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A Class of Convergent Rational Runge-Kutta Schemes for solution Of Ordinary Differential Equations (ODEs)

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Abstract

In this paper, a class of convergent implicit Rational Runge-Kutta schemes using Taylor and binomial series expansion, are developed, analysed and computerized to solve ODEs.

Numerical results arising from the new schemes compare favourably with the existing Euler's method. Furthermore, the results show that the schemes are effective and efficient.

Keywords: Implicit, Rational, Runge-Kutta, effective, efficient and convergent.

1.0 Introduction:

In the field of Science, Technology and Engineering, the rate of change of one variable in relation to another is called a derivative. Any equation which connects the derivatives of a differentiable function of one independent variable with respect to itself is called ordinary differential equations (ODEs).

The most general form of an ODE is

$$y' = f(x, y),\ y(x_o) = y_o \tag{1}$$

where y is the dependent variable.

In an attempting to solve this, it will be assumed that f(x, y), satisfies the following conditions

(i) f(x, y) is a real vector function.

(ii) f(x, y) is defined and continuous in the region

$$D = \{x, y / a \le x \le b, -\infty < y < \infty\} \tag{2}$$

(iii) There exist a real constant L such that for any $x \in [a, b]$ and numbers y_1 and y_2 in D.

$$\left|\left|f(x, y) - f(x, y_2)\right|\right| \le L\left|y_1 - y_2\right| \tag{3}$$

where L is the Lipschitz constant of order 1.

Research in techniques for solvingODEs have generated a lot of interest because of the difficult nature of the solution process of ODEs.Popular methods include conventional R – K schemes such as implicit, semi-implicit and explicit schemes.

In 1982 Hong Yuanfu introduced a Rationalized Runge-Kutta scheme of the general form

$$y_{n+1} = \frac{y_n + \sum_{i=1}^{R} W_i K_i}{1 + y_n \sum_{i=1}^{R} V_i H_i} \tag{4}$$

where , $K_1 = hf(x_n, y_n)$

$$K_i = hf(x_n + c_i h, y_n + \sum_{j=1}^{i} a_{ij} K_j)$$

$$H_1 = hg\ (x_n, z_n)$$

$$H_i = hg\left(x_n + d_i h,\ z_n + \sum_{j=1}^{i} b_{ij} H_j\right) \tag{5}$$

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$$g(x_n, z_n) = -Z_n^2 f(x_n, y_n) = -\frac{1}{y_n^2} f(x_n y_n) \qquad (6)$$

In his development, $a_{ij} = b_{ij} = 0$ for $j \geq i$. He developed families of methods of orders two and three of these schemes. During analysis, he discovered that the schemes are A-stable. Perhaps, this A-stability property and simplicity of programming of explicit Rational R – K scheme stimulated [6] in extending the schemes to family of order four.

However, experience with the conventional R-K schemes have shown that implicit R – K scheme have better resolution properties (than explicit ones). This expectation is the chief motivation of the present consideration.

2. 0. Derivation of the Scheme

Recall from (4) that an R-stage implicit Rational R – K scheme is

$$y_{n+1} = \frac{y_n + \sum_{i=1}^{R} W_i K_i}{1 + y_n \sum_{L=1}^{R} V_i H_i} \qquad (7)$$

where,

$$K_i = hf\left(x_n + c_i h,\ y_n + \sum_{j=1}^{L} a_{ij} k_j\right)$$

$$H_i = hg\left(x_n + d_i h,\ z_n + \sum_{j=1}^{L} b_{ij} H_j\right) \qquad (8)$$

$$g(x_n, z_n) = -Z_n^2 f(x_n, y_n)$$

$$Z_n = 1/y_n \qquad (9)$$

with the constraints

$$c_i = \sum_{j=1}^{i} a_{ij}, i = 1(1)R$$

$$d_i = \sum_{j=1}^{i} b_{ij}, \quad i = 1(1)R \qquad (10)$$

The parameters V_i, W_i, C_i, d_i, a_{ij} and b_{ij} are to be determined from the system of non-linear equation generated by adopting the following steps;

(i) obtained the Taylor series expansion of K_i's and H_i's about point $(x_n y_n)$ for i=1(1)R.
(ii) Insert the series expansion into (6).
(iii) Compare the final expansion with Taylor series expansion of y_{n+1} about (x_n, y_n) in the power of h.

