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Variable Thermal Conductivity on Compressible Boundary Layer Flow over a Circular Cylinder

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Abstract

In this paper, variable thermal conductivity on heat transfer over a circular cylinder is presented. The concept of assuming constant thermal conductivity on materials is however not efficient. Hence, the governing partial differential equation is reduced using non-dimensionless variables into a system of coupled non-linear ordinary differential equation, which is solved numerically. While the analysis on the stability and existence and uniqueness for different cases of variable thermal conductivity are shown, and as the temperature increases, the points of separation at surface temperature, decreases to an asymptotic value.

Keywords: Thermal conductivity, compressible boundary layer, Boundary conditions, stability and Numerical solution.

1.0 Introduction:

Thermal conductivities of materials vary dramatically both in magnitude and temperature from one material to another due to differences in sample sizes. Though, literature has shown that successful studies had been carried out, but with a limitation of assuming constant thermal conductivity on the effect of heat transfer on forced convection boundary layer flow past a circular cylinder in a viscous compressible fluid, several authors have studied the effect of heat transfer on compressible boundary layer flow of various kinds of dynamic systems. Brown [1] studied the effect of heat transfer on the growth of the boundary layer in the impulsive motion of a cylinder in a viscous compressible fluid. Whereas in [4] Hossain et al investigated the effect of heat transfer on compressible boundary layer flow over a circular cylinder, where they showed that heat transfer parameters has effect in moving the boundary layer upstream, Milena et al [6] examined the thermal conductivity and specific heat capacity of several types of granular agricultural products, namely of spring oat and soybean and were measured in dependence on moisture content from the dry state to the water fully saturated state, bearing in mind that the obtained results will find use in the selection of suitable methods for processing of agricultural products, in a qualified assessment of optimal modes of technological processes, and the development of modern fully automatic agricultural equipments. Dominguez-Muñoz et al [2] highlighted that increasing attention is being paid to the application of uncertainty and sensitivity analysis methods to model validation and building simulation by presenting polynomial fits for the average thermal conductivity and its standard deviation as functions of density for typical insulation materials. They explained further that insulation materials are extensively used to reduce the heat losses (or gains) from thermal systems like buildings, pipes and ducts. Xinwei Wang et al. [10] studied thermal conductivity of Nanoparticle-fluid mixture, where the effective thermal conductivity of mixtures of fluids and nanometer-size particles were measured by steady state parallel-plate method, using two types of Nanoparticle (Al_2O_3 and CuO), dispersed in water, vacuum pump fluid, engine oil and ethylene glycol and the result showed that the thermal conductivities of Nanoparticle-fluid mixtures are higher than those of the base fluids. In this research, we develop sufficient base on the analysis varying different thermal conductivity and heat transfer through a laminar boundary layer in the flow of a viscous fluid over a body of arbitrary shape and arbitrary specified surface temperature constitutes a very important problem in the field of heat transfer. The difference in the temperature initiates the physical contact between the particles, creating kinetic energy and momentum.

2 Formulation of the Problem

The equations describing the steady flow of compressible, laminar two-dimensional boundary layer flow under the assumption that the viscosity (μ) is proportional to the absolute temperature (T) and the Prandtl number (σ) is unity [4],

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is given as

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)=0 \tag{2.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2.2}$$

$$\rho C_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)-u\frac{\partial p}{\partial x}=\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{2.3}$$

with

$$p=\rho RT \tag{2.4a}$$

$$\mu=\mu_0\left(\frac{T}{T_o}\right) \tag{2.4b}$$

subject to the following boundary conditions;

$$\begin{aligned} &u=v=0, \qquad T=T_w \qquad at \quad y=0 \\ &u=U_1, \qquad T=T_1 \qquad at \quad y=\infty \end{aligned} \tag{2.5}$$

where T_w is the constant wall temperature, (x,y) are the Cartesian coordinates with $x-$and $y-$axes along and normal to the surface of the cylinder respectively, (u,v) are the velocity components along $x-$and $y-$axes, p is the pressure, ρ is the density, k is the thermal conductivity, C_p is the specific heat at constant pressure, R is the gas constant and the suffix o, refers to some standard state, say, $x=0$. The main stream velocity U_1 is taken as the velocity in the irrotational motion of an incompressible fluid. Thus, if a is the radius of the cylinder, then,

$$U_1(x)=U_\infty\sin(x/a) \tag{2.6}$$

3 Method of Solution

In obtaining a solution describing the flow and heat transfer equations, equations (2.1) - (2.3) are further reduced to almost an incompressible form by applying the Stewartson's transformations [9].

$$Y=\frac{a_1}{a_0\sqrt{v_0}}\int_0^y\frac{\rho}{\rho_0} \tag{3.1}$$

$$\rho u=\rho_0\sqrt{v_0}\,\frac{\partial\psi}{\partial y} \tag{3.2}$$

For an incompressible two-dimensional steady laminar flow with $\rho=$ constant, we deduce from equation (2.1) by transformation (3.1) and (3.2), the values of

$$u=\left(\frac{a_1}{a_0}\right)\frac{\partial\psi}{\partial y} \tag{3.3}$$

$$v=-\frac{\rho_0}{\rho}\sqrt{v_0}\,\frac{\partial\psi}{\partial x} \tag{3.4}$$

$$\frac{\partial u}{\partial y}=\frac{a_1^{\,2}}{a_0^{\,2}\sqrt{v_0}}\frac{\partial^2\psi}{\partial Y^2} \tag{3.5}$$

$$\mu\frac{\partial u}{\partial y}=\frac{\mu_0 T}{T_0}\frac{a_1^{\,2}}{a_0^{\,2}\sqrt{\mu_0}}\frac{\partial^2\psi}{\partial Y^2} \tag{3.6}$$

