.50000D-02	1.75000D-01	2.51100D-08	1.97436D-02	5.96046D-08	0.00000D+00
1.25000D-02	1.87500D-01	7.19412D-09	1.68208D-02	1.11022D-16	-5.96046D-08
6.25000D-03	1.93750D-01	3.85074D-09	1.24074D-02	-5.96046D-08	5.96046D-08
3.12500D-03	1.96875D-01	2.81726D-09	7.84083D-03	0.00000D+00	5.96046D-08
1.56250D-03	1.98438D-01	2.40973D-09	4.43528D-03	5.96046D-08	0.00000D+00
7.81250D-04	1.99219D-01	2.22864D-09	2.36270D-03	-5.96046D-08	-1.11022D-16
3.90625D-04	1.99609D-01	2.14326D-09	1.21590D-03	1.11022D-16	0.00000D+00
1.95313D-04	1.99805D-01	2.10180D-09	6.19892D-04	1.11022D-16	0.00000D+00
9.76563D-05	1.99902D-01	2.08138D-09	**Not Stated**	2.22045D-16	0.00000D+00

Example 2: [3]

$$\begin{bmatrix} x^1 \\ y^1 \end{bmatrix} = \begin{bmatrix} 5x & -2y \\ 3x & \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Exact Solutions is given as

:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2e^{2t} & -e^{3t} \\ 3e^{2t} & -e^{3t} \end{bmatrix}$$

Table 2a: Error in Numerical Integration First Component (Uniform Mesh size h = 0.03)

Values of TI	Adaptive Implicit Runge – Kutta [3]	Our new Integrators	
		Order Seven integrator	Order Eight integrator
3.00000000D-02	0.3238292541D-04	-6.15099061D-09	-6.15099061D-09
6.00000000D-02	0.2995808673D-04	-1.11073899D-07	-1.11073899D-07
9.00000000D-02	0.2787104097D-04	-1.15763126D-07	-1.15763126D-07
1.20000000D-01	0.2605580892D-04	-1.05589737D-09	-1.05589737D-09
1.50000000D-01	0.2446254221D-04	-1.62302885D-07	-1.62302885D-07
1.80000000D-01	0.2305287807D-04	3.51731264D-08	3.51731264D-08
2.10000000D-01	0.2179681180D-04	-4.33673466D-08	-4.33673466D-08
2.40000000D-01	0.2067053833D-04	-6.53996413D-08	-6.53996413D-08
2.70000000D-01	0.196542975D-04	-1.62053748D-08	-1.62053748D-08
3.00000000D-01	0.1873444048D-04	4.25262905D-08	4.25262905D-08
3.30000000D-01	0.1789630619D-04	-1.37350189D-08	-1.37350189D-08
3.60000000D-01	0.1712994847D-04	-8.42760395D-08	-8.42760395D-08
3.90000000D-01	0.1712994847D-04	-9.86133362D-08	-9.86133362D-08
4.20000000D-01	0.1642652616D-04	-9.38846805D-08	-9.38846805D-08
4.50000000D-01	0.1577859276D-04	4.70074202D-10	4.70074202D-10
4.80000000D-01	0.1517983194D-04	-9.95356952D-08	-9.95356952D-08
5.10000000D-01	0.1462485089D-04	-5.62501530D-08	-5.62501530D-08
5.40000000D-01	0.1410901759D-04	2.95151725D-09	2.95151725D-09
5.69999900D-01	0.13628331113D-04	-2.83948820D-08	-2.83948820D-08
5.99999900D-01	0.1275894717D-04	1.67908905D-08	1.67908905D-08
6.29999900D-01	0.1236456325D-04	2.52512509D-08	2.52512509D-08
6.59999800D-01	0.1199382870D-04	-1.15564580D-08	-1.15564580D-08
6.89999800D-01	0.1164467828D-04	1.53459467D-11	1.53459467D-11
7.19999800D-01	0.1131528043D-04	4.31058034D-09	4.31058034D-09
7.49999800D-01	0.1070939677D-04	-1.91763583D-08	-1.91763583D-08
7.79999700D-01	0.1016509888D-04	-1.72803656D-08	-1.72803656D-08
8.09999700D-01	0.9913183923D-05	-4.51632189D-08	-4.51632189D-08
8.39999700D-01	0.9673451145D-05	1.03405924D-08	1.03405924D-08
8.69999600D-01	0.9445040210D-05	-1.83549945D-07	-1.83549945D-07
8.99999600D-01	0.1043015177D-04	-4.18202735D-08	-4.18202735D-08

