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PADÉ – TYPE INTEGRATORS FOR INITIAL VALUE PROBLEMS

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Abstract

A formal direct conversion of a general Padé Approximant into its corresponding general rational integrator is made through the use of a real operator and its power series. The research work proved that the derived integrators are consistent and convergent. Experiments carried out reveal that the polynomial degree of the numerator and denominator can be chosen freely. Results of our experiments also indicate good performance of the integrators on the selected problems.

Keywords: Power series, Padé-Type integrators, underlying interpolants, zero entries, linear algebraic equations.

1. Introduction:

The object is to solve the initial valued problem (ivp)

$$\mathbf{y}^{(1)} = \mathbf{f}(x,\mathbf{y}),\ \mathbf{y}(0) = y_0,\quad a \le x \le b \tag{1.1}$$

The choice being made is Padé Approximant as the underlying interpolant for the numerical integrator. The desire for Padé comes from the unwieldy nature and analytic difficulties associated with non-Padé rational integrators (see [13]) as compared with the encouraging works of Padé Types found in [1], [14], [17], [18], [6], [7] , [8].

Essentially, this research work is designed to serve as a formal direct explicit definition, converting the Padé Approximant through a real operator and power series definition to Padé Integrator. In section three of the paper we highlight well known occurrences by formally postulating them and proving the postulations under theorems and a lemma. This has become necessary because practitioners who are aware of them need their formal proofs. New comers would by virtue of this work be placed in better position to know what they are in for. The research work is completed with applications.

2. Preliminaries:

Consider the Padé operator

$U : \mathbf{R} \to \mathbf{R}$ defined by the identity

$$U(x)Q_M(x) \equiv P_L(x) \tag{2.1}$$

where $Q_M(x)$, $P_L(x)$ are real polynomials defined by,

$$Q_M(x) = 1 + \sum_{r=1}^{M} q_r x^r,\ Q_M(0) = q_0 \equiv 1, \tag{2.2}$$

$$P_L(x) = \sum_{r=0}^{L} p_r x^r \tag{2.3}$$

The Padé Approximant of the function y(x) to be functionally approximated is given by,

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$$U_{L,M}(x) = \frac{\sum_{r=0}^{L} p_r x^r}{1+\sum_{r=1}^{M} q_r x^r} \qquad (2.4)$$

The Approximant $U_{L,M}(x)$ is expressible by the truncated real power series,

$$U_{L,M}(m) = \sum_{r=0}^{L+M} C_r x^r , \qquad (2.5)$$

from its parent operator power series,

$$U(x) = \sum_{r=0}^{\infty} C_r x^r \qquad (2.6)$$

To evaluate the L + M + 1 unknown parameters

$$[p_0, p_1, p_2,..., p_L] \; ,[q_0, q_1, q_2,..., q_M]$$

the operator U is taken to be the function y whose approximant $\left[\sum_{r=0}^{L} p_r x^r\right]\left[1+\sum_{r=1}^{M} q_r x^r\right]^{-1}$ is sought. As a result of the number, L + M + 1 unknown parameters, the real power series (2.6) is needed up to the first L + M + 1 terms only. Consequently we write,

$$\left[\sum_{r=0}^{L+M+1} C_r x^r\right]\left[1+\sum_{r=1}^{M} q_r x^r\right] \equiv \sum_{r=0}^{L} p_r x^r , \qquad (2.7)$$

giving rise to

$$p_r = \sum_{\alpha=0}^{r} C_\alpha q_{r-\alpha}, \quad r = 0, 1, 2, \; . \; . \; . \qquad (2.8)$$

The range of the parameter p_r, q_r then ensures that,

$p_r = 0$ whenever r = L + 1, L + 2, L + 3, L + 4, ...

$q_{r-\alpha} = 0$ whenever $r - \alpha = M + 1, M + 2, M + 3$, . . .