Let $\mu_0\left(\frac{T}{T_0}\right)\equiv\mu_0\left(\frac{p}{p_0}\right)$, and equation (3.6) becomes

$$\mu\frac{\partial u}{\partial y}=\frac{\mu_0 p}{p_0}\frac{a_1^{\ 2}}{a_0^{\ 2}\sqrt{\mu_0}}\frac{\partial^2\psi}{\partial Y^2} \tag{3.7}$$

$$\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)=\left(\frac{\rho}{\rho_0}\right)\frac{a_1^{\ 3}}{a_0^{\ 3}}\frac{\partial^3\psi}{\partial Y^3} \tag{3.8}$$

furthermore, from equation (2.3),

$$\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)=\left(\frac{a_1^{\ 2}}{a_0^{\ 2}v_0}\right)\frac{\partial k}{\partial Y}\frac{\partial T}{\partial Y}+k\left(\frac{a_1^{\ 2}}{a_0^{\ 2}v_0}\right)\frac{\partial^2 T}{\partial Y^2}$$

$$\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)=\frac{a_1^{\ 2}}{a_0^{\ 2}v_0}\left(\frac{\partial k}{\partial Y}\frac{\partial T}{\partial Y}+k\frac{\partial^2 T}{\partial Y^2}\right) \tag{3.9}$$

By the Eulerian equation of motion of an inviscid flow, assuming nobody force and steady flow

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{3.10}$$

If there be a constant flow along the x-direction, and the pressure gradient term is assumed to be known from Bernoulli's equation and applied to the outer inviscid flow, we have

$$\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}\right)=U\frac{dU}{dx}+\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=U_1\frac{dU_1}{dx}+\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{3.11}$$

with $\left(u=U_1, T=T_1\right)$ from the boundary conditions in (2.5). Using the stream function, equation (3.11) becomes

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y}=\frac{p}{p_0}\left(\frac{a_1^{\ 3}}{a_0^{\ 3}}\right)\frac{\partial^3\psi}{\partial Y^3}+U_1\frac{dU_1}{dx} \tag{3.12}$$

and by the power law of Isentropic process,

$$\left(\frac{p}{p_0}\cdot\frac{a_1^{\ 3}}{a_0^{\ 3}}\right)^{-1}\left(\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y}\right)=\frac{\partial^3\psi}{\partial Y^3}+\left(\frac{p}{p_0}\cdot\frac{a_1^{\ 3}}{a_0^{\ 3}}\right)^{-1}U_1\frac{dU_1}{dx}$$

$$\left(\frac{T}{T_0}\cdot\frac{a_1^{\ 3}}{a_0^{\ 3}}\right)^{-1}\left(\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y}\right)=\frac{\partial^3\psi}{\partial Y^3}+\left(\frac{p}{p_0}\cdot\frac{a_1^{\ 3}}{a_0^{\ 3}}\right)^{-1}U_1\frac{dU_1}{dx}$$

$$\left(\left(\frac{a_1}{a_0}\right)^3\cdot\left(\frac{a_1}{a_0}\right)^{\gamma/\gamma-1}\right)^{-1}\left(\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y}\right)=\frac{\partial^3\psi}{\partial Y^3}+\left(\frac{p}{p_0}\cdot\frac{a_1^{\ 3}}{a_0^{\ 3}}\right)^{-1}U_1\frac{dU_1}{dx}$$

$$\left(\frac{a_1}{a_0}\right)^{3+\left(\gamma/_{\gamma-1}\right)-1}\left(\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y}\right)=\frac{\partial^3\psi}{\partial Y^3}+\left(\left(\frac{a_1}{a_0}\right)^3\cdot\left(\frac{a_1}{a_0}\right)^{2\gamma/(\gamma-1)}\right)^{-1}U_1\frac{dU_1}{dx}$$

$$\left(\frac{a_1}{a_0}\right)^{3+\left(\gamma/_{\gamma-1}\right)-1}\left(\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y}\right)=\frac{\partial^3\psi}{\partial Y^3}+\left(\left(\frac{a_1}{a_0}\right)^{(5\gamma-3)/(\gamma-1)}\right)^{-1}\times\left(\frac{T_0}{T}\right)^{-1}=\left(\frac{a_1}{a_0}\right)^{(5\gamma-3)/(\gamma-1)}\cdot\frac{T_0}{T}U_1\frac{dU_1}{dx}$$

$$\left(\frac{a_1}{a_0}\right)^{3\gamma-2/(\gamma-1)}\left(\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y}\right)=\frac{\partial^3\psi}{\partial Y^3}+\left(\frac{a_1}{a_0}\right)^{5\gamma-3/(\gamma-1)}\frac{T}{T_0}U_1\frac{dU_1}{dx} \tag{3.13}$$