The numbers of parameters normally exceeds the number of equations, but in the spirit of [2, 3, 5] these parameters are chosen to ensure that one or more of the following conditions are satisfied.

1. Minimum bound of local truncation error exists.
2. Adequate order of accuracy of the scheme is achieved.
3. The method has maximum interval of absolute stability .
4. Minimum computer storage facilities are utilized.

By equation (6), the general one-stage implicit Rational R-K scheme of order two is of the form

$$y_{n+1} = \frac{y_n + W_1 K_1}{1 + y_n V_1 H_1} \qquad (10)$$

where,

$$K_1 = hf(x_n + c_1 h,\ y_n + a_{11} K_1)$$

$$H_1 = hg(x_n + d_1 h,\ z_n + b_{11} H_1) \qquad (11)$$

$$g(x_n\ z_n) = -Z_n^2 f(x_n, y_n)$$

with the constraints

$$c_1 = a_{11}$$

$$d_1 = b_{11} \qquad (12)$$

Adopting binomial expansion theorem on the right hand side of equation (10) and ignoring terms of order higher than one , we get

$$y_{n+1} = y_n + W_1K_1 - y_n^2\, V_1H_1 + (\text{higher orderr term}) \quad (13)$$

The Taylor series expansion of y_{n+1} about y_n gives

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y_n^{(3)} + \frac{h^4}{4!}\, y_n^{(4)} + 0h^5 \quad (14)$$

Now

$$y'_n = f(x_n, y_n) = f_n$$

$$y''_n = f_x + f_n f_y = Df_n$$

$$y_n^{(3)} = f_{xx} + 2f_n f_{xy} + f_n^2 f_{yy} + f_y(f_x + f_n f_y) = D^2 f_n + f_y Df_n$$

$$y_n^{(4)} = f_{xxx} + 3f_n f_{xyy} + 3f_n^2 f_{xyy} + f_n^3 f_{yyy} + f_y\left(f_{xx} + 2f_n f_{xy} + f_n^2 f_{yy}\right) + \left(f_x + f_n f_y\right)\left(3f_{xy} + 3f_n f_y + f_y^2\right)$$

$$= D^3 f_n + f_y D^2 f_n + 3Df_n Df_y + f_y^2 Df_n \quad (15)$$

Substituting (15) into (14) gives

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2!}Df_n + \frac{h^3}{3!}\left(D^2 f_n + f_y Df_n\right) + \frac{h^4}{4!}\left(D^3 f_n + f_y D^2 f_n + 3Df_n Df_y + f_y^2 Df_n\right) + 0(h^5) \quad (16)$$

Similarly expanding K_1 about (x_n, y_n) we have,

$$K_1 = h\left(f_n + \left(C_1 hf_x + a_{11}k_1 f_y\right) + \frac{1}{2}\left(C_1^2 h^2 f_{xx} + 2c_1 ha_{11}k_1 f_{xy} + a_{11}^2 k_1^2 f_{yy}\right)\right) + 0h^4 \quad (17)$$

$$\therefore K_1 = hA_1 + h^2 B_1 + h^3 D_1 + 0h^4 \quad (18)$$

where,

$A_1 = f_n$, $B_1 = C_1(f_x + f_n f_y) = C_1 Df_n$

$$D_1 = C_1 B_1 f_y + \tfrac{1}{2}C_1^2\left(f_{xx} + 2f_n f_{xy} + f_n^2 f_{yy}\right)$$

$$= C_1^2\left(Df_n f_y + \tfrac{1}{2}D^2 f_n\right) \quad (19)$$

In a similar manner, expansion of H_1 about (x_n, z_n) yields

$$H_1 = hN_1 + h^2M_1 + h^3R_1 + 0h^4 \quad (20)$$

where,

$N_1 = g_n$, $M_1 = d_1Dg_n$,

$$R_1 = d_1^2\left(g_z Dg_n + \tfrac{1}{2}D^2 g_n\right) \quad (21)$$

Expressing g and its partial derivatives in terms of f to facilitate the comparison of coefficients leads to