Observe that the range $L + 1 \le r \le L + M$ is an M-system of simultaneous linear algebraic equations which we represented by the matrix equation,

A q = b (2.9)

where,

A = $[a_{ij}]$ with $a_{ij} = \begin{cases} C_{i+j+D-1} & \text{whenever } L \ge M \\ C_{i+j+D^*-1} & \text{whenever } L \le M \end{cases}$ (2.10)

q = $[q_M, q_{M-1}, q_{M-2} \; . \; . \; . \; q_2, q_1]^T$

b = $[b_i]$ with $b_i = \begin{cases} C_{M+D+i} & \text{whenever } L \ge M \\ C_{M-D+i} & \text{whenever } L \le M \end{cases}$ (2.11)

with i, j = 1, 2, 3, . . . , M

$D = L - M$ (2.12)

$D^* = M - L$ (2.13)

Combining the two distinct sets (2.10) and (2.11) with the understanding that,

$D = L - M = -(M - L) = -D^*$ we obtain irrespective of $L \ge M$ or $M \ge L$

$a_{ij} = C_\gamma$ where $\gamma = i + j + D - 1$ (2.14)

$b_i = -C_\beta$ where $\beta = L + i$ (2.15)

with $D = L - M$

3. Construction of the Rational Integrators

To use the operator U in carrying out the solution process to our initial value problems, we note that the definition that makes U(x) = y(x) in the interpolant design is not enough to serve our needs for the required solutions to the initial value problem (ivp). We therefore subject the Padé operator to the additional constraints which are, for each non – negative integer, n.

$U(x_{n+i}) = y_{n+i}$, i = 0, 1.

In this case the y_{n+i} are the computed values of the theoretical solutions y(x) at the mesh points $x = x_{n+i}$. Consequently, we have the operator integration constraints defined by,

$$U(x_{n+i}) = \begin{cases} y(x_{n+i}), & \text{for } i = 0 \text{ only} \\ y_{n+i}, & \text{for } i = 0,1 \end{cases} \tag{3.1}$$

Observe that the real series (2.6) implies that,

$$U(x_{n+i}) = \sum_{r=0}^{\infty} C_r x_{n+1}^r \quad \text{with } x_{n+1} = (n+1)h \tag{3.2}$$

Hence we obtain,

$$\sum_{r=0}^{\infty} C_r x_{n+1}^r = U(x_{n+1}) \equiv y_{n+1} = \sum_{r=0}^{\infty} \frac{h^r y_n^r}{r!} \tag{3.3}$$

implying,

$$C_r = \frac{h^r y_n^{(r)}}{r! x_{n+1}^r}, \quad r = 0, 1, 2, 3, \quad . \ . \ . \tag{3.4}$$

Consequently,
$$p_r = \sum_{\alpha=0}^{r} \frac{h^\alpha y_n^{(\alpha)}}{\alpha! x_{n+1}^\alpha} q_{r-\alpha}, \quad r = 0, 1, 2, \ . \ . \ ., L \tag{3.5}$$

The needed Padé-Type integration formula is therefore given by,

$$y_{n+1} = \left[\sum_{r=0}^{M} p_r x_{n+1}^r\right]\left[1 + \sum_{r=1}^{L} q_r x_{n+1}^r\right]^{-1} \tag{3.6}$$

The $q_M, q_{M-1}, \ . \ . \ ., q_2, q_1$ are obtained from the M-system of simultaneous linear algebraic equations **Aq** = **b** where **A,b** are henceforth specified by,

$$a_{ij} = \frac{h^\gamma y_n^{(\gamma)}}{\gamma! x_{n+1}^\gamma}, \qquad \gamma = i + j + D - 1 \tag{3.7}$$

$$b_i = -\frac{h^\beta y_n^{(\beta)}}{\beta! x_{n+1}^\beta}, \qquad \beta = L + i \tag{3.8}$$

$$D = L - M \tag{3.9}$$

Remarks:

1. The application of the Padé Approximant to obtain the integrator makes this researcher to use the name Padé-Type Integrator.
2. The state L ≤ M gives rise to the matrix **A** having zeros in some of its entries a_{ij}. These are well known occurrences to approximant practitioners. They are caused by the natural form of the denominator polynomial coefficients in the underlying approximant. In the next few postulations a formal study and report are made of the locations and the number of zero entries.

Armed with the results given by the theorems, users would find work on **A** more convenient to carry out.

Theorem 1(Existence of Zero Entry)

Let the integrator (3.6) – (3.9) be given. From the set of all L < M, one can find a pair [L, M] resulting in the M x M matrix **A** having exactly one of its entries a_{ij} being identically zero.

Proof

By considering D = L – M, take the case D = -2. Thus we have the pair [L, L + 2] satisfying the requisite condition L < M.

