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On The Analysis of The Stability Functions Of Order Three And Four Numerical Integrator for Initial Value Problems

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Abstract

The Stability functions of numerical integrators for the solution of initial value problems (ivps) in Ordinary Differential Equations (ODEs) are of paramount interest as it helps to determine the suitability of the method for obtaining the solutions required.

In this paper we analyze the stability functions of two new integrators of order three and four we derived. Our results shows that they are both A – stable and L – stable and hence can cope effectively with stiff problems.

Keywords: Rational Integrator, Stability, Cartesian Coordinates, Initial Value Problems.

1. Introduction:

Many of the problems in real life from physical situations, chemical kinetics, engineering work, population models, electrical networks, biological simulations, mechanical oscillations, process control often results in Initial Value Problems in Ordinary Differential equations that are either Stiff, Singular or Oscillatory, [1 – 3, 5, 6, 8].

To solve stiff systems, we need numerical integrators that possess special stability properties such as A – stability and L – stability.

In this paper, we desire to analyze the order three and four rational integrators of [5, 6] in order to determine their stability properties.

The next section, section 2 of this paper will be on derivation of the integrators. In section 3 we will derive the stability functions of the integrators while section 4 will be devoted to the Analysis of the Stability functions of the new integrators. The remarks and conclusions will be on section 5.

2. The Order three and four Integrators

In [5, 6] the order three and four rational integrators were derived using the operator:
$U : R \rightarrow C^{m+2}(x)$ be defined by

$$U(x)\ \left[1+q_1x+q_2x^2+q_3x^3\right] \equiv p_m(x) \tag{2.1}$$

where,

U(x) has at least 1st, 2nd, 3rd . . . (m + 2)th derivatives

$$U(x_{n+i}) = \begin{cases} y(x_{n+i}) & \text{for } i = 0 \\ y_{n+i} & \text{for } i = 0, 1, 2 \end{cases} \tag{2.2}$$

With the indicator m = 0 and comparing equations (2.1) and (2.2) we obtain the order three integrator as

$$y_{n+1} = \frac{y_n}{1 + A + B + C} \tag{2.3}$$

where

$$A = q_1 x_{n+1} = \frac{-hy_n^{(1)}}{y_n} \tag{2.4}$$

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$$\text{B} = q_2 x_{n+1}^2 = \frac{h^2\left[2y_n^{(1)^2} - y_n y_n^{(2)}\right]}{2y_n^2} \tag{2.5}$$

$$\text{C} = q_3 x_{n+1}^3 = \frac{h^3\left[6y_n^{(2)} y_n^{(1)} y_n - 6y_n^{(1)^3} - y_n^{(3)} y_n^2\right]}{6y_n^3} \tag{2.6}$$

Similarly with our indicator as m = 1 and comparing equations (2.1) and (2.2) we obtain our order four numerical integrator as

$$y_{n+1} = \frac{\sum_{r=0}^{1} \frac{h^r y_n^{(r)}}{r!} + Ay_n}{1 + A + B + C} \tag{2.7}$$

where

$$\text{A} = q_1 x_{n+1} = \frac{h\left[y_n^2 y_n^{(4)} - 6y_n y_n^{(2)^2} - 4y_n y_n^{(1)} y_n^{(3)} + 12y_n^{(2)} y_n^{(1)^2}\right]}{4\left[-y_n^2 y_n^{(3)} + 6y_n y_n^{(1)} y_n^{(2)} - 6y_n^{(1)^3}\right]} \tag{2.8}$$

$$\text{B} = q_2 x_{n+1}^2 = \frac{h^2\left[2y_n y_n^{(2)} y_n^{(3)} - y_n y_n^{(1)} y_n^{(4)} - 6y_n^{(1)} y_n^{(2)^2} + 4y_n^{(1)^2} y_n^{(3)}\right]}{4\left[-y_n^2 y_n^{(3)} + 6y_n y_n^{(1)} y_n^{(2)} - 6y_n^{(1)^3}\right]} \tag{2.9}$$

$$\text{C} = q_3 x_{n+1}^3 = \frac{h^3\left[4y_n y_n^{(3)^2} - 24y_n^{(1)} y_n^{(3)} y_n^{(3)} + 18y_n^{(2)^3} - 3y_n y_n^{(2)} y_n^{(4)} + 6y_n^{(1)^2} y_n^{(4)}\right]}{24\left[-y_n^2 y_n^{(3)} + 6y_n y_n^{(1)} y_n^{(2)} - 6y_n^{(1)^3}\right]} \tag{2.10}$$

3. **Derivation of the Stability Functions**

To effectively solve ivps that are stiff and singular we need integrators that possess special qualities such as A – stability. For an integrator to possess L – stability property it guarantees the effectiveness of the integrator in handling ivps. For easy understanding we state the following definitions

Definition: [4]: A numerical method is said to be A-STABLE if its Region of Absolute Stability (RAS) contains the whole of the left-hand, half of the complex plane i.e. $\text{Re}(\bar{\text{h}}) < 0$.