$$g_n = \frac{-f_n}{y_n^2},\ g_x = \frac{-f_x}{y_n^2},\ g_{xx} = \frac{-f_{xx}}{y_n^2}$$

$$g_z = \frac{-2f_n}{y_n} + f_y,\ g_{xz} = \frac{-2f_x}{y_n} + f_{xy}$$

$$g_{xxz} = \frac{-2f_{xx}}{y_n} + f_{xxy},\ g_{zz} = -2f_n + y_n^2 f_{yy}$$

$$g_{xzz} = -2f_x - y_n^2 f_{xyy}$$

$$= 4y_n^2 f_y + 6y_n^2 f_{yy} + y_n^4 f_{yyy} \quad (22)$$

Substituting (22) into (21) , to get

$$N_1 = \frac{-f_n}{y_n^2},\ M_1 = \frac{-d_1}{y_n^2}\left(Df_n + \frac{2f_n^2}{y_n}\right)$$

$$R_1 = \frac{-d_1^2}{y_n^2}\left[\left(\frac{-2f_n}{y_n} + f_y\right)\left(Df_n + \frac{f_n^2}{y_n}\right)\right] + ½\left(D^2 f_n - \frac{2f_n}{y_n}\left(f_{nn}^2 + f_x\right)\right)\Bigg] \quad (23)$$

Adopting (18) and (20) in (13) gives

$$y_{n+1} = y_n + W_1\left(hA_1 + h^2B_1 + h^3D_1 + 0h^4\right) - y_n^2\left(V_1\left(hN_1 + h^2M_1 + h^3R_1 + 0h^4\right)\right.$$

$$= y_n\left(W_1A_1 - y_n^2V_1N_1\right)h + \left(W_1B_1 - y_n^2V_1M_1\right)h^2 + \left(W_1D_1 - y_n^2V_1R_1\right)h^3 + 0h^4 \qquad (24)$$

Taking the coefficient of h and h^2 into consideration we obtained the following system of equation for family of one-stage scheme of order two.

$$W_1+V_1 = 1$$
$$W_1c_1 + V_1d_1 = ½ \qquad (25)$$

With the constraints

$$a_{11} = c_1$$
$$b_{11} = d_1 \qquad (26)$$

We can now obtain the following results

(i) with $W_1 = 0, V_1 = 1, c_1 = d_1 = ½ , a_{11} = b_{11} = ½$

equation (10) yields

$$y_{n+1} = \frac{y_n}{1 + y_nH_1} \qquad (27)$$

where,

$$H_1 = hg(x_n + ½ h, z_n + ½ H_1) \qquad (28)$$

(ii) With $V_1 = W_1 = ½ , c_1 = a_{11} = ¾ , d_1 = b_{11} = ¼$

equation (10) yields

$$y_{n+1} = \frac{y_n + ½K_1}{1 + \frac{y_n}{2}H_1} \qquad (29)$$

where $K_1 = hf(x_n + ¾ h, y_n + ¾ K_1)$

$$H_1 = hg(x_n + ¼ h, z_n + ¼ H_1) \qquad (30)$$

(iii) With $W_1 = ¼, V_1 = ¾, C_1 = d_1 = ½ , a_{11} = b_{11} = ½$

equation (10) yields

$$y_{n+1} = \frac{y_n + \frac{1}{4}K_1}{1 + \frac{3}{4}y_nH_1} \qquad (31)$$

where

$$K_1 = hf(x_n + ½ h, y_n + ½ K_1)$$
$$H_1 = hg(x_n + ½ h, z_n + ½ H_1)$$

(iv) With $W_1 = \frac{1}{3}, V_1 = \frac{2}{3}, a_{11} = c_1 = \frac{1}{3}$

$b_{11} = d_1 = 7/12$

Equation (10) becomes

$$y_{n+1} = \frac{y_n + \frac{1}{3}K_1}{1 + \frac{2}{3}y_nH_1} \qquad (32)$$

where

$$K_1 = hf(x_n + \tfrac{1}{3}h, y_n + \tfrac{1}{3}K_1)$$
$$H_1 = hf(x_n + 7/12\ h, x_n + 7/12\ H_1)$$