For this pair [L, L + 2] the finite polynomials coefficients are the parameters,
$p_0, p_1, \ldots, p_l$ for the numerator while $q_1, q_2, \ldots, q_L, q_{L+1}, q_{L+2}$ are the parameter for the denominator.
By (3.2) we have,

$$p_L = \sum_{\alpha=0}^{L} \frac{h^\alpha y_n^{(\alpha)}}{\alpha! x^\alpha_{n+1}} q_{r-\alpha},$$

As a consequence, the next set of L + 2 equations that are used to compute the denominator coefficient, $\{q_r, r = 1, 2, \ldots, L + 2\}$ are given by,

$$\sum_{\alpha=0}^{L+i} \frac{h^\alpha y_n^{(\alpha)}}{\alpha! x^\alpha_{n+1}} q_{L+i-\alpha} = \frac{h^{L+i} y_n^{(L+i)}}{(L+i)! x^{L+i}_{n+1}}, \ (q_{L+i-\alpha} = 0 \quad \text{for } i-\alpha > 2) \qquad (3.10)$$

where row i = 1, 2, . . ., L + 2
The first row of the L + 2 by L + 2 matrix **A** becomes,

$$\sum_{\alpha=0}^{L+i} \frac{h^\alpha y_n^{(\alpha)}}{\alpha! x^\alpha_{n+1}} q_{L+i-\alpha} = \frac{h^{L+i} y_n^{(L+i)}}{(L+i)! x^{L+i}_{n+1}} \qquad (3.11)$$

Observe that the coefficient of q_{L+2} along this first row of **A** is zero. But then the coefficient of q_{L+2} in row 1 is the entry a_{11}. Hence we have established the existence of a matrix **A** with an entry a_{11} that is identically zero. The remaining L + 1 equations,

$$\sum_{\alpha=0}^{L+i} \frac{h^\alpha y_n^{(\alpha)}}{\alpha! x^\alpha_{n+1}} q_{L+i-\alpha} = \frac{h^{L+i} y_n^{(L+i)}}{(L+i)! x^{L+i}_{n+1}}, \ q_{L+i-\alpha} = 0 \quad \text{for } i-\alpha > 2 \qquad (3.12)$$

where Rows i = 2, 3, . . ., L + 2, have no coefficients that are identically zero. Hence, we have established a case where A has exactly one entry identically zero.

Lemma 1

Given the integrator (3.6) – (3.9), we have $y_n^{(r)} \equiv 0$ whenever r < 0 and $y_n^r \neq 0$ whenever $r \geq 0$.

Proof

By the definition of the operator power series (2.6), we have no member of the set $\{C_r, r = 0, 1, 2, \ldots\}$ that is identically zero. The definition also meant that $C_r \equiv 0$ whenever r < 0. Consequently the integration relation deduced in (3.4) yields $\frac{h^r y_n^{(r)}}{r! x_{n+r} r} = C_r \equiv 0$ for r < 0. But then for every real r, $h \neq 0$, $x_{n+1} \neq 0$ we have $h^r \neq 0$, $x^r_{n+1} \neq 0$ and $r \geq 0$ gives $r \geq 1 > 0$.

Hence $y_n^{(r)} \not\equiv 0$ whenever $r \geq 0$. Hence the lemma ie $y_n^{(r)} \begin{cases} \equiv 0 & for \ r < 0 \\ \not\equiv 0 & for \ r \geq 0 \end{cases}$ (3.13)

Remarks

1. The above simple Lemma serves mainly as a basis upon which the next theorem (theorem 2) is established.
2. Arising from theorem 2 below would be the issue of the possibility of varying D(= L-M) while either L or M is fixed under the structured condition in which the matrix **A** exists. The answers to the possibility are expressed in theorem 3.

Theorem 2 (Zero entries of A)

Let the integrator (3.6) – (3.9) be given.
then, (i). $\{a_{ij} \neq 0, i, j = 1,2, \ldots, M$ whenever $L \geq M - 1\}$
(ii). $\{a_{ij} \equiv 0$ for $i = 1,2, \ldots, (M - L - j), j = 1,2, \ldots, (M - L - i)$ if $L \leq M - 2\}$
(iii). The number of zero entries equals to $\frac{\left[\,|L-M|-1\right]|L-M|}{2}$

Proof

(i). The set of all $L \geq M - 1$ is the same as the set $\{L - M \geq -1\}$. By the definition D = L-M. This part (i) would have being proven if we established that,