Table 2b: Error in Numerical Integration Second Component (Uniform Mesh size h = 0.03)

Values of TI	Adaptive Implicit Runge – Kutta [3]	Our new Integrator	
		Order Seven integrator	Order Eight integrator
3.00000000D-02	0.000000000D+00	2.97718588D-07	2.97718588D-07
6.00000000D-02	0.000000000D+00	2.11451257D-08	2.11451257D-08
9.00000000D-02	0.000000000D+00	-9.78790471D-08	-9.78790471D-08
1.20000000D-01	0.000000000D+00	-2.58062087D-08	-2.58062087D-08
1.50000000D-01	0.000000000D+00	1.22857956D-07	1.22857956D-07
1.80000000D-01	0.000000000D+00	1.05937690D-07	1.05937690D-07
2.10000000D-01	0.000000000D+00	-5.75347849D-08	-5.75347849D-08
2.40000000D-01	0.000000000D+00	-1.06070213D-07	-1.06070213D-07
2.70000000D-01	0.000000000D+00	2.09359915D-07	2.09359915D-07
3.00000000D-01	0.1096978240D-05	-1.54657053D-07	-1.54657053D-07
3.30000000D-01	0.1109145273D-05	-1.97948710D-07	-1.97948710D-07
3.60000000D-01	0.1109145273D-05	1.89917504D-08	1.89917504D-08
3.90000000D-01	0.1121585230D-05	-1.53889994D-08	-1.53889994D-08
4.20000000D-01	0.1134307399D-05	-1.47247327D-07	-1.47247327D-07
4.50000000D-01	0.1147321494D-05	4.04549403D-08	4.04549403D-08
4.80000000D-01	0.1160637680D-05	3.69060326D-07	3.69060326D-07
5.10000000D-01	0.1174266597D-05	2.75495756D-07	2.75495756D-07
5.40000000D-01	0.1188219395D-05	-6.38247699D-08	-6.38247699D-08
5.69999900D-01	0.1217143933D-05	-1.37702192D-08	-1.37702192D-08
5.99999900D-01	0.1232140783D-05	-1.16802906D-07	-1.16802906D-07
6.29999900D-01	0.1247511805D-05	1.91263741D-07	1.91263741D-07
6.59999800D-01	0.1263271177D-05	-2.49093537D-08	-2.49093537D-08
6.89999800D-01	0.1279433808D-05	2.02886157D-08	2.02886157D-08
7.19999800D-01	0.1296015373D-05	-7.65913257D-08	-7.65913257D-08
7.49999800D-01	0.1313032376D-05	4.66411394D-08	-4.04818419D-08
7.79999700D-01	0.1330502196D-05	7.59433001D-08	2.21412357D-07
8.09999700D-01	0.1348443152D-05	1.09458130D-07	2.99723895D-07
8.39999700D-01	0.1366874563D-05	1.47845335D-07	2.23335017D-07
8.69999600D-01	0.1385816818D-05	1.91262203D-07	-1.04686314D-07
8.99999600D-01	0.1405291455D-05	2.40932396D-07	-1.40836560D-07

5. Remarks and Conclusion

In this paper, we have derived two rational integrators of Order Seven and Order Eight for solving initial value problems in ordinary differential equations that is consistent and stable. Examining closely the Tables 1, 2a and 2b we observe that the new integrators gives a better approximation when compared with the Maximum order second derivative hybrid multi – step methods of [5] and the Adaptive implicit and Classical Runge – Kutta methods of [4] since the resulting solutions of the new integrators are closer to the analytic solutions.

The use of numerical integrators in solving first order initial value problem have been demonstrated using the order seven and order eight rational integrators and have proved more attractive. Tables 1 shows encouraging results as our order four and order five rational integrators convergence rate is quick and increases as each mesh point increases than the Maximum order second derivative hybrid multi – step methods. Tables 2a and 2b show that our integrators converge more quickly to the analytic solution at each mesh point than the Adaptive implicit and Classical Runge – Kutta methods of Ademiluyi et al (2007). This makes our rational numerical integrators more efficient than the ones compared with.

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