3. Error, Convergence, Consistent and Stability Properties

3.1. Error Analysis

Error of numerical approximation techniques for ODEs arises from different causes that can be majorly classified into discretization, truncation, and round–off errors respectively.

Discretization error is the error introduced as a result of transforming a differential equation into difference equation. Mathematically the discretization error e_{n+1} associated with the formular (10) is the difference between the exact solution and numerical solution y_{n+1} generated by (10) at point x_{n+1}. That is

$$e_{n+1} = y_{n+1} – y(x_{n+1}) \qquad (33)$$

Truncation error is the error introduced as a result of ignoring some of the higher terms of the power series (Taylor and binomial series) during the development of the new formular. Mathematically it can be defined as

$$T_{n+1} = \mathrm{y}(\mathrm{x}_{n+1}) - \frac{\mathrm{y}(\mathrm{x}_n) + \sum_{i=1}^{R} \mathrm{W_i K_i}}{1 + \mathrm{y}(\mathrm{x}_n)\sum \mathrm{V_i H_i}} \tag{34}$$

where $K_i = hf(x_n + c_i h, y(x_n) + \sum_{j=1}^{i} a_{ij} K_j)$

$$H_i = hg(x_n + d_i h, z(x_n) + \sum_{j=1}^{i} b_{ij} H_j)$$

For example, the local truncation error for the family of one-stage scheme of order two is

$$\mathrm{T}_{n+1} = \left(D^2 f_n + f_y.Df_n\right)\left(\tfrac{1}{6} - \tfrac{1}{2}\mathrm{W}_1 c_1^2 - \tfrac{1}{2}\mathrm{V}_1 d_1^2\right) - \mathrm{V}_1 d_1^2\left(2\mathrm{f}_n y_n\left(\mathrm{Df}_n - \frac{2f_n^2}{y_n}\right) - 2\mathrm{f}_y \frac{f_n}{y_n}\right)\mathrm{h}^3 \tag{35}$$

Round-off error is the error introduced as a results of the computing devices. Mathematically it can be expressed as

$$Y_{n+1} = \mathrm{y}_{n+1} - P_{n+1} \tag{36}$$

where y_{n+1} is the expected solution of the difference equations while P_{n+1} is the computer output at the $(n+1)^{+h}$ iteration.

The Convergent Property

The numerical scheme (10) for solving ODE (1) isl be said to be convergent, if the numerical approximation y_{n+1} that is generated by it tends to the exact solution $y(x_{n+1})$ of the ODE (1) as the step size tends to zero.
That is

$$\lim_{\substack{n\to\infty \\ h\to o}} \left[y(\mathrm{x}_{n+1}) - y_{n+1}\right]0 \tag{37}$$

To analyze the convergence of the propose scheme, we consider the following standard theorem which we state without proof.

__**Theorem 1:**__ Let $\{e_j,\ \mathrm{j} = \mathrm{o}\,(1)\mathrm{n}\}$ be the set of real numbers, If there exist finite constants R and S such that

$$|e_j| < R|e_{j-1}| + \mathrm{S},\quad \mathrm{j} = \mathrm{o}\,(1)\mathrm{n}\text{-}1 \tag{38}$$

then
$$|e_j| \le \left(\frac{\mathrm{R}^j - 1}{R-1}\right)\mathrm{S} + \mathrm{R}^j|e_o|,\ \mathrm{R} \ne 1 \tag{39}$$