$\{a_{ij} \neq 0, i, j = 1, 2, \ldots, M$ whenever $D \geq -1\}$

Consider the set $\wedge = \{-1, 0, 1, 2, \ldots\}$ which is bounded below by -1. Set D = -1 then relation (3.7) gives $\gamma = i + j + D -$

1 = i + j – 2 ≥ 0 for all i, j = 1, 2, . . ., M. This in turn implies that for all D in ∧, $\gamma \geq 0$ for every i, j = 1, 2, . . ., M. Applying the above Lemma on relation (3.7) namely for every i, j = 1, 2, . . ., M we conclude

$$a_{ij} = \frac{h^{\gamma} y_n^{(\gamma)}}{\gamma! x_{n+r}^{\gamma}} \neq 0$$

Hence {$a_{ij} \neq 0$, i, j = 1, 2, . . ., M whenever D ≥ - 1}
This completes part (i) of the Theorem.
(ii) Let $|D|$ represent the usual absolute value of D as a real number. The set {L ≤ M – 2} is the same as the set {D ≤ -2, where D = L – M}. Consequently to establish part (ii) of the theorem we must prove that, {$a_{ij} \equiv 0$ for i = 1, 2, . . ., (|D|-j), j = 1, 2, . . . (|D|-i) whenever D ≤ - 2} and on a similar note to prove part (iii) of the theorem we must prove that the number of zero entries equals, $\frac{(|D|-1)|D|}{2}$

By relation (3.7), for all i, j = 1, 2, . . ., M

$$a_{ij} = \frac{h^{\gamma} y_n^{(\gamma)}}{\gamma! x_{n+r}^{\gamma}} \quad \gamma = i + j + D - 1$$

Employing the above Lemma we obtain,

$$a_{ij} \begin{cases} \equiv 0 & whenever \;\; \gamma < 0 \\ \not\equiv 0 & whenever \;\; \gamma \geq 0 \end{cases} \tag{3.14}$$

Let us note trivially that under this condition wherein D ≤ -2

$$\gamma = i + j + D - 1 \begin{cases} < 0 & whenever \;\; i + j < -D + 1 \\ \geq 0 & whenever \;\; i + j \geq -D + 1 \end{cases} \tag{3.15}$$

Hence, for every i,j = 1,2, . . ., M, the inequalities on γ in (3.15), whenever D ≤ -2 become,

$$\gamma = i + j + D - 1 \begin{cases} < 0 & whenever \;\; 2 \leq i + j \leq |D| \\ \geq 0 & whenever \;\; i + j \geq |D| + 1 \end{cases} \tag{3.16}$$

We are concerned with γ < 0 part of (3.16) in order to prove our stand.

For clarity, the set {i + j : 2 ≤ i + j ≤ |D|} is expressed as array as shown hereunder.

$$\begin{bmatrix} i = 1: & j = 1, & 2, & 3, & 4, & 5, & \ldots & |D|-3, & |D|-2, & |D|-1 \\ i = 2: & j = 1, & 2, & 3, & 4, & 5, & \ldots & |D|-3, & |D|-2 & \\ i = 3: & j = 1, & 2, & 3, & 4, & 5, & \ldots & |D|-3 & & \\ . & . & . & . & . & . & . & . & . & . \\ i = |D|-3: & j = 1, & 2, & 3 & & & & & & \\ i = |D|-2: & j = 1, & 2 & & & & & & & \\ i = |D|-1: & j = 1 & & & & & & & & \end{bmatrix} \tag{3.17}$$

which may be re – expressed as

{i = 1, 2, . . ., (|D|-j), j = 1, 2, . . ., (|D|-i} (3.18)

Consequently, we have the < 0 part of (3.14) written as
{$a_{ij} \equiv 0$ for i = 1, 2, . . ., (|D|-j), j = 1, 2, . . ., (|D|-i) whenever D ≤ 2} which is what we are required to prove as part (ii) of the theorem.
(iii). By direct counting from row 1 to row |D|-1 or from column 1 to column |D| - 1 as given in array (3.17) above the number of entries that are identically zero

$$= \sum_{i=1}^{|D|-1} (|D| - i) = \frac{(|D|-1)|D|}{2} = \sum_{j=1}^{|D|-1} (|D| - j) \tag{3.19}$$

The proof of theorem 2 is complete.