Definition: [7, 8]: A numerical integrator is said to be Absolutely Stable if the absolute value of the stability function $\varsigma(\bar{h})$ is less than unity. That is,

$$\left|\varsigma(\bar{\text{h}})\right| = \left|\ \varsigma(\text{u} + \text{iv})\ \right| < 1, \quad \text{i} = \sqrt{-1} \tag{3.1}$$

Definition: Region of Absolute Stability (RAS) [9]: A region D of the complex plane is said to be a Region of Absolute Stability (RAS) of a given method, if the method is absolutely stable for $\bar{\text{h}} \in \text{D}$.

Definition: [7]: A given one-step method is said to be L-stable if it is A-stable and in addition,

$$\underset{\text{Re}(\bar{h})\to-\infty}{Limit} \left|s\left(\bar{h}\right)\right| = 0 \tag{3.2}$$

In this section, we shall be interested in investigating the stability properties of our new methods, we begin by applying the integrator to the usual test equation $y^1 = \lambda y$.

3.1 Stability function of the Order Three Numerical Integrator

Our integrator is given as

$$y_{n+1} = \frac{y_n}{1 + A + B + C}$$

where A, B, C are as specified by (2.4), (2.5), and (2.6) respectively.
On applying the integrator on the test equation $y^1 = \lambda h$ we simply have,

$$A(\bar{h}) = -\bar{h}$$

$$B(\bar{h}) = \frac{\bar{h}^2}{2}$$

$$C(\bar{h}) = -\frac{\bar{h}^3}{6}$$

Substituting and simplifying we have

$$\varsigma(\bar{h}) = \frac{y_{n+1}}{y_n} = \frac{6}{6 - 6\bar{h} + 3\bar{h}^2 - \bar{h}^3}$$

$$\therefore \quad \varsigma(\bar{h}) = \frac{6}{6 - 6\bar{h} + 3\bar{h}^2 - \bar{h}^3} \qquad (3.3)$$

3.2 Stability function of the Order Four Numerical Integrator

Our integrator is given as,

$$y_{n+1} = \frac{\sum_{r=0}^{1} \frac{h^r y_n^{(r)}}{r!} + Ay_n}{1 + A + B + C}$$

where A, B, C are as specified by (2.8), (2.9), and (2.10) respectively.
Also on applying them to the test equation we simply have,

$$A(\bar{h}) = \frac{-3\bar{h}}{4}$$

$$B(\bar{h}) = \frac{\bar{h}^2}{4}$$

$$C(\bar{h}) = \frac{-\bar{h}}{24}$$

Substituting into the integrator and simplifying we have our stability function as

$$\varsigma(\bar{h}) = \frac{y_{n+1}}{y_n} = \frac{24 + 6\bar{h}}{24 - 18\bar{h} + 6\bar{h}^2 - \bar{h}^3} \qquad (3.4)$$

4. Analysis of the Stability Function of the Numerical Integrator

We now wish to analysis the stability function of our two numerical integrator using Cartesian coordinate forms.

4.1 The Case of Order three

Our Stability function is given as

$$\zeta(\bar{h}) = \frac{6}{6 - 6\bar{h} + 3\bar{h}^2 - \bar{h}^3}$$

By setting $\bar{h} = u + iv,\ i^2 = -1$, we get

$$\left|\ \zeta(\bar{h})\right| \le 1 \Leftrightarrow$$
$$|\,6| \le \left|6 - 6(u+iv) + 3(u+iv)^2 - (u+iv)^3\right|$$
$$\Leftrightarrow \left|6 - 6(u+iv) + 3(u+iv)^2 - (u+iv)^3\right| \ge |\,6|$$

Now then,

$$6 - 6(u+iv) + 3(u+iv)^2 - (u+iv)^3 = A(u,v) + i\,B(u,v)$$
$$where$$
$$A(u,v) = 6 - 6u + 3u^2 - 3v^2 - u^3 + 3uv^2$$
$$B(u.v) = -6v + 6uv - 3u^2v + v^3$$
$$A(u,v)^2 = 36 - 72u + 72u^2 - 36v^2 - 48u^3 + 72uv^2 + 21u^4 - 54u^2v^2 - 6u^5$$
$$+ 24u^3v^2 + 9v^4 - 18uv^4 + u^6 - 6u^4v^2 + 9u^2v^4$$
$$B(u,v)^2 = 36v^2 - 72uv^2 + 72u^2v^2 - 12v^4 - 36u^3v^2 + 12uv^4 + 9u^4v^2 - 6u^2v^4 - v^6$$

