Further Remarks on Vaccination for Non-Fatal Susceptible-Invective-Susceptible (SIS) Disease

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Abstract

We modify existing SIS model to reflect permanent immunity due to vaccination or previous non-fatal attacks of the disease. The model incorporates information on current and past states of the disease needed by families in deciding on whether to vaccinate or not their children. Of particular interest in the analysis are the criteria for stable endemic equilibrium.

1.0 Introduction:

It is believed that the more we know how diseases spread, the more we will be able to control the spread. We know that vaccines have helped to curtail the spread of diseases and the typical example is the eradication of smallpox [2]. It is known that availability of vaccines may not be sufficient for people to take the necessary advantage. So in some cases, families have to be motivated to take advantage of the vaccines. The history of many vaccinations shows that the progress toward increasing degrees of disease control is inter-mixed by episodes of coverage decrease due to the tension between public health targets and individual freedom i.e. between compulsory vaccination and conscientious or philosophical exception [2, 4]. Buonomo et al. [2] further pointed out that societies are increasingly facing move complex challenge of rational exception, consisting in the family decision to not vaccinate children as a consequence of pseudo-rational comparison between the risk of infection and the risk of vaccine-related side effects. What is clear is that under a voluntary vaccination policy, rational exception makes eradication impossible and may trigger childhood diseases to oscillate [2].

Because of the importance of vaccines in the eradication of diseases and the possibility of side effects of the vaccination, the search for better knowledge of how the vaccines work continues and existing mathematical models are modified in the process.

In this paper we investigate an SIS model with information-dependent function that measures how information on current and past states of the disease is used by families in deciding on whether to vaccinate or not their children.

2. Mathematical Model

As in [2], we envisage a fully effective vaccine providing life-long immunity:

$$S' = \mu(1 - p_0) - \frac{\beta(t)SI}{\phi(I)} - \mu S + \gamma I \tag{1}$$

$$I' = \frac{\beta(t)SI}{\phi(I)} - (\mu + \gamma)I, \tag{2}$$

where

S = susceptible individuals
I = infective individuals
μ = recruitment rate of individuals (newborn and immigrants). It is also the death rate
p_0 = fraction of individuals that are vaccinated.
$\beta(t)$ = rate at which susceptible individuals become infected.
γ = rate at which infected individual are treated on recover.

$\phi(I)$ summarizes how information on current and past states of the disease is used by families in deciding in whether to vaccinate or not their children.

We assume that the continues function $\phi(I)$ satisfies

$$\phi(I) = 1 + \phi_1(I), \quad \phi_1(I) \text{ is a function } I, \phi_1(0) = 0, \phi_1'(I) \geq 0.$$

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3. **Mathematical Analysis**

The population N(t) is constant. So if V(t) is the fraction of the individuals that have been vaccinated, then

$$V(t) + S(t) + I(t) = 1. and$$

$$S + I \leq 1 - p_0$$

Following [2], we let

$$\overline{\beta} = \beta, \text{ in case of constant rate, and}$$

otherwise

$$\overline{\beta} = \frac{1}{\theta} \int_0^\theta \beta(u)du,$$

where θ is the periodicity of the rate of transmission β(t).

3.1 **Disease free equilibrium (DFE) and its properties**

$$DFE = (1 - p_0, 0)$$

Let $R_l = \dfrac{(1 - p_0)\overline{\beta}}{\mu + \gamma}$, $R_0 = \dfrac{\beta}{\mu + \gamma}$ is the basic reproduction number.

So, it suffices to take β (t) as constant.

Proposition 1

(i) Stability of DFE does not depend on $\phi(I)$.

(ii) If $R_l < 1$ then DFE is globally asymptotically stable

(iii) if $R_l \geq 1$ then DFE is unstable.

Proof:

DFE $= (1 - p_0, 0) = (S_0, 0)$

Let $x = S - S_0$, then

$$x' = -\mu x - \overline{\beta}S_0 I + \gamma I + \text{non linear terms}$$

$$I' = \overline{\beta}S_0 I - (\mu + \gamma)I + \text{non linear terms}$$

The nonlinear terms tend to zero as $I \to 0$, $x \to 0$,

So the relevant matrix is

$$A = \begin{pmatrix} -\mu & -\overline{\beta}S_0 + \gamma \\ 0 & \overline{\beta}S_0 - \mu + \gamma \end{pmatrix}$$

Hence stability of DFE does not depend on $\phi(I)$ proving (i)

Also $|A - \lambda I| = 0$ gives

$$\lambda_1 = -\mu, \ \lambda_2 = \overline{\beta}S_0 - \mu - \gamma \text{ and } S_0 = 1 - p_0.$$

So DFE is globally asymptotically stable

if $R_l < 1$ and unstable if $R_l \geq 1$.

3.2 Endemic Equilibrium

We assume that $\phi(I) = 1 + a_1 I + a_2 I^2 + ... + a_n I^n$ (3)

is a polynomial of degree n, the coefficients $a_1, a_2 ... a_n$ are non-negative.