Theorem 3
Let the integrator (3.6) – (3.9) be given. If D is allowed to vary while:
i. M is fixed then $-M \le D \le -2$ whenever $L \le M - 2$
ii. L is fixed, then $-1 \le D \le L$ whenever $L \ge M - 1$

Proof
By definition of D, L = M + D
i. If M is fixed, the definition of the polynomial degree L makes $L \ge 0$ and so $M + D \ge 0$ meaning that $-M \ge D$. (3.20)
If in addition we have $L \le M - 2$ then $D = L - M \le 2$ (3.21)
Combine (3.20) and (3.21) and we are at home i.e. for a fixed M while D varies, we have $-M \le D \le -2$ whenever $L \le M - 2$.
ii. Similarly, if we fix L while varying D, then we must write $M = L - D$.
The polynomial degree $M \ge 0$ then gives $L - D \ge 0$ implying
$D \le L$ (3.22)
The additional constraint $L \ge M - 1$ yields
$D \ge -1$. (3.23)
By (3.22) and (3.23) part (ii) of this theorem is established. We are done.

4. Consistency and Convergence
The Padé-Type integrators studied here are one step method. Consequently their consistency and convergence are being looked into by using established results for any one-step methods. Symbolically, one-step methods are written in normal form by,

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\Phi\left(x_n, y_n; h\right) \tag{4.1}$$

where,
$\Phi(x_n, y_n; h)$ is called the increment function, x_n the usual mesh point and h the mesh size.

Definition (Consistency) [9]
A one-step method (4.1) is said to be consistent if $\Phi\left(x_n, \mathbf{y}_n; h\right) = \mathbf{f}\left(x_n, \mathbf{y}_n\right)$

Definition (Convergence) [9]
A one-step method (4.1) is said to be convergent if for every arbitrary initial vector $\mathbf{y}_0$, an arbitrary point $x_n \in [a, b]$, the global error.

$$\mathbf{e}_n = \mathbf{y}(x_n) - \mathbf{y}_n \tag{4.2}$$

satisfies the following relationship,

$$\underset{h\to 0}{Limit}\left(\max_n \, e_n\right) = 0 \tag{4.3}$$

provided x_n is always a mesh point.
The following result is available in [5].

Theorem 4 (Consistency and Convergence) [11]
A one-step numerical integrator of the form (4.1) is consistent if and only if it is convergent.

Theorem 5
The Padé-Type integrators (3.6) – (3.10) are consistent and convergent.

Proof
Case (i) $L \le M$
From (2.2) and (3.6) we write,

$$y_{n+1} - y_n = \left[\sum_{r=1}^{L}\left(p_r - q_r y_n\right)x_{n+1}^r - \sum_{r=L+1}^{M} q_r y_n x_{n+1}^r\right]Q_m^{-1}(x_{n+1}) \tag{4.4}$$

$$= h\Phi\left(x_n, y_n; h\right) \tag{4.5}$$

where,

$$\Phi\left(x_n, y_n; h\right) = \left[\sum_{r=1}^{L}\left(p_r - q_r y_n\right)(n+1)^r h^{r-1} - \sum_{r=L+1}^{M} q_r y_n (n+1)^r h^{r-1}\right]Q_m^{-1}(x_{n+1}) \tag{4.6}$$

Thus,

$$\Phi\left(x_n, y_n; h\right) = \left(p_1 - q_1 y_n\right)(n+1) + \left[\sum_{r=2}^{L}\left(p_r - q_r y_n\right)(n+1)^r h^{r-1} - \sum_{r=L+1}^{M} q_r y_n (n+1)^r h^{r-1}\right] Q_m^{-1}(x_{n+1}) \qquad (4.7)$$

So that, $$\Phi\left(x_n, y_n; h\right) = \text{Limit } \Phi\left(x_n, y_n; h\right)$$

$$= \left(p_1 - q_1 y_n\right) \frac{x_{n+1}}{h} \qquad (4.8)$$

By (3.7) $$p_r = \sum_{\alpha=0}^{r} \frac{h^\alpha y_n^{(\alpha)}}{\alpha! x_{n+1}^\alpha} q_{r-\alpha}, \quad r = 0(1)L \qquad (4.9)$$