Hence our inequality above becomes,

$$\left|\ \zeta(\bar{h})\right| \le 1 \Leftrightarrow \left|\ A(u,v) + i\,B(u,v)\right| \ge |6|$$
$$ie\ \Leftrightarrow A(u,v)^2 + B(u,v)^2 \ge 36$$

which holds after expansion and rearrangement $\Leftrightarrow$

$$u^6 + 3u^4v^2 + 3u^2v^4 + v^6 - 6u^5 - 12u^3v^2 - 6uv^4 + 21u^4 + 18u^2v^2 - 3v^4 - 48u^3 + 72u^2 - 72u + 36 \ge 36$$

which in turn holds $\Leftrightarrow$

$$u^6 + 3u^4v^2 + 3u^2v^4 + v^6 - 6u^5 - 12u^3v^2 - 6uv^4 + 21u^4 + 18u^2v^2 - 3v^4 - 48u^3 + 72u^2 - 72u \ge 0$$

After rearranging and collecting terms the relation holds $\Leftrightarrow$

$$(u^2+v^2)^3 - 6u(u^2+v^2)^2 + 9(u^2+v^2) + 12(u^2+v^2)(u^2-v^2) - 24u\ (2u^2 - 3u + 3) \ge 0$$

Our preference is for: $\left(u^2+v^2\right)^3,\ 9\left(u^2+v^2\right)^2$

$$Observe\ that : \forall\ u,\ v\ we\ have$$

$$(u^2+v^2)^3 \ge 0,\ 9(u^2+v^2)^2 \ge 0$$

Hence Region of Instability (RIS) from this set is empty by this contribution as our RAS covers the whole of the left and right hand of the complex plane.

For the preference: $-6u\left(u^2+v^2\right)^2\ and\ 24u\ \left(2u^2 - 3u + 3\right)$

$$\forall\ u \le 0, and\ \ \forall\ v$$
$$-6u\left(u^2+v^2\right)^2 \ge 0$$
$$-24u(2u^2 - 3u + 3) \ge 0$$

Hence our RIS from this set is the right half of the complex plane, by this our Region of Absolute Stability for this group covers the entire left half of the complex plane

Also for the preference: $12\left(u^2+v^2\right)\left(u^2-v^2\right)$

$(i)\ \forall\ u,\ \ v\ we\ have:$

$$\left(u^2+v^2\right)\geq 0,$$

$$(ii)\ \ \left(u^2-v^2\right)\geq 0$$

$$\Leftrightarrow u\geq v\geq 0\ or\ u\leq v\leq 0,$$

$$Hence\ \ 12\left(u^2+v^2\right)\left(u^2-v^2\right)\geq 0\ \ \forall\ \ u\leq 0$$

Therefore our inequality

$$\left(u^2+v^2\right)^3-6u\left(u^2+v^2\right)^2+9\left(u^2+v^2\right)+12\left(u^2+v^2\right)\left(u^2-v^2\right)-24u\ \left(2u^2-3u+3\right)\geq 0\ u\leq 0$$

Hence our Integrator is A – Stable and by direct substitution of $\zeta\left(\bar{h}\right)$,

$$\underset{\mathrm{Re}\left(\bar{h}\right)\to-\infty}{Limit}\ \left|\zeta\left(\bar{h}\right)\right|=0$$

Which shows that our integrator is L- Stable.

4.2 The Case of Order four

Our Stability function is given as

$$\zeta\left(\bar{h}\right)=\frac{24+6\bar{h}}{24-18\bar{h}+6\bar{h}^2-\bar{h}^3}$$

By setting $\bar{h}=u+iv,\ i^2=-1$, we get

$$\left|\ \zeta\left(\bar{h}\right)\right|\leq 1\Leftrightarrow$$

$$\left|\ 24+6\left(u+iv\right)\right|\leq\left|\ 24-18\left(u+iv\right)+6\left(u+iv\right)^2-\left(u+iv\right)^3\right|$$