Multiplying equation (2.2) by u, and adding it to equation (2.3), taking $C_p=C$, with the boundary condition (2.5) and taking the function S as relating to the absolute temperature T, with Mach number relation

$$\left(1+\frac{\gamma-1}{2}M_1^{\,2}\right)S=\frac{T}{T_1}-\frac{\gamma-1}{2}M_1^{\,2}\left(1-\frac{u^2}{U_1^{\,2}}\right)-1 \tag{3.14}$$

$$\frac{T}{T_1}=1+\frac{\gamma-1}{2}M_1^{\,2} \quad ; \quad M_1=\frac{v}{a}$$

we have

$$\rho\left(\left(\frac{a_1}{a_0}\right)\frac{\partial\psi}{\partial Y}\frac{\partial S}{\partial x}-\left(\frac{\rho_0}{\rho}\right)\sqrt{v_0}\,\frac{\partial\psi}{\partial x}\cdot\frac{\partial S}{\partial Y}\left(\frac{a_1}{a_0}\cdot\frac{\rho}{\rho_0\sqrt{v_0}}\right)\right)=\frac{a_1^{\,2}}{a_0^{\,2}v_0}\left(\frac{\partial k}{\partial Y}\frac{\partial S}{\partial Y}+k\frac{\partial^2 S}{\partial Y^2}\right)$$

$$\rho\left(\left(\frac{a_1}{a_0}\right)\frac{\partial\psi}{\partial Y}\frac{\partial S}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial S}{\partial Y}\left(\frac{a_1}{a_0}\right)\right)=\rho\frac{a_1^{\,2}}{a_0^{\,2}}\left(\frac{\partial k}{\partial Y}\frac{\partial S}{\partial Y}+k\frac{\partial^2 S}{\partial Y^2}\right)$$

$$\left(\frac{a_1}{a_0}\right)\left(\frac{\partial\psi}{\partial Y}\frac{\partial S}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial S}{\partial Y}\right)=\frac{a_1^{\,2}}{a_0^{\,2}}\left(\frac{\partial k}{\partial Y}\frac{\partial S}{\partial Y}+k\frac{\partial^2 S}{\partial Y^2}\right) \tag{3.15}$$

We sufficiently consider a flow in which the Mach number <<1, replacing the factor a_0/a_1 by unity, hence, the equations describing the flow and heat transfer are

$$\frac{\partial\psi}{\partial Y}\frac{\partial^2\psi}{\partial Y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial Y^2}=\frac{\partial^3\psi}{\partial Y^3}+U_1\frac{dU_1}{dx}(1+S) \tag{3.16}$$

$$\frac{\partial\psi}{\partial Y}\frac{\partial S}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial S}{\partial Y}=\frac{\partial k}{\partial Y}\frac{\partial S}{\partial Y}+k\frac{\partial^2 S}{\partial Y^2} \tag{3.17}$$

subject to the boundary conditions

$$\begin{aligned}&\psi=\frac{\partial\psi}{\partial Y}=0 \qquad S=\frac{T_w}{T_1}-1=S_w \qquad at \qquad Y=0\\ &\frac{\partial S}{\partial Y}=U_1(x) \qquad S\to 0 \qquad as \qquad Y=\infty\end{aligned} \tag{3.18}$$

The computational constraints involved in solving differential equations, is usually cumbersome, hence, we introduce Merkin's [5] non-dimensionless variables

$$\xi=\frac{x}{a},\quad \eta=Y\left(\frac{\sqrt{v_0}}{R}\right)\mathrm{Re}^{1/2} \tag{3.19}$$

$$\psi=\sqrt{v_0}\,\mathrm{Re}^{1/2}\,\xi f(\xi,\eta),\quad S(x,Y)=S(\xi,\eta) \tag{3.20}$$

We non-dimensionalize equation (3.16), (3.17) using (3.19) and (3.20), to get

$$\left[\begin{aligned}&\frac{v_0}{R}\mathrm{Re}\,\xi f'(\eta)\left(\frac{1}{a}\right)\cdot\frac{v_0}{R}\mathrm{Re}\big(f'(\xi,\eta)+\xi f''(\xi,\eta)\big)-\left(\frac{1}{a}\right)v_0^{\frac{1}{2}}\,\mathrm{Re}^{\frac{1}{2}}\big(f(\xi,\eta)+\xi f'(\xi,\eta)\big)\cdot\frac{v_0^{\frac{3}{2}}}{R^2}\mathrm{Re}^{\frac{3}{2}}\,\xi f''(\eta)\\ &\qquad\qquad=\frac{v_0^{\,2}}{R^3}\mathrm{Re}^2\,\xi f'''(\eta)+U_1\frac{dU_1}{dx}(1+S)\end{aligned}\right]$$