We obtain, since $q_1 = Q_M(0) \equiv 1$

$$p_1 = y_n q_1 + \frac{h y_n^{(1)}}{x_{n+1}} \qquad (4.10)$$

Combine (4.8) and (4.10) to get the required result, that is,

$$\Phi\left(x_n, y_n; 0\right) = f\left(x_n, y_n\right)$$

Case (ii) L > M

By similar arguments or reasoning as we did in case (i) above, we get

$$y_{n+1} - y_n = h\Phi\left(x_n, y_n; h\right)$$

where, $$\Phi\left(x_n, y_n; h\right) = \sum_{r=1}^{L}\left(p_r - q_r y_n\right)\left(n+1\right)^r h^{r-1} Q_M^{-1}\left(x_{n+1}\right) \qquad (4.11)$$

and so,

$$\Phi\left(x_n, y_n; 0\right) = \underset{h\to 0}{Limit}\ \Phi\left(x_n, y_n; h\right) = f\left(x_n, y_n\right)$$ as required.

Conclusively, for any pair (L, M) the integrators are consistent and hence convergent.

5. Applications

Throughout, N_f = number of functional evaluations.

Test Problem 1: Van der Pol's oscillator problem

$$y^1 = \begin{bmatrix} 0 & 1 \\ -1 & 5(1-y_1^2) \end{bmatrix} y, \quad y(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad 0 \le x \le 1,$$ Exact Solution Unknown

There exists no known theoretical (EXACT) solution for this problem. Hence we compare our results with the best results established so far. The solutions are given in tables 1(a) and 1(b) at the point x = 1 using the stepsize h as indicated beginning from x = 0.0. From tables 1(a) and 1(b) one concludes that the Padé – Types compare favourably with [7], [2], [3] and [16].

Test Problem 2:

$$y^1 = \begin{bmatrix} -2000 & 1000 \\ 1 & -1 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad y(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad 0 \le x \le 5$$

The theoretical solution is given by

$$y = \begin{bmatrix} -4.9975\ (-4) & -5.0025\ (-4) \\ 2.4994\ (-7) & -1.0002\ (-3) \end{bmatrix} \begin{bmatrix} E(\lambda_1 t) \\ E(\lambda_2 t) \end{bmatrix} + \begin{bmatrix} 1(-3) \\ 1(-3) \end{bmatrix}$$

where: $a\left(-b\right) = a \times 10^{-b}$, $E\left(\lambda_i t\right) = \exp\left(\lambda_i t\right),\ i = 1,2$

The characteristic equation of the Jacobian matrix is given by

$$\lambda^2 + 2001\lambda + 1000 = 0$$

With an initial estimate of:

$$\lambda_1 = -2000.5$$

$$\lambda_2 = -0.50$$

the first step of Newton – Raphson iterative method yields:

$$\lambda_1 = -2000.500125$$

$$\lambda_2 = -0.499875$$

With these approximate values of the eigenvalues, the theoretical results were computed. Obviously, increased iterative steps using Newton – Raphson method should produce better approximations to the eigenvalues. In effect better results are expected depending on the level of accuracy of the eigenvalues. Tables 2(a) and 2(b) show the performance of the Padé – Type integrators against [3] and [10].

Test Problem 3:

$y^{(1)} = 1 + y^2, y(0) = 1, \quad 0 \le x \le 1,$ Exact Solution $y = \tan\left(x + \frac{\pi}{4}\right)$

Problem Type: Scalar with Singularity at $x = \frac{\pi}{4}$.

The point of singularity of the solution is $x = \frac{\pi}{4}$ radians. Consequently, table 3(a) shows clearly a fall in the accuracies as we approach this point on the table. This fall is more pronounced in the interval (0.75, 0.80) radians. Observe that as we move away from the interval, our accuracies increase. In table 3(b) we bring into sharper focus, the performance of selected integrators at the point x = 0.75 radians. The point of joy here comes from the good performance of the Padé – Type integrators against the selected ones indicated on the table 3(a) and 3(b).

In table 3(b), the theoretical solution at x = 0.75 is given by $\tan\left(x + \frac{\pi}{4}\right) = 28.23825285014$. Our table 3(b) highlights the errors in the computed solution given by our Padé – Types and the selected ones at this point x = 0.75

6. Conclusion

In this paper we presented a general Padé-Type integrator whose numerator and denominator degrees can be selected freely. The kinds of matrices where zero entries exist in the governing matrix equations were highlighted by pointing out their locations and number of them in existence per such matrices. The Padé-Type integrators were proven theoretically to be convergent and consistent. Results arising from our experiments confirm the suitability of the Padé-Type of integrators for use, at least on those classes of problems they were tested on practically.