Now then,

$$24+6\left(\mathrm{u}+\mathrm{iv}\right)=\mathrm{A}\left(\mathrm{u},\mathrm{v}\right)+\mathrm{i}\,\mathrm{B}\left(\mathrm{u},\mathrm{v}\right)$$

where

$$\mathrm{A}\left(\mathrm{u},\mathrm{v}\right)=24+6\mathrm{u}$$

$$\mathrm{B}\left(\mathrm{u.v}\right)=6\mathrm{v}$$

and

$$24-18\left(\mathrm{u}+\mathrm{iv}\right)+6\left(\mathrm{u}+\mathrm{iv}\right)^2-\left(\mathrm{u}+\mathrm{iv}\right)^3=\mathrm{C}\left(\mathrm{u},\mathrm{v}\right)+\mathrm{i}\,\mathrm{D}\left(\mathrm{u},\mathrm{v}\right)$$

where

$$\mathrm{C}\left(\mathrm{u},\mathrm{v}\right)=24-18\mathrm{u}+6\mathrm{u}^2-6\mathrm{v}^2-\mathrm{u}^3+3\mathrm{uv}^2$$

$$\mathrm{D}\left(\mathrm{u},\mathrm{v}\right)=-18\mathrm{v}+12\mathrm{uv}-3\mathrm{u}^2\mathrm{v}+\mathrm{v}^3$$

Hence our inequality above becomes,

$$\left|\ \zeta\left(\bar{h}\right)\right|\leq 1\Leftrightarrow\left|\ A\left(u,v\right)+i\,B\left(u,v\right)\right|\leq\left|\ C\left(u,v\right)+i\ D\left(u,v\right)\right|$$

$$ie\ \Leftrightarrow A\left(u,v\right)^2+B\left(u,v\right)^2\leq C\left(u,v\right)^2+D\left(u,v\right)^2$$

which holds after expansion $\Leftrightarrow$

$$576+288u+36u^2+36v^2\leq 576-864u+612u^2+36v^2-264u^3-72uv^2+72u^4+72u^2v^2-12u^5$$
$$-24u^3v^2-12u+u^6+3u^4v^2+3u^2v^4+v^6$$

which in turn holds $\Leftrightarrow$

$$\left(u^2+v^2\right)^3-12u\left(u^2+v^2\right)^2+72u^2\left(u^2+v^2\right)-72u\left(u^2+v^2\right)-48u\left(4u^2-12u+24\right)\geq 0$$

$$Observe\ that:$$

$$\left(u^2+v^2\right)^3\geq 0,\ 72u^2\left(u^2+v^2\right)\geq 0,\ \forall\ u,v$$

$$-12u\left(u^2+v^2\right)\geq 0,\ -72u\left(u^2+v^2\right)\geq 0\ and -48u\left(4u^2-12u+24\right)\geq 0\ \ \forall v\ and\ u<0$$

$$\therefore F(u,v)\geq 0\ \ where$$

$$F(u,v)=\left(u^2+v^2\right)^3-12u\left(u^2+v^2\right)^2+72u^2\left(u^2+v^2\right)-72u\left(u^2+v^2\right)-48u\left(4u^2-12u+24\right)>0\ \forall\ u<0,\forall v$$

$$Observe\ that:\forall\ u,\ v\ we\ have$$

$$\left(u^2+v^2\right)^3\geq\ 0\ \text{and}\ \ 72u^2\left(u^2+v^2\right)\ ^3\geq 0$$

Observe also that $\forall\ u\leq 0\ and\ \ \forall v$ we have that

$$-12u\left(u^2+v^2\right)^2\ \geq\ 0,$$

$$-72u\ \left(u^2+v^2\right)^2\ \geq\ 0$$

$$-48u\left(4u^2-12u+24\right)\geq\ 0$$

And so the method is A – Stable.
Consequently, the inequality

$$\left(u^2+v^2\right)^3-12u\left(u^2+v^2\right)^2+72u^2\left(u^2+v^2\right)-72u\left(u^2+v^2\right)-48u\left(4u^2-12u+24\right)\geq 0\quad\forall\quad u<0$$

$$ie\ \left|\ \varsigma\left(u,v\right)\right|\leq 1\ \ \forall\ u,v\in\left\{u+iv:u<0\right\}$$

Hence the integrator is A – stable and so the Region of Absolute Stability of the integrator is the entire left – half of the complex plane.

The Integrator is L – Stable as by direct substitution of $\zeta\left(\bar{h}\right)$

$$\underset{\mathrm{Re}\left(\bar{h}\right)\to-\infty}{Limit}\ \left|\zeta\left(\bar{h}\right)\right|=0$$

5. Remarks and Conclusion

In this paper we have presented the analysis of two new order three and four numerical integrators that are both A – stable and L – stable. We have also been able to create interest in the study of the stability nature of any method we hope to adopt in solving initial value problems in ordinary differential equations.

The stability properties possessed by our Order three and four numerical integrator makes them more efficient and effective is solving stiff problems in Ordinary differential equations.

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