Let e_{n+1} and T_{n+1} denote the discretization and truncation errors generated by (10) respectively.
Adopting binomial expansion and ignoring higher terms in equation (10) and (33), we obtain

$$\mathrm{y}((\mathrm{x}_{n+1}) = y(x_n) + \mathrm{h}\psi_2(x_n, \mathrm{y}(\mathrm{x}_n); h) + h\phi_1(x_n, y(x_n); h) + \text{higher term} + \mathrm{T}_{n+1} \tag{40}$$

where $\phi_1,\ and\,\psi_2$ are continuous in the domain $a \le x \le b$, $|y| < \infty$, $0 < h \le h_o$ define as

$$h\phi_1(x_n, \mathrm{y}(\mathrm{x}_n); h) = \sum_{i=1}^{R} W_1 H_1 \tag{41}$$

$$\psi_2(x_n, \mathrm{y}(\mathrm{x}_n); h) = 1 + \mathrm{y}(\mathrm{x}_n) + \sum_{j=1}^{R} b_{ij} H_j) \tag{42}$$

Similarly (7) yields

$$y_{n+1} = y_n + h\psi_2(x_n, \mathrm{y}_n; h) + h\phi_1(x_n, y_n; h) + \text{higher terms} \tag{43}$$

Subtract equation (40) from (43) and use equation (33) to get

$$e_{n+1} = e_n + h[\psi_2(x_n, y(x_n); h] - \psi_2(x_n, \mathrm{y}_n; h] + h(\phi[x_n, y(x_n); h\ ) \text{ - } \phi_1(x_n, \mathrm{y}_n; h] + \mathrm{T}_{n+1} \tag{44}$$

By taking the absolute values on both sides of equation (44), we have the inequality

$$|e_{n+1}| \le e_n + Kh|e_n| + hL|e_n| + T \tag{45}$$

where L and K are Lipschitz constant for $\phi_1(x, \mathrm{y}; \mathrm{h})$, and $\psi_2(x, \mathrm{y}; \mathrm{h})$ respectively and

$$\mathrm{T} = \sup_{a \le x \le b} |\mathrm{T}_{n+1}| \tag{46}$$

By setting N = L + K
Inequality (44) becomes

$$|e_{n+1}| \le |e_n|\left(1+\mathrm{hN}\right)+T, \quad n = 0, 1 \ldots\ldots. \tag{47}$$

From theorem 1, expression (47) becomes

$$|e_n| \le \frac{\left((1+hN)^n - 1\right)}{hN}T + (1+\mathrm{hN})^n|e_o| \tag{48}$$

Since $(1+hN)^n = e^{nhN} = e^{N(x_n - a)}$
and $x_n \le b$, then $x_n - a \le b - a$

Consequently $e^{N(x_n-a)} \le e^{N(b-a)}$ (49)

$$e_n \le \frac{\left(e^{N(b-a)} - 1\right)T}{hN} + e^{N(b-a)}|e_o| \tag{50}$$

$$\mathrm{T}_{n+1} = h[\psi_2(x_n+\theta h, \mathrm{y}(\mathrm{x_n}+\theta h) - \psi_2(x_n, y(x_n)] + h[\phi_1(x_n+\theta h, \mathrm{y}(\mathrm{x_n}+\theta h) - \phi_1(x_n, \mathrm{y}(\mathrm{x_n})]$$
$$= h[\psi_2(x_n+\theta h), \mathrm{y}(\mathrm{x_n}+\theta h) - \psi_2(x_n+\theta h, \mathrm{y}(\mathrm{x_n}) + \psi_2(x_n+\theta h, \mathrm{y}(\mathrm{x_n})]$$
$$+ h[\phi_1, (x_n+\theta h, y(x_n+\theta h) - \phi_1(x_n+\theta h, \mathrm{y}(\mathrm{x_n}) + \phi_1(x_n+\theta h, \mathrm{y}(\mathrm{x_n}) + \phi_1(x_n, y(x_n)] \tag{51}$$

By taking the absolute value of (51) on both sides and taking equation (45) into consideration, we get

$$\mathrm{T} = \mathrm{hL}|\mathrm{y}(\mathrm{x_n}+\theta h) - y(x_n)| + \mathrm{jh}^2\theta + hK|\mathrm{y}(\mathrm{x_n}+\theta h) - y(x_n) + Mh^2\theta$$
$$\mathrm{T} = h^2\theta N y'(\xi) + (J+M)h^2\theta, \ \mathrm{x_n} \le \xi \le x_{n+1} \tag{52}$$