$$\left[\begin{aligned}&\left(\frac{v_0}{R}\mathrm{Re}\,\xi f'(\eta)\cdot\left(\frac{1}{a}\right)\frac{v_0}{R}\mathrm{Re}\big(f'(\xi,\eta)+\xi f''(\xi,\eta)\big)\right)-\left(\frac{1}{a}\right)v_0^{\frac{1}{2}}\,\mathrm{Re}^{\frac{1}{2}}\big(f(\xi,\eta)+\xi f'(\xi,\eta)\big)\\ &\qquad\cdot\frac{v_0^{\frac{3}{2}}}{R^2}\mathrm{Re}^{\frac{3}{2}}\,\xi f''(\eta)=\frac{v_0^{\,2}}{R^3}\mathrm{Re}^2\,\xi f'''(\eta)+U_\infty\sin\xi\cos\xi(1+S)\end{aligned}\right]$$

$$\left[\begin{array}{l}\frac{v_0^{\ 2}}{R^3}\mathrm{Re}^2\,\xi f'''(\eta)+\frac{v_0^{\ 2}}{aR^2}\mathrm{Re}^2\,\xi f(\xi)\,f''(\eta)+\frac{v_0^{\ 2}}{aR^2}\mathrm{Re}^2\,\xi^2 f'(\xi)\,f''(\eta)-\frac{v_0^{\ 2}}{aR^2}\mathrm{Re}^2\,\xi f'^2(\eta)\\ \qquad -\frac{v_0^{\ 2}}{aR^2}\mathrm{Re}^2\,\xi^2 f'(\eta)\,f''(\xi)+U_\infty\sin\xi\cos\xi(1+S)=0\end{array}\right]$$

$$\left[\begin{array}{l}\frac{v_0^{\ 2}}{R^3}\mathrm{Re}^2\ f'''(\eta)+\frac{v_0^{\ 2}}{aR^2}\mathrm{Re}^2\ f(\xi)\,f''(\eta)-\frac{v_0^{\ 2}}{aR^2}\mathrm{Re}^2\ f'^2(\eta)+U_\infty\frac{\sin\xi\cos\xi}{\xi}(1+S)\\ \qquad =\frac{v_0^{\ 2}}{aR^2}\mathrm{Re}^2\,\xi f'(\eta)\,f''(\xi)-\frac{v_0^{\ 2}}{aR}\mathrm{Re}^2\,\xi f'(\xi)\,f''(\eta)\end{array}\right]$$

$$f'''(\eta)+f(\xi)\,f''(\eta)-f'^2(\eta)+\frac{\sin\xi\cos\xi}{\xi}(1+S)=\xi\left(f'(\eta)\frac{\partial f'}{\partial\xi}-f''(\eta)\frac{\partial f}{\partial\xi}\right) \tag{3.21}$$

From equation (3.17),

$$\left[\begin{array}{l}\frac{v_0}{R}\mathrm{Re}\,\xi f'(\eta)\cdot\left(\frac{1}{a}\right)\frac{\partial S}{\partial\xi}-\left(\frac{1}{a}\right)v_0^{\frac{1}{2}}\,\mathrm{Re}^{\frac{1}{2}}\left(f(\xi,\eta)+\xi f'(\xi,\eta)\right)\cdot\frac{v_0^{\frac{1}{2}}}{R}\mathrm{Re}^{\frac{1}{2}}\,\frac{\partial S}{\partial\eta}=\frac{v_0^{\frac{1}{2}}}{R}\mathrm{Re}^{\frac{1}{2}}\,\frac{\partial k}{\partial\eta}\cdot\frac{v_0^{\frac{1}{2}}}{R}\mathrm{Re}^{\frac{1}{2}}\,\frac{\partial S}{\partial\eta}\\ \qquad +k\frac{v_0}{R^2}\mathrm{Re}\frac{\partial^2 S}{\partial\eta^2}\end{array}\right]$$

$$\left[\begin{array}{l}\frac{v_0}{R}\mathrm{Re}\,\xi f'(\eta)\cdot\left(\frac{1}{a}\right)\frac{\partial S}{\partial\xi}-\left(\frac{1}{a}\right)v_0^{\frac{1}{2}}\,\mathrm{Re}^{\frac{1}{2}}\left(f(\xi,\eta)+\xi f'(\xi,\eta)\right)\cdot\frac{v_0^{\frac{1}{2}}}{R}\mathrm{Re}^{\frac{1}{2}}\,\frac{\partial S}{\partial\eta}\\ \qquad =\frac{v_0^{\frac{1}{2}}}{R}\mathrm{Re}^{\frac{1}{2}}\,\frac{\partial k}{\partial\eta}\cdot\frac{v_0^{\frac{1}{2}}}{R}\mathrm{Re}^{\frac{1}{2}}\,\frac{\partial S}{\partial\eta}+k\cdot\frac{v_0}{R^2}\mathrm{Re}\frac{\partial^2 S}{\partial\eta^2}\end{array}\right]$$

$$\frac{v_0}{aR}\mathrm{Re}\,\xi f'(\eta)\frac{\partial S}{\partial\xi}-\frac{v_0}{aR}\mathrm{Re}\,f(\xi)\frac{\partial S}{\partial\eta}-\frac{v_0}{aR}\mathrm{Re}\,\xi f'(\xi)\frac{\partial S}{\partial\eta}=\frac{v_0}{R^2}\mathrm{Re}\frac{\partial S}{\partial\eta}\frac{\partial k}{\partial\eta}+\frac{v_0}{R^2}\mathrm{Re}\,k\frac{\partial^2 S}{\partial\eta^2}$$

$$k\frac{\partial^2 S}{\partial\eta^2}+f(\xi)\frac{\partial S}{\partial\eta}+\frac{\partial S}{\partial\eta}\frac{\partial k}{\partial\eta}=\xi f'(\eta)\frac{\partial S}{\partial\xi}-\xi f'(\xi)\frac{\partial S}{\partial\eta}$$

$$kS''(\eta)+f(\xi)S'(\eta)+k'(\eta)S'(\eta)=\xi\left(f'(\eta)\frac{\partial S}{\partial\xi}-S'(\eta)\frac{\partial f}{\partial\xi}\right) \tag{3.22}$$

