Journal of the Nigerian Association of Mathematical Physics
Volume **18 (May, 2011), pp 261 – 266**

Three-Step Hybrid Linear Multistep Method for Solution of First Order Initial Value Problems in Ordinary Differential Equations.

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Abstract

In this article, a three-step hybrid linear multistep method for solution of first order initial value problems (IVPs) in ordinary differential differential equations (ODEs) is proposed.

Essentially, the method is based on collocation of the differential system and the interpolation of the approximate solution at the grid and off-grid points. Evaluation of the proposed method at $x = x_{n+k}$ and $x = x_{n+5/2}$ yields a class of two discrete schemes of order 7 which are not self-starting. Furthermore, a self-starting method is adopted. The proposed method is consistent, zero stable and convergent. Finally, the accuracy of the method is tested on some standard IVPs.

Keywords: , linear multistep, initial value problems, collocation, interpolation, consistent, zero stable and convergent.

1.0 Introduction:

Many problems encountered in the various branches of science, engineering and management give rise to differential equations of the form:

$$y' = f(x, y),\ y(x_0) = y_0,\ a \le x \le b \qquad (1.1)$$

where f is assumed to be Lipschiz constants[8].

The solution of (1.1) has been discussed by various researchers (see [1, 2, 3, 4, 5, 7, 8]). However, experience has shown [3, 4] that the traditional multistep methods including the hybrid ones can be made continuous through the idea of multistep collocation. These earlier works have focused on the construction of continuous multistep methods by employing the multistep collocation. The continuous multistep methods produce piecewise polynomial solutions over k –steps [x_n, x_{n+k}] for the first order systems of ordinary differential equation (ODEs). Sirisena [8] developed a continuous new Butcher type two-step block hybrid multistep method for problem (1.1). The results obtained showed a class of discrete schemes of order 5 and error constants ranging from $C_6 = 1.45 \times 10^{-5}$ to $C_6 = 1.790 \times 10^{-4}$. In this paper, we propose a continuous Butcher type three-step block hybrid method employing multistep collocation approach, which yields a class of 2 discrete schemes of order 7 with error constants.

$$C_8 = -\frac{27}{777420} \text{ and } C_8 = \frac{155525}{1273724928}$$ for solving problem (1.1).

2.0 DEVELOPMENT OF THE METHODS

In this section we discussed the development of continuous scheme and its discrete schemes using [5] where a K-step multistep collocation method with m collocation points was obtained as follows:

$$\bar{y}(x) = \sum_{j=o}^{t-1} \alpha_j(x) y(x_{n+j}) + h_j \sum_{j=o}^{m-1} \beta_j(x) f(\bar{x}_j, \bar{y}(\bar{x}_j)) \qquad (2.1)$$

where,

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$$\alpha_j(x) = \sum_{i=o}^{t+m-1} \alpha_{j,i+1} x^i \qquad (2.2)$$

$$h\beta_j(x) = \sum_{i=o}^{t+m-1} h\beta_{j,i+1}\ x^i \qquad (2.3)$$

are the continuous coefficients of the method and x_{n+j}, j = 0, 1...t-1 in (2.1) are t (0<t≤ k) arbitrary chosen interpolation points from (x_n, … x_{n+k}) and $\overline{x}_j$, j = o, 1…, m-2 are the m collocation points belonging to {x_n…, x_{n+k}}.

To determine α_j (x) and β_j (x), we use a matrix equation of the form

$$DC = I \qquad (2.4)$$

Where,
I is an identity matrix,
D and C are matrices defined as in [5].

$$D = \begin{bmatrix} 1 & x_n & x_n^2 \dots & x_n^{t+m-2} \\ 1 & x_{n+1} & x_{n+1}^2 \dots & x_{n+1}^{t+m-2} \\ : & : & : & : \\ 1 & x_{n+t-1} & x_{n+T-1}^2 \dots & x_{n+t-1}^{t+m-2} \\ 0 & 1 & 2\overline{x}_o \dots & (t+m-2)\overline{x}^{t+m-3} \\ : & : & : & : \\ . & . & . & . \\ .0 & 1 & 2\overline{x}_{m-1} \dots & (t+m)\overline{x}_{m-1}^{t+m-3} \end{bmatrix} \qquad (2.5)$$

and

$$C = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1}\dots & \alpha_{t-1,1} & h\beta_{o,1}\dots & h\beta_{n-1,1} \\ \alpha_{o,2} & \alpha_{1,2}\dots & \alpha_{t-1,2} & h\beta_{0,2}\dots & h\beta_{m-1,2} \\ . & . & . & . & . \\ : & : & : & : & : \\ \alpha_{o,t+m} & \alpha_{1,t+m}\dots & \alpha_{t-1,t+m} & h\beta_{o,t+m}\dots & h\beta_{m+1,t+m} \end{bmatrix} \qquad (2.6)$$

The columns of the matrix C = D^{-1} consists of the continuous coefficients, i.e.