Proposition 2: If $R_l > 1$, there is a unique endemic equilibrium EE = (S_e, I_e), where $S_e = R_0^{-1}\phi(I_e)$ and I_e is a unique positive solution of the equation

$$\mu(1 - p_0) - \mu I - \mu R_0^{-1}\phi(I) = 0$$ (4)

Proof: Clearly, if I≠0, then equation (2) gives $S_e = \dfrac{(\mu + \gamma)}{\beta}\phi(I_e) = R_0^{-1}\phi(I_e)$ substituting S into (1), we obtain

$$\mu(1 - p_0) - \mu I - \mu R_0^{-1}\phi(I) = 0 \tag{5}$$

Equation (5)   implies: $a_n I^n + a_{n-1}I^{n-1} + ... + a_2 I^2 + R_o I - R_o(1 - p_0) = 0$ (6)

Since a_i's i=1,2…n are non-negative then by Descartes' rule of signs, (6) has a unique positive root I_e.

Theorem 1: The endemic positive equilibrium (S_e, I_e) is globally asymptotically stable if $\dfrac{\phi'(I)}{\phi(I)} > \dfrac{\mu}{\mu + \gamma}$ (7)

<u>Proof</u>: Let   x=S-S_e, y=I-I_e
Then

$$\frac{dx}{dt} = -\frac{\beta I_e x}{\phi(I_e)} - \mu x - \frac{\beta S_e y}{\phi(I_e)} + \gamma y + \frac{\beta S_e I_e}{(\phi(I_e))^2} y(\phi'(I_e)) + non\, linear\, term$$

$$\frac{dy}{dt} = \frac{\beta I_e x}{\phi(I_e)} + \frac{\beta S_e y}{\phi(I_e)} - (\mu + \gamma)y - \frac{\beta S_e I_e}{(\phi(I_e))^2} y(\phi'(I_e)) + non\, linear\, term,$$

The linear part is $\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -\dfrac{\beta I_e}{\phi(I_e)} - \mu & -\dfrac{\beta S_e}{\phi(I_e)} + \gamma + \dfrac{\beta S_e I_e}{(\phi(I_e))^2}\phi'(I_e) \\ \dfrac{\beta I_e}{\phi(I_e)} & \dfrac{\beta S_e}{\phi(I_e)} - (\mu + \gamma) - \dfrac{\beta S_e I_e}{(\phi(I_e))^2}\phi'(I_e) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

This gives the characteristic roots $\lambda = \dfrac{(\alpha + \Gamma + 2\mu) \pm \sqrt{\alpha\Gamma + \Gamma^2 + \mu^2}}{2}$

(8)

Clearly $\alpha > 0, \mu > 0\, and\, \Gamma > 0$ if $\dfrac{\phi'(I)}{\phi(I)} > \dfrac{\mu}{\mu + \gamma}$

That is the two $\lambda's$ are negative.
Hence the EE is GAS.
This completes the proof.

3.3   Appropriate Lyapunov function for endemic equilibrium
Let $\phi(I) = e^{\alpha I}$, $\alpha > 0$

Assume that the endemic positive equilibrium is (S_e, I_e).
Then if $x = S - S_e$, $y = I - I_e$
Then

$$\frac{dx}{dt} = -\beta I_e x e^{-\alpha I_e - \alpha y} - \beta S_e y e^{-\alpha I_e - \alpha y} - \mu x + \gamma y + \text{nonlinear term} \tag{9}$$

$$\frac{dy}{dt} = \beta I_e x e^{-\alpha I_e - \alpha y} + \beta S_e y e^{-\alpha I_e - \alpha y} - (\mu + \gamma)y + \text{ nonlinear term} \tag{10}$$

Theorem 2: Let
$$V(x, y) = x^2 + axy + by^2, \tag{11}$$

$$such\, that\, b - \frac{a^2}{4} > 0. \tag{12}$$

Then v(x, y) is a Lyapunov function for the system (9) and (10) if
$$a(1 + I_e) < 2I_e,\, b < 1,\, I_e = S_e,\, a\gamma - 2b(\gamma + \mu) < 0\, and\, 2\gamma - a(\mu + \gamma) = 0. \tag{13}$$

Proof:

$$\frac{dV}{dt} = xy\, e^{-\alpha I_e - \alpha y}\left[2\beta I_e + a\beta I_e - 2\beta S_e - a\beta I_e\right]$$
$$+\left[a\beta I_e + a\beta - 2\beta I_e\right] x^2\, e^{-\alpha I_e - \alpha y}$$
$$+\left[2\beta b S_e - a\beta S_e\right] y^2\, e^{-\alpha I_e - \alpha y}$$
$$-2\mu x^2 + \left(a\gamma - 2\mu b - 2\gamma b\right) y^2$$
$$+\left(2\gamma - \mu_a - a\gamma\right) xy$$

Thus $\frac{dV}{dt} < 0\ if\ (13)\ holds$

This completes the proof

Theorem 3: $\left(S_e, I_e\right)$ is asymptotically stable if (13) holds.

Proof:
This follows from the fact that V(x, y) is a Lyapunov function.

Discussion of Results

When the information on current and past states of the disease is sufficient $t\left(\alpha \to \infty\ and\ \phi = e^{\alpha I}\right)$ the endemic equilibrium is unconditionally stable. When $\alpha = 0,$ no information on the history of the disease, then the endemic equilibrium is not automatically stable. In fact the stability of the endemic equilibrium depends on the value of the equilibrium point, the recovery rate γ and the recruitment rate μ of the newborn. This study shows that information on past and present states of the disease in vital in the eradication of the disease.

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