At the stagnation point $\xi=0$, equation (3.21) and (3.22) becomes

$$f'''+ff''-f'^2+S+1=0 \tag{3.23}$$

$$kS''+k'S'+fS'=0 \tag{3.24}$$

with the boundary conditions

$$\begin{array}{ll}f(0)=f'(0)=0, & S(0)=S_w\\ f'(\infty)=1, & S(\infty)=0\end{array} \tag{3.25}$$

Case I

If we consider thermal conductivity that is linear in S, i.e.

$$k=1+\alpha S \tag{3.26}$$

then the resulting coupled ordinary differential equation is

$$f'''+ff''-f'^2+S+1=0 \tag{3.27}$$

$$(1+\alpha S)\,S''+(\alpha S'+f)\,S'=0 \tag{3.28}$$

subject to the boundary conditions of (3.25)

Case II

If the thermal conductivity is quadratic in S , i.e.

$$k = 1 + \alpha^2 S^2 \tag{3.29}$$

then, we have

$$f''' + ff'' - f'^2 + S + 1 = 0 \tag{3.30}$$

$$\left(1 + \alpha^2 S^2\right) S'' + \left(2\alpha^2 SS' + f\right) S' = 0 \tag{3.31}$$

subject to the boundary conditions of (3.25). Equations (3.27), (3.28) and (3.30), (3.31) along with the boundary conditions (3.25) are solved numerically using the Equilibrium (Boundary-Value) method [3] for the different values of surface temperature and surface curvature parameter at two different Mach angles $\alpha = 0.3 \; and \; \alpha = 0.075$.

4. Properties of Solution

Existence and Uniqueness of Solution

Theorem 1

There exists a unique solution for problems (3.27) and (3.28), satisfying the boundary conditions (3.25).

***Proof*:**

We want to show that equations (3.27) and (3.28) have a unique solution.

Now,

Let
$$\frac{\partial f_i}{\partial x_j} \le k; \quad such \; that \quad i, j = 1(1)6 \tag{3.32}$$

There exist a constant $k < \infty$, where

$$k = \max\{k_{ij}\} \tag{3.33}$$

Let
$$x_1 = \eta, \quad x_2 = f, \quad x_3 = f', \quad x_4 = f'', \quad x_5 = S, \quad x_6 = S' \tag{3.34}$$

then

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \\ x_6' \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, x_3, x_4, x_5, x_6) \\ f_2(x_1, x_2, x_3, x_4, x_5, x_6) \\ f_3(x_1, x_2, x_3, x_4, x_5, x_6) \\ f_4(x_1, x_2, x_3, x_4, x_5, x_6) \\ f_5(x_1, x_2, x_3, x_4, x_5, x_6) \\ f_6(x_1, x_2, x_3, x_4, x_5, x_6) \end{pmatrix} = \begin{pmatrix} 1 \\ x_3 \\ x_4 \\ -x_2 x_4 + (x_3)^2 - x_5 - 1 \\ x_6 \\ -\dfrac{(\alpha x_6 + x_2) x_6}{(1 + \alpha x_5)} \end{pmatrix} \tag{3.35}$$

by equations (3.32),

$$\left|\frac{\partial f_i}{\partial x_j}\right| \le k; \quad \forall \quad i, j = 1(1)6$$

$$\left|\frac{\partial f_1}{\partial x_1}\right| = \left|\frac{\partial f_1}{\partial x_2}\right| = \left|\frac{\partial f_1}{\partial x_3}\right| = \left|\frac{\partial f_1}{\partial x_4}\right| = \left|\frac{\partial f_1}{\partial x_5}\right| = \left|\frac{\partial f_1}{\partial x_6}\right| = 0$$

$$\left|\frac{\partial f_2}{\partial x_1}\right| = \left|\frac{\partial f_2}{\partial x_2}\right| = \left|\frac{\partial f_2}{\partial x_4}\right| = \left|\frac{\partial f_2}{\partial x_5}\right| = \left|\frac{\partial f_2}{\partial x_6}\right| = 0; \quad \left|\frac{\partial f_2}{\partial x_3}\right| = 1$$

$$\left|\frac{\partial f_3}{\partial x_1}\right| = \left|\frac{\partial f_3}{\partial x_2}\right| = \left|\frac{\partial f_3}{\partial x_3}\right| = \left|\frac{\partial f_3}{\partial x_5}\right| = \left|\frac{\partial f_3}{\partial x_6}\right| = 0; \quad \left|\frac{\partial f_3}{\partial x_4}\right| = 1$$