$\alpha_j(x)$; j =o, 1…k-1 and β_j (x); j = 0, 1…. k-1.

In this paper k = t = 3, m = 6, $\overline{x}_o = x_n$, $\overline{x}_n$, $\overline{x}_1 = x_{p+1}$, $\overline{x}_2 = x_{n+2}$. Then equation (2.1) becomes

$$\overline{y}(x) = \alpha_o(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y + h\left[\beta_o(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3} + \beta_{5/2}(x)\ f_{n+5/2}\right] \qquad (2.7)$$

Thus, the matrix D in (2.5) becomes

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 \\ 0 & 1 & 2x_{n+5/2} & 3x_{n+5/2}^2 & 4x_{n+5/2}^3 & 5x_{n+5/2}^4 & 6x_{n+5/2}^5 & 7x_{n+5/2}^6 \end{bmatrix} \qquad (2.8)$$

We obtained C =D^{-1} in (2.8) to determine $\alpha_i(x)$; i = 0(1)2 and $h\beta_i$ (x); i = 0, 1, 2, 3, 5/2 in (2.7) as follows:

$$\alpha_o(x) = \frac{1}{2468h^7}\left[-20115h^5(x-x_n)^2 + 39189h^4(x-x_n)^3 - 32910h^3(x-x_n)^4 + 14169h^2 (x-x_n)^5 - 3065h(x-x_n)^6 + 264(x-x_n)^7 + 2468h^2\right] \quad (2.9)$$

$$\alpha_1(x) = \frac{1}{617h^7}\left[-180h^5(x-x_n)^2 + 6148h^4(x-x_n)^3 - 10005h^3(x-x_n)^4 + 6156h^2(x-x_n)^5 - 1670h(x-x_n)^6 + 168(x-x_n)^7\right] \quad (2.10)$$

$$\alpha_2(x) = \frac{1}{2468h^7}\left[20835h^5(x-x_n)^2 - 6378h^4(x-x_n)^3 + 72930h^3(x-x_n)^4 - 38793h^2(x-x_n)^5 + 9745h(x-x_n)^6 - 936(x-x_n)^7\right] \quad (2.11)$$

$$h\beta_o(x) = \frac{1}{37020h^6}\left[37020h^6(x-x_n) - 136364h^5(x-x_n)^2 + 200531h^4(x-x_n)^3 - 150680h^3(x-x_n)^4 + 61199h^2(x-x_n)^5 - 12782h(x-x_n)^6 + 1076(x-x_n)^7\right] \quad (2.12)$$

$$h\beta_1(x) = \frac{1}{7404h^6}\left[-68220h^5(x-x_n)^2 + 182932h^4(x-x_n)^3 + 1861147h^3(x-x_n)^4 - 90946h^2(x-x_n)^5 - 21483h(x-x_n)^6 - 1972(x-x_n)^7\right] \quad (2.13)$$

$$h\beta_2(x) = \frac{1}{2468h^6}\left[-11250h^5(x-x_n)^2 + 35645h^4(x-x_n)^3 - 42556h^3(x-x_n)^4 + 23805h^2(x-x_n)^5 - 6272h(x-x_n)^6 + 628(x-x_n)^7\right] \quad (2.14)$$

$$h\beta_3(x) = \frac{1}{7404h^6}\left[-980h^5(x-x_n)^2 + 3308h^4(x-x_n)^3 - 4289h^3(x-x_n)^4 + 2666^2(x-x_n)^5 - 797h(x-x_n)^6 + 92(x-x_n)^7\right] \quad (2.15)$$

$$h\beta_{5/2}(x) = \frac{1}{9255h^6}\left[9216h^5(x-x_n)^2 - 30464h^4(x-x_n)^3 + 38400h^3(x-x_n)^4 - 22976h^2(x-x_n)^5 + 6528h(x-x_n)^6 - 704(x-x_n)^7\right] \quad (2.16)$$

Putting equations (2.9) – (2.16) into equation (2.7), we obtained a continuous scheme.