Table 1a. Solutions for the First Component

H	Padé – Type Integrator	[3]	[2]	[16]	[7]	N_f
0.0125	(3,4): 1.8694389 (2,3): 1. 8694389 (1,2): 1. 8694386	1.8694388	1.8694389	N = 10:1.8692929 N = 5:1.8692926 N = 3:1.8692926	K = 2: 1.86966552 K = 1: 1.86934576	80
0.0250	(3,4): 1.8694388 (2,3): 1. 8694387 (1,2): 1. 8694386	1.8694389	1.8694387	N = 10:1.8691420 N = 5:1.8691415 N = 3:1.8691415	K = 2: 1.87024893 K = 1: 1.86937097	40
0.0500	(3,4): 1.8694384 (2,3): 1. 8694846 (1,2): 1. 8694369	1.8694380	1.8694357	N = 10:1.8688365 N = 5:1.8688354 N = 3:1.8688354	K = 2: 1.87243397 K = 1: 1.87020864	20
0.1000	(3,4): 1.8694344	1.8705973	1.8693953	N = 10:1.8682119		10

	(2,3): 1. 8695057 (1,2): 1. 8692889			N = 5:1.8682097 N = 3:1.8682097	K = 2: 1.87757840 K = 1: 1.87481214	

(L,M) = (Numerator degree, Denominator degree)

Table 1b. Solutions for the Second Component

H	Padé – Type Integrator	[3]	[2]	[16]	[7]	N_f
0.0125	(3,4): – 0.14823588 (2,3): – 0.14823588 (1,2): – 0.14823591	-0.14823588	-0.14823587	N = 10: – 0.1495262 N = 5: – 0.1495268 N = 3: – 0.1495268	K = 2: –0.148204777 K = 1: – 0.1482486	80
0.0250	(3,4): – 0.14823588 (2,3): – 0.14823589 (1,2): – 0.14823615	-0.14823587	-0.14823589	N = 10: – 0.1497297 N = 5: – 0.1497309 N = 3: – 0.1497309	K = 2: – 0.1481248 K = 1: – 0.1482451	40
0.0500	(3,4): – 0.14823594 (2,3): – 0.14822960 (1,2): – 0.14823825	-0.14823599	-0.14823631	N = 10: – 0.1501089 N = 5: – 0.1501113 N = 3: – 0.1501112	K = 2: – 0.1478670 K = 1: – 0.1484301	20
0.1000	(3,4): – 0.14823650 (2,3): – 0.14822671 (1,2): – 0.14826017	-0.14610294	-0.14824187	N = 10: – 0.1507556 N = 5: – 0.1507630 N = 3: – 0.1507629	K = 2: – 0.1471289 K = 1: – 0.1484933	10

(L,M) = (Numerator degree, Denominator degree)

Table 2a. Each solution is multiplied by 10^4. Solution for the first component at h = 0.01

x	THEORETICAL SOLUTION	Padé – Type Integrators				[3]	[10]
		(3,4)	(2,3)	(1,2)	(0,1)	h=0.5	h=0.01
0.5	6.1038	6.1038	6.1046	6.1027	6.1079	6.1038	6.0731
1.0	6.9655	6.9655	6.9661	6.9646	6.9686	6.9655	6.9427
1.5	7.6365	7.6365	7.6370	7.6359	7.6390	7.6365	7.6175
2.0	8.1592	8.1592	8.1596	8.1587	8.1611	8.1592	8.1444
2.5	8.5663	8.5663	8.5663	8.5659	8.5678	8.5663	8.5555
3.0	8.8834	8.8834	8.8836	8.8830	8.8845	8.8834	8.8741

3.5	9.1303	9.1303	9.1305	9.1301	9.1312	9.1303	9.1263
4.0	9.3226	9.3226	9.3226	9.3224	9.3233	9.3226	9.3238
4.5	9.4724	9.4724	9.4726	9.4723	9.4730	9.4724	9.4765
5.0	9.5891	9.5891	9.5891	9.5890	9.5895	9.5891	9.5945

(L,M) = (Numerator degree, Denominator degree)