$$\left|\frac{\partial f_4}{\partial x_1}\right| = \left|\frac{\partial f_4}{\partial x_6}\right| = 0; \quad \left|\frac{\partial f_4}{\partial x_2}\right| = \left|-x_4\right| = n_1; \quad \left|\frac{\partial f_4}{\partial x_3}\right| = \left|2x_3\right| = n_2; \quad \left|\frac{\partial f_4}{\partial x_4}\right| = \left|-x_2\right| = n_3;$$

$$\left|\frac{\partial f_4}{\partial x_5}\right| = \left|-1\right| = 1$$

$$\left|\frac{\partial f_5}{\partial x_1}\right| = \left|\frac{\partial f_5}{\partial x_2}\right| = \left|\frac{\partial f_5}{\partial x_3}\right| = \left|\frac{\partial f_5}{\partial x_4}\right| = \left|\frac{\partial f_5}{\partial x_5}\right| = 0; \quad \left|\frac{\partial f_5}{\partial x_6}\right| = 1$$

$$\left|\frac{\partial f_6}{\partial x_1}\right| = \left|\frac{\partial f_6}{\partial x_3}\right| = \left|\frac{\partial f_6}{\partial x_4}\right| = 0; \quad \left|\frac{\partial f_6}{\partial x_2}\right| = \left|\frac{-x_6}{1+\alpha x_5}\right| = n_4; \quad \left|\frac{\partial f_6}{\partial x_5}\right| = \left|\frac{\alpha(\alpha x_6 + x_2)x_6}{(1+\alpha x_5)^2}\right| = n_{5;}$$

$$\left|\frac{\partial f_6}{\partial x_6}\right| = \left|\frac{-((\alpha x_6 + x_2)+\alpha x_6)}{1+\alpha x_5}\right| \le \left|\frac{\alpha x_6 + x_2}{1+\alpha x_5}\right| + \left|\frac{\alpha x_6}{1+\alpha x_5}\right| < 2\alpha\left|x_6\right| + \left|x_2\right| < n_6$$

Clearly, $\left|\frac{\partial f_i}{\partial x_j}\right|_{i,j=1(1)6}$ is bounded and there exist k such that $k = \max\{0,1,n_1,n_2,n_3,n_4,n_5,n_6\}$ and $0 < k < \infty$.

Therefore, $f_i\left(x_1,x_2,x_3,x_4,x_5,x_6\right)$ are Lipchitz continuous. Hence, there exist a unique solution for problems (3.27) and (3.28).

Theorem 2
There exists a unique solution for problems (3.30) and (3.31), satisfying equation (3.25).
***Proof*:**
The condition for existence and uniqueness is established as stated in theorem 1. Hence,

$$\begin{pmatrix} f_1\left(x_1,x_2,x_3,x_4,x_5,x_6\right) \\ f_2\left(x_1,x_2,x_3,x_4,x_5,x_6\right) \\ f_3\left(x_1,x_2,x_3,x_4,x_5,x_6\right) \\ f_4\left(x_1,x_2,x_3,x_4,x_5,x_6\right) \\ f_5\left(x_1,x_2,x_3,x_4,x_5,x_6\right) \\ f_6\left(x_1,x_2,x_3,x_4,x_5,x_6\right) \end{pmatrix} = \begin{pmatrix} 1 \\ x_3 \\ x_4 \\ -x_2x_4 + \left(x_3\right)^2 - x_5 - 1 \\ x_6 \\ -\dfrac{\left(2\alpha^2 x_5 x_6 + x_2\right)x_6}{\left(1+\alpha^2\left(x_5\right)^2\right)} \end{pmatrix} \qquad (3.36)$$

applying (3.32), we get

$$\left|\frac{\partial f_1}{\partial x_1}\right| = \left|\frac{\partial f_1}{\partial x_2}\right| = \left|\frac{\partial f_1}{\partial x_3}\right| = \left|\frac{\partial f_1}{\partial x_4}\right| = \left|\frac{\partial f_1}{\partial x_5}\right| = \left|\frac{\partial f_1}{\partial x_6}\right| = 0$$

$$\left|\frac{\partial f_2}{\partial x_1}\right| = \left|\frac{\partial f_2}{\partial x_2}\right| = \left|\frac{\partial f_2}{\partial x_4}\right| = \left|\frac{\partial f_2}{\partial x_5}\right| = \left|\frac{\partial f_2}{\partial x_6}\right| = 0; \quad \left|\frac{\partial f_2}{\partial x_3}\right| = 1$$

$$\left|\frac{\partial f_3}{\partial x_1}\right| = \left|\frac{\partial f_3}{\partial x_2}\right| = \left|\frac{\partial f_3}{\partial x_3}\right| = \left|\frac{\partial f_3}{\partial x_5}\right| = \left|\frac{\partial f_3}{\partial x_6}\right| = 0; \quad \left|\frac{\partial f_3}{\partial x_4}\right| = 1$$

$$\left|\frac{\partial f_4}{\partial x_1}\right| = \left|\frac{\partial f_4}{\partial x_6}\right| = 0; \quad \left|\frac{\partial f_4}{\partial x_2}\right| = \left|-x_4\right| = m_1; \quad \left|\frac{\partial f_4}{\partial x_3}\right| = \left|2x_3\right| = m_2; \quad \left|\frac{\partial f_4}{\partial x_4}\right| = \left|-x_2\right| = m_3;$$