Where M and J are partial derivative of ϕ_1 *and* ψ_2 with respect to x respectively.
By setting Q = J + M and

$$\mathrm{Y} = \mathrm{Sup}|\mathrm{y}'(\mathrm{x})|$$
$$a \le x \le b \tag{53}$$

Therefore, equation (52) yields

$$\mathrm{T} = h^2\theta\ (\mathrm{NY} + \mathrm{Q}) \tag{54}$$

By substituting (54) into (50), we have

$$|e_n| \le \frac{\mathrm{h}^2\theta e^{N(b-a)(}[NY+Q]}{hN} + e^{N(b-a)}|e_o| \tag{55}$$

Assuming no error in the input data. That is $e_o = 0$, then in the limit as $h \to 0$.
we obtain

$$\underset{h\to 0,\ n\to\infty}{\mathrm{limit}} |\mathrm{e_n}| = 0 \tag{56}$$

which implies that

$$\underset{h\to 0,\ n\to\infty}{\mathrm{limit}}\ y_n = y(x_n) \tag{57}$$

Consistency
The one-step method is said to be consistent, if

$$\lim_{h\to o}\left[\frac{\mathrm{y}_{n+1} - y_n}{h}\right] = \mathrm{f}(\mathrm{x_n}, y_n) \tag{58}$$

To show the consistency of this scheme
Recall that

$$\mathrm{y}_{n+1} = y_n + W_1K_1 - \mathrm{y}_n^2\mathrm{V}_1H_1 + (\text{ higher order terms}) \tag{59}$$

Subtract y_n from both sides of equation (59) to get

$$\mathrm{y}_{n+1} - y_n = W_1K_1 - \mathrm{y}_n^2\mathrm{V}_1H_1 + (\text{ higher order terms}) \tag{60}$$

But $K_1 = hf(x_n + c_1h,\ y_n + a_{11}K_1)$
$H_1 = hg\ (x_n + d_1h,\ \ z_n + b_{11}\ H_1)$ (61)

Substitute (61) into (60) and divide by h and taking the limit as h $\rightarrow$ 0, gives

$$\lim_{h\to o}\left[\frac{y_{n+1}-y_n}{h}\right]=f(x_n,y_n) \tag{62}$$

Stability Properties

To analyze the stability properties.

Recall that general one stage implicit rational R – K scheme is

$$y_{n+1}=\frac{y_n+W_1K_1}{1+y_n\ V_1H_1} \tag{63}$$

where

$K_1 = hf(x_n+c_1h,\ y_n + a_{11} K_1)$

$H_1 = hg\ (x_n, + d_1h,\ z_n + b_{11} H_1)$

Applying (63) to the stability equation

$$y' = \lambda y,\ y(x_n) = y_o \tag{64}$$

We obtain the recurrent relation

$$y_{+1}=\left(\frac{1+W_1^T(1-a_{11}P)^{-1}}{1-V_1^T(1+b_{11}P)^{-1}}\right)y_n \tag{65}$$

That is $y_{n+1} = \mu(P)\ y_n$

For example, the associated stability function for (27) to (32) is

$$\mu(p)=\frac{1+\frac{1}{2}P}{1-\frac{1}{2}P} \tag{66}$$

It is A –stable

Since $\left|\mu(P)\right|<1$ at P$\in$ [-∞, 0]

<u>Numerical Computation and Results</u>

In order to demonstrate the accuracy of this scheme some sample problems were considered.

<u>Problem 1:</u> Consider initial value problem

$$y' = -1000(y-x^3)+3x^2,\ y(o)=1 \tag{67}$$

The theoretical solution is

$$y(x) = x^3 + e^{-1000x} \tag{68}$$

The numerical results of problem1 which compare the accuracy of the scheme and Euler's scheme are shown in Table 1.