Table 2b. Each solution is multiplied by 10^4. Solution for the second component at h = 0.01

x	THEORETICAL SOLUTION	Padé – Type Integrators				[3]	[10]
		(3,4)	(2,3)	(1,2)	(0,1)	h=0.5	h=0.01
0.5	2.2099	2.2099	2.2100	2.2098	2.2108	2.2096	2.1467
1.0	3.9327	3.9327	3.9327	3.9326	3.9330	3.9324	3.8854
1.5	5.2745	5.2745	5.2745	5.2744	5.2747	5.2743	5.2371
2.0	6.3195	6.3195	6.3195	6.3194	6.3196	6.3194	6.2901
2.5	7.1335	7.1335	7.1335	7.1332	7.1335	7.1333	7.1106
3.0	7.7674	7.7674	7.7674	7.7664	7.7675	7.7673	7.7492
3.5	8.2612	8.2612	8.2612	8.2615	8.2612	8.2610	8.2592
4.0	8.6457	8.6457	8.6457	8.6458	8.6457	8.6456	8.6525
4.5	8.9452	8.9452	8.9452	8.9453	8.9452	8.9452	8.9564
5.0	9.1785	9.1785	9.1785	9.1785	9.1784	9.1734	9.1915

(L,M) = (Numerator degree, Denominator degree)

Table 3a. Errors in Numerical Integrators on test problem 3 (Uniform mesh – size h = 0.05)

X	**Theoretical Solution**	**PADÉ–TYPE INTEGRATOR, ORDER = L+M**						**N_f**
		(5,6)	**(4,5)**	**(3,4)**	**(2,3)**	**(1,2)**	**(0,1)**	
0.10	1.22304888	-2.474(-16)	-2.537(-17)	9.523(-15)	-1.570(-10)	1.897(-6)	-5.810(-3)	2
0.20	1.50849765	-5.378(-17)	-5.378(-17)	1.571(-14)	-1.673(-10)	3.037(-6)	-6.134(-3)	4
0.30	1.89576512	-2.671(-16)	-4.510(-17)	2.682(-14)	-1.880(-10)	5.258(-6)	-6.831(-3)	6
0.40	2.46496276	3.701(-16)	-7.394(-17)	5.322(-14)	-2.260(-10)	1.024(-5)	-8.132(-3)	8

0.50	3.40822344	-1.242(-16)	3.198(-16)	1.260(-13)	-3.000(-10)	2.405(-5)	-1.067(-2)	10
0.60	5.33185522	8.587(-16)	-2.950(-17)	4.219(-13)	-4.765(-10)	7.980(-5)	-1.664(-2)	12
0.65	7.34043658	3.625(-15)	2.737(-15)	1.002(-12)	-6.886(-10)	1.868(-4)	-2.370(-2)	13
0.70	11.6813738	4.130(-15)	4.130(-15)	3.402(-12)	-1.233(-9)	6.246(-4)	-4.144(-2)	14
0.75	28.2382520	8.947(-14)	8.947(-14)	3.054(-11)	-4.352(-9)	5.420(-3)	-1.386(-1)	15
0.80	-68.4796683	5.922(-13)	5.922(-13)	3.906(-10)	-9.207(-9)	6.256(-2)	-2.191(-1)	16
0.90	-8.68762955	6.055(-15)	6.055(-15)	-4.703(-12)	3.811(-10)	-1.105(-3)	1.551(-2)	18
1.00	-4.58803782	-1.974(-15)	-1.974(-15)	-4.931(-13)	2.647(-10)	-1.022(-4)	1.000(-2)	20

(L,M) = (Numerator degree, Denominator degree)

Table 3b. Errors in Numerical Integrators

Integrator (Method)	**Error at x = 0.75**
[12]	1 (– 4)
[13]	1 (– 1)
[4]	1 (– 2)
[15]	2 (– 2)
[8] M1	6.2 (–7)
[8] M2	1.9 (–8)
[8] M3	1.22 (–8)
[8] M4	1.99 (– 9)
Padé – Type (5,6)	8.9 (– 14)
Padé – Type (4,5)	8.9 (– 14)
Padé – Type (3,4)	3.0 (– 11)
Padé – Type (2,3)	4.3 (– 9)
Padé – Type (1,2)	5.4 (–3)
Padé – Type (0,1)	1.4 (– 1)

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