$$\left|\frac{\partial f_4}{\partial x_5}\right| = \left|-1\right| = 1$$

$$\left|\frac{\partial f_5}{\partial x_1}\right| = \left|\frac{\partial f_5}{\partial x_2}\right| = \left|\frac{\partial f_5}{\partial x_3}\right| = \left|\frac{\partial f_5}{\partial x_4}\right| = \left|\frac{\partial f_5}{\partial x_5}\right| = 0; \quad \left|\frac{\partial f_5}{\partial x_6}\right| = 1$$

$$\left|\frac{\partial f_6}{\partial x_1}\right| = \left|\frac{\partial f_6}{\partial x_3}\right| = \left|\frac{\partial f_6}{\partial x_4}\right| = 0; \quad \left|\frac{\partial f_6}{\partial x_2}\right| = \left|\frac{-x_6}{1+\alpha^2(x_5)^2}\right| = m_4;$$

$$\left|\frac{\partial f_6}{\partial x_5}\right| = \left|\frac{4\alpha^4 x_5 x_6^{\;2}}{\left(1+\alpha(x_5)^2\right)^2}\right| = m_5;$$

$$\left|\frac{\partial f_6}{\partial x_6}\right| = \left|\frac{-\left(\left(2\alpha^2 x_5 x_6 + x_2\right) + 2\alpha^2 x_5 x_6\right)}{\left(1+\alpha^2(x_5)^2\right)}\right| \le \left|\frac{-2\alpha^2 x_5 x_6 - x_2}{1+\alpha^2(x_5)^2}\right| + \left|\frac{-2\alpha^2 x_5 x_6}{1+\alpha^2(x_5)^2}\right|$$

$$< \left|\frac{2\alpha^2 x_5 x_6}{1+\alpha^2(x_5)^2}\right| - \left|\frac{x_2}{1+\alpha^2(x_5)^2}\right| + \left|\frac{2\alpha^2 x_5 x_6}{1+\alpha^2(x_5)^2}\right|$$

$$< 4\alpha^2\left|x_5 x_6\right| - \left|x_2\right| < m_6$$

$\left|\frac{\partial f_i}{\partial x_j}\right|_{i,j=1(1)6}$ is bounded and there exist k such that $k = \max\{0,1,m_1,m_2,m_3,m_4,m_5,m_6\}$ and $0 < k < \infty$. Therefore, $f_i\left(x_1,x_2,x_3,x_4,x_5,x_6\right)$ are Lipchitz continuous and hence there exist a unique solution.

Stability

We examine the stability of the model by the Liapounov method and to achieve the desired result, we streamline the eigenvalues to be $\lambda = 0 \; or \; -1$, so as to avoid multiple solutions.

Theorem 3

Suppose $x = 0$ is a stationary point for $\dot{x} = f(x)$, let the Jacobian matrix A be a Liapounov function such that

$$\dot{A}(x) \le 0, \;\; \forall \quad x \tag{3.37}$$

then, equation (3.27), (3.28) and (3.30), (3.31) is asymptotically stable when $x \le 0$.

Proof:

Using the system of differential equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -x_1 x_3 + x_2^2 - y_1 - 1 \\ \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -(1+\alpha y_1)^{-1}(\alpha y_2 + x_1) y_2 \end{aligned} \tag{3.38}$$

with stationary points of $\dot{x}_3 = 0 \; and \; \dot{y}_2 = 0$, we employed the four stationary points $(x_1,x_2,x_3,y_1,y_2) = (0,0,0,0,0), (1,0,1,0,2), (1,-2,1,1,-2)$ and $(1,1,1,1,1)$, such that for different values of Mach angle α, $\lambda \le 0$. Hence, our Liapounov function is asymptotically stable.

5 Results and Discussion

The result is presented as temperature, separation parameter of thermal conductivity over a circular cylinder in figures 1 to 4. And the results show that at $\alpha = 0.3 \; and \; \alpha = 0.075$ for both cases of non-linear ODE, as the temperature increases, the points of separation at surface temperature decreases to an asymptotic value.

Generally, an increase in temperature S_w, makes its temperature coefficients increase also, thus, it is higher at the initial stagnation point $\xi = 0$ and later break down at the separation point ξ for values of S_w.

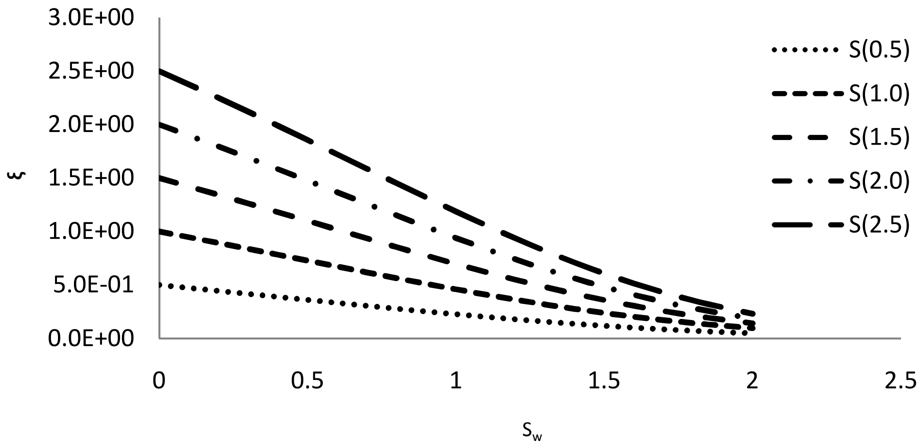

Figure 1: Variation of Surface temperature with separation Parameter at $\alpha = 0.3$ for $k = 1 + \alpha S$