<u>Problem 2</u>: Consider the initial value problem

$$y' = 2x + y,\quad y(o)=1 \tag{69}$$

whose theoretical solution is

$$y(x) = -2(x+1) + 3e^x$$

The numerical results of problem2 which compare the accuracy and convergency of both the scheme and Euler's scheme are shown in Table 2.

Table 1**: Results of a new convergent Implicit Rational R-K scheme and Euler's Scheme**

H	**YEXACT**	**PROPOSED ONE STAGE R-K METHOD OF ORDER TWO Y_N**	**E1**	**EULER'S SCHEME OF Y_N**	**E2**
.10000000D+00	.36887944D+00	.36940452D+00	.52508172D-02	.37603125D+00	.71518088D-02
.50000000D-01	.60665566D+00	.60668197D+00	.26309584D-04	.60689665 D+00	.24098742D-03
.25000000D-01	.77881641D+00	.77881743D+00	.10234944D-06	.77882424D+00	,78335668D-05
.12500000D-01	.88249886D+00	.88249889D+00	.36809087D-07	.88249911D+00	.24978363D-06
.62500000D-01	.96923326D+00	.96941331D+00	.12297566D-08	.96941331D+00	.78857242D-08
.31250000D-02	.98449644D+00	.96923326D+00	.23411650D-09	.96923326D+00	.24769542D-09
.15625000D-02	.99221794D+00	.98449644D+00	.75289774D-11	.98449644D+00	.77603479D-11
.78125000D-02	.99619137D+00	.99221794D+00	.15305257D-13	.99221794D+00	.24280578D-11
.39062500D-03	.99804878D+00	.99610137D+00	.96332942D-14	.99610137D+00	.75495166D-14
.19531250D-03	.99902391D+00	.99804878D+00	.60418337D-15	.99804878D+00	.22204460D-12
.97656250D-04	.99951184D+00	.99902391D+00	.15495538D-09	.99902391D+00	.0000000D+00
.48828125D-04	.99951184D+00	.99951184D+00	.19385937D-15	.99951184D+00	.11102230D-10
.24414063D-04	.99975589D+00	.99975589D+00	.24242830D-11	.99975589D+00	0000000D+00
.12207031D-04	.99987794D+00	.99987794D+00	.30320191D-15	.99987794D+00	.11102230D-12

TABLE 2 **NUMERICAL SOLUTIONS OF PROBLEM USING A NEW CONVERGENT IMPLICIT RATIONAL RUNGE- KUTTA SCHEMES AND EULER'S SCHEME**

H	YEXACT $y(x_n)$	EULER'S y_n	IMPLICIT y_n
.10000000D-01	0.10101515D+01	0.10101514D+01	0.10101511D+01
.50000000D-02	0.10050386D+01	0.10050385D+01	0.10050381D+01
.25000000D-02	0.10025096D+01	0.10025095D+01	0.10025094D+01
.12500000D-02	0.10012527D+01	0.10012526D+01	0.10012521D+01
.62500000D-03	0.10006267D+01	0.10006266D+01	0.10006266D+01
.31250000D-03	0.10003138D+01	0.10003137D+01	0.10003134D+01
.15625000D-03	0.10001568D+01	0.10001567D+01	0.10001561D+01
.78125000D-04	0.10000788D+01	0.10000787D+01	0.10000784D+01
.39062500D-04	0.10000399D+01	0.10000398D+01	0.10000396D+01
.19531250D-04	0.10000209D+01	0.10000208D+01	0.1000206D+01
.97656250D-05	0.10000109D+01	0.10000108D+01	0.1000105D+01
.48828120D-05	0.10000059D+01	0.1000058D+01	0.1000054D+01
.24414060D-05	0.10000029D+01	0.10000028D+01	0.1000021D+01
.12207030D-05	0.10000019D+01	0.1000018D D+01	0.10000015D+01

Discussion

A cursory observation of results in Tables 1 and 2 show that the new convergent implicit rational R-K schemes produce more accurate results than those produced by Euler's scheme of the same stage.

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