$$\bar{y}(x) = \frac{y_n}{2468h^7}\left[-20115h^5(x-x_n)^2 + 39189h^4(x-x_n)^3 - 32910h^3(x-x_n)^4 + 14169h^2(x-x_n)^5 - 3065h(x-x_n)^6 + 264(x-x_n)^7 + 2468h^7\right]$$
$$+ \frac{y_{n+1}}{617h^7}\left[-100h^5(x-x_n)^2 + 6148h^4(x-x_n)^3 - 10005h^3(x-x_n)^4 + 6156h^2(x-x_n)^5 - 1670h(x-x_n)^6 + 168(x-x_n)^7\right]$$

$$+ \frac{y_{n+2}}{2468h^7}\Big[20835h^5(x-x_n)^2 - 63781h^4(x-x_n)^3 + 72930h^3(x-x_n)^4 - 38793h^2(x-x_n)^5 + 9745h(x-x_n)^6 + 936(x-x_n)^7 + \frac{f_n}{37020h^6}\Big(37020h^6(x-x_n)\Big)\Big]$$
$$- 136364h^5(x-x_n)^2 + 200531h^4(x-x_n)^3 - 150680h^3(x-x_n)^{-4} + 61199h^2(x-x_n)^5$$
$$- 12782h^5(x-x_n)^6 + 1076(x-x_n)^7 + \frac{f_{n+1}}{7404h^6}\Big[-62280h^5(x-x_n)^2 + 182932h^4(x-x_n)^3$$
$$- 186147h^3(x-x_n)^4 + 90946(x-x_n)^5 - 21483h(x-x_n)^6 + 1972(x-x_n)^7\Big]$$
$$+ \frac{f_{n+2}}{2468h^6}\Big[-11250h^5(x-x_n)^2 + 35645h^4(x-x_n)^3 - 42556h^3(x-x_n)^4 + 23805h^2(x-x_n)^5$$
$$- 6272h(x-x_n)^6 + 628(x-x_n)^7\Big] + \frac{f_{n+3}}{7404h^6}\Big[-980h^5(x-x_n)^2$$
$$+ 3308h^4(x-x_n)^3 + 4289h^3(x-x_n)^4 + 2666h^2(x-x_n)^5 - 797h(x-x_n)^6 + 92(x-x_n)^7\Big]$$

$$+\frac{f_{n+5/2}}{9255h^6}\left[921h^5(x-x_n)^2-30464h^4(x-x_n)^3+38400h^3(x-x_n)^4-22976h^2(x-x_n)^5 +6528h(x-x_n)^6-704(x-x_n)^7\right] \quad (2.17)$$

On evaluating (2.17) at $x = x_{n+3}$ and $x = x_{n+5/2}$, we obtained the following 2 discrete equations.

$$y_{n+3}-\frac{783}{617}y_{n+2}+\frac{135}{617}y_{n+1}+\frac{31}{617}y_n=\frac{h}{18510}\left[-234f_n-2970f_{n+1}-810f_{n+2}+2790f_{n+3}+13824f_{n+5/2}\right] \quad (2.18)$$

and

$$y_{n+5/2}-\frac{4077}{157952}y_n-\frac{29000}{157952}y_{n+1}+\frac{124875}{157952}y_{n+2}=\frac{h}{157952}\left[-990f_{n+1}+16125f_{n+1}+67500f_{n+2}-1125f_{n+3}+32640f_{n+5/2}\right] \quad (2.19)$$

The schemes (2.17) and (2.18) has order p = 7, error constants.

$$C_8=\frac{15525}{1273724928} \text{ and } C_8=\frac{-27}{777420} \text{ respectively.}$$

Since the order p > 1, then the equations (2.18) and (2.19) are consistent as in [2].

Equations (2.18) and (2.19) are two equations with four unknowns. For the two equations to constitute two member block hybrid method, we need to eliminate two unknowns either by using existing standard one-step method or using the analytical solution or develop two more equations. What we adopted is discussed in the next section.

1.0 STARTING VALUES

We adopted the explicit sixth order Runge-Kutta scheme in [2] to evaluate y_{n+j}; j=1 and 2; n = o i.e.

$$y_{n+j}=y_{n+j-1}+\frac{h}{840}\left[41k_1+216k_3+27k_4+272k_5+27k_6+216k_7+41k_8\right] \quad (3.1)$$

where,

$$k_1\ f(x_n, y_{n+j-1})$$
$$k_2=f(x_n+h/9,\ y_{n+j-1}+h/9k_1)$$
$$k_3=f\{x_n+h/6,\ y_{n+j-1}+h/24_1(k_1+3k_2)\}$$
$$k_4=f\{x_n+h/3,\ y_{n+j-1}+h/6(k_1+3k_2+4k_3)\}$$
$$k_5=f\{x_n+h/2,\ y_{n+j-1}+h/8(-5k_1+27k_2-24k_3-6k_4)\}$$
$$k_6=f\{x_n+2h/2,\ y_{3+j-1}+h/9(22k_1-981k_2+867k_3-102k_4+k_5)\}$$
$$k_7=f\{x_n+5h/6,\ y_{3+j-1}+h/48(-183k_1+678k_2-472k_3-66k_4+80k_5+3k_6)\} \quad (3.2)$$

4.0 CONVERGENCE AND STABILITY ANALYSIS

In this section, we discuss the stability and convergence properties of the schemes (2.18) and (2.19).