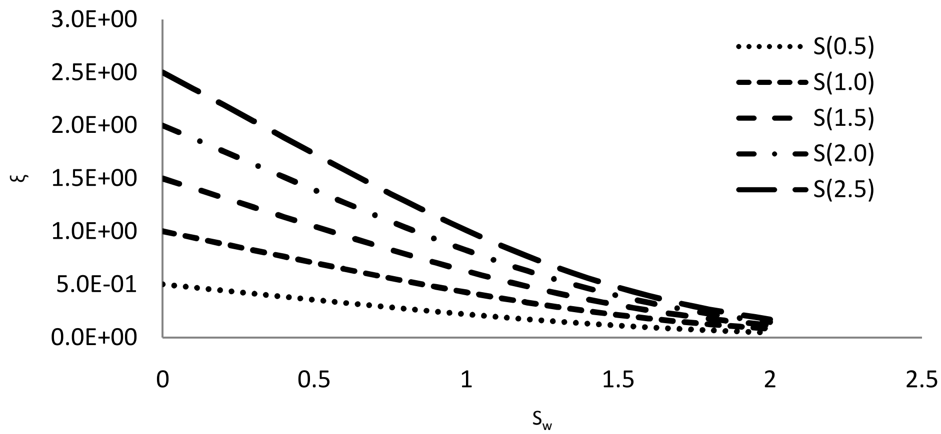

Figure 2: Variation of Surface temperature with separation Parameter at $\alpha = 0.075$ for $k = 1 + \alpha S$

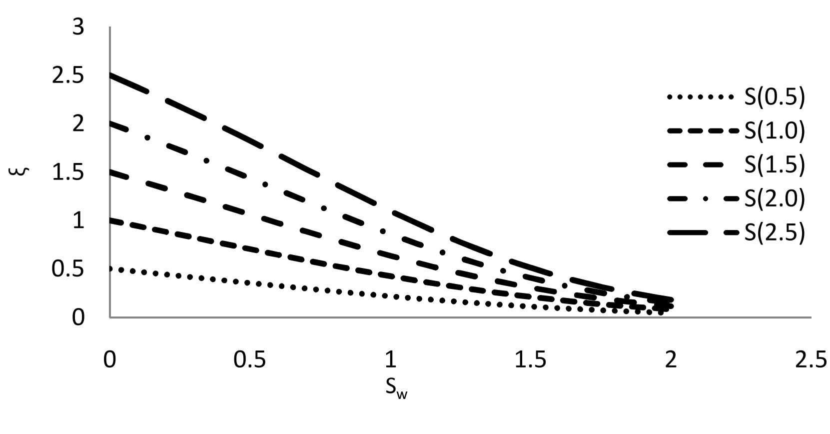

Figure 3: Surface temperature profile with separation Parameter at $\alpha = 0.3$ $k = 1 + \alpha^2 S^2$

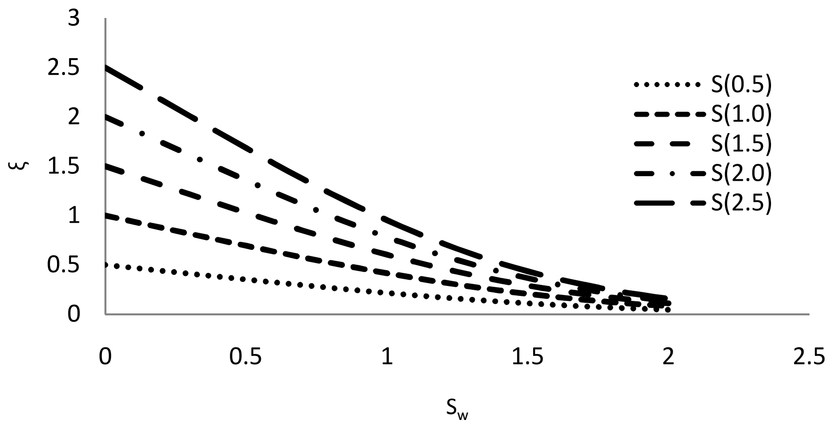

Figure 4: Surface temperature profile with separation Parameter at $\alpha = 0.075$ $k = 1 + \alpha^2 S^2$

6 Conclusion

The problem of heat transfer effect over a cylindrical cylinder at different values of thermal conductivity establishes that thermal conductivity cannot be assumed to be constant. The reason being that thermal conductivity varies dramatically both in magnitude and temperature from one material to another due to differences in sample sizes. Consequently, whenever the property of any material is considered, thermal conductivity should not be regarded as a constant.

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