Zero stability of (2.18)

$$P|\xi|=\xi^3-\frac{783}{617}\xi+\frac{135}{617}+\frac{31}{617}=0$$

$\xi_1=1,\ \xi_2=0.4285$ and $\xi_3=0.1587$

Since $\xi_1\neq\xi_2\neq\xi_3$ and

$|\xi_1|\leq 1|\xi_2|\leq$ and $|\xi_3|\leq 1,$ then the method is zero stable according to [2]

Zero stability of (2.19)

$$\ell(\xi)=\xi^{5/2}-\frac{124875}{15795}\xi^2-\frac{29000}{157952}\xi^3-\frac{4077}{157952}=0$$

$\xi_1=1, \xi_2=0.0258$ and $\xi_3=0.2094$

Since $\xi_1\neq\xi_2\neq\xi_3$

Therefore, the method is zero stable according to [2]. According to [2], the necessary and sufficient condition for a linear multistep method to be convergent are that it be consistent and zero stable. Therefore, proposal schemes are convergent.

5.0 NUMERICAL EXPERIMENT

In this section, we use the proposed schemes (2.18) and (2.19) with equations (3.1) and (3.2) as starting values to solve the examples stated below. The errors arising from the computed and theoretical values are compared with Sirisena et al [9] as shown in Tables 5.1, 5.2 and 5.3 below.

Example 5.1 $y' = -y,\ y(o) = 1,\ 0 \le x \le 1,\ h = 0.1$

$y(x) = e^{-x}$

Example 5.2 $y' = x - y,\ y(o) = 0,\ 0 \le x \le 1,\ h = 0.1$

$y(x) = x + e^{-x} - 1$

Example 5.3 $y' = 8(y - x) + 1,\ y(o) = 2,\ 0 \le x \le 1,\ h = 0.1$

$y(x) = x + 2e^{-8x}$

Table 5.1.: Result of Example 5.1

X	Sirisena et al [9]	Proposed scheme
0.1	2.0×10^{-9}	2.1×10^{-10}
0.2	2.0×10^{-9}	2.2×10^{-10}
0.3	1.0×10^{-9}	6.0×10^{-10}
0.4	2.0×10^{-9}	1.0×10^{-10}
0.5	1.0×10^{-9}	4.1×10^{-9}
0.6	3.0×10^{-9}	7.0×10^{-10}
0.7	2.0×10^{-9}	1.5×10^{-9}
0.8	3.0×10^{-9}	7.0×10^{-10}
0.9	3.0×10^{-9}	1.4×10^{-9}
1.0	3.0×10^{-9}	8.0×10^{-10}

Table 5.2. : Comparison of Errors

X	Sirisena et al [9]	Our New Method
0.1	2.0×10^{-9}	0.0
0.2	2.1×10^{-9}	0.0
0.3	1.7×10^{-9}	6.0×10^{-10}
0.4	0.0	2.0×10^{-11}
0.5	6.7×10^{-9}	7.0×10^{-10}
0.6	0.0	1.0×10^{-10}
0.7	1.0×10^{-9}	8.0×10^{-10}
0.8	0.0	2.0×10^{-10}
0.9	0.0	9×10^{-10}
1.0	0.0	4.0×10^{-10}

Table 5.3. : Comparison of Errors

X	Sirisena et al [9]	Our method
0.1	3.6×10^{-4}	1.7×10^{-5}
0.2	1.5×10^{-4}	1.6×10^{-5}
0.3	5.9×10^{-5}	9.3×10^{-6}
0.4	1.6×10^{-5}	4.6×10^{-6}
0.5	4.3×10^{-5}	1.8×10^{-6}
0.6	2.1×10^{-5}	4.2×10^{-7}
0.7	5.7×10^{-7}	1.8×10^{-7}
0.8	1.6×10^{-6}	2.3×10^{-6}
0.9	5.1×10^{-6}	3.8×10^{-7}
1.0	2.8×10^{-6}	3.2×10^{-7}

6.0 DISCUSSION / CONCLUSION

From the above presented tables, our new method is more accurate when compared with Sirisena et al [9]. The proposed method uses two difference equations per step while Sirisena et al [9] used three difference equations per step.

Our new proposed scheme is consistent and convergent. And it compares favourably with the existing scheme.

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