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Motion of Moving Concentrated Loads on a Simply-Supported Non-Uniform Rayleigh Beam with Non-Classical Boundary Conditions

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Abstract

This paper investigates the transverse response of simply-supported non-uniform Rayleigh beams resting on a constant elastic foundation. The beams properties: moment of inertial I(x) and mass per unit length of the beam miu vary along the span L of the beam. The Mindlin and Goodman's technique is used to transform the governing non-homogeneous forth order partial differential equations with non-homogeneous boundary conditions into non-homogeneous forth order partial differential equations with homogeneous boundary conditions.

The resultant transformed equation is then further treated using the versatile Generalized Galerkins's method with the series representation of the Dirac Delta function, a modification of Struble's asymptotic methods and the integral transformation techniques in conjunction with the convolution theory. Analytical solution was obtained for the transverse displacement response of the non-uniform Rayleigh beam. Analytical and Numerical results reveal that the deflection profile of the non-uniform Rayleigh beam decreases as the value of the foundation stiffness K increases. It is also found that the increase of the foundation stiffness K causes increase in the critical velocity of the dynamical system, thereby reducing the risk of resonance.

Keywords: , Rayleigh beam, non-uniform, axial force, non-classical boundary, rotatory-inertia, Foundation-modulli, simply supported.

1. Introduction:

The vibration analysis of structural elements has been and continuous to be the subject of numerous researchers, since it embraces a wide class of problem with immense importance in Engineering Science. In recent years, such important engineering problems as the vibration of turbines, hulls of ships and bridge girders or variable depth e.t.c, involving the theory of vibration of structures of variable cross-section have intensified the need for the study of the response of non-uniform elastic systems under the action of moving loads.

Until recently,[1] the literature on one-dimensional structures such as non-uniform Rayliegh beams subjected to dynamic loads is very meager. In most cases, even in the absence of inertia effects of the moving loads, exact expressions for the dynamic response and frequencies of non-uniform beams executing flexural vibration cannot be determined, even for lower modes of vibration. Aside the flexural rigidity and mass per unit length of the beam that are certain functions of the spatial coordinates, x, in the governing equation, one important problem that arises when the inertia effects of the masses are considered is the singularity which occurs in the spartial coordinate of the inertia terms of the governing differential equation of motion.

Several investigations have been carried out on the dynamics of structures under moving loads. Such researchers include. [4], [3], [5], [6], [7], [8], [10], [14]and [17].

These works, though impressive, have neglected the practical cases where the elastic systems are of variable cross-section. Very recently, [,16], investigated the Dynamic Behaviour of non-Uniform Bernoulli-Euler Beams Subjected to Concentrated

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Loads Travelling at Varying Velocities. They obtained an analytical solution to the dynamical problem. For the illustrative classical boundary conditions considered, they found that for the same natural frequency, the critical speed for moving mass problem is smaller than that of the moving force problem. Hence, resonance is reached earlier in moving mass problem.

In some of my previous papers, the problem of the flexural motion of a uniform Rayleigh beam resting on an elastic foundation and traversed by masses moving with uniform speed was investigated. In the mathematical model, the beam properties do not vary along the span L of the beam. However, in many practical problems involving dynamics of structures (beams or plates) under moving loads, the structures have variable cross-sections. Thus, the problem of simply-supported non-uniform Rayleigh beam under the action of loads moving with variable speeds is considered in this paper.

When other methods commonly in used to solve dynamical problem broke down or incapable to handle this problem, we resort to a modification of an approximate method best suited for solving diverse problems in dynamics of structures generally referred to as Galerkin' method. This we term Generalised Galerkin's method GGM. This method is employed to simplify the governing fourth order partial differential equation with singular and variable coefficients. The resulting Galerkin's equations are solved via the modified struble's asymptotic technique. Struble's asymptotic technique is used to simplify the coupled second order partial differential equation into a second order ordinary differential equation.

2.0 Governing Equations.

In this paper, the dynamic behavior of a non-uniform Rayleigh beam resting on a constant elastic foundation where the beams properties such as the moment of inertial I , and the mass per unit length of the beam μ vary along the span L of the beam is considered. R^0 is the Rotatory inertial, K is the elastic foundation Moduli; x is the spatial coordinate and t is the time. The transverse displacement $U(x,t)$ of the beam when it is under the action of a moving load of mass M which is moving with velocity C is governed by the fourth order partial differential equation given by

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2}{\partial x^2}U(x,t)\right]+\mu(x)\frac{\partial^2 U(x,t)}{\partial t^2}-\frac{\partial}{\partial x}\left[\mu(x)R^0\frac{\partial^3 U(x,t)}{\partial x\partial t^2}\right]$$
$$+M\delta(x-ct)\left(\frac{\partial^2}{\partial t^2}+\frac{2c\partial^2}{\partial x\partial t}+\frac{c^2\partial^2}{\partial x^2}\right)U(x,t)+KU(x,t)=Mg\delta(x-ct) \qquad (1)$$

Where g is the acceleration due to gravity. It is remarked here that, since the Rayleigh beam is non-uniform, I and μ are no longer constants but vary with the spatial coordinate along the span of the beam .in particular, adapting the example in Fryba. L :Noordhoff 1972,[13].Let $\mathrm{I}(\mathrm{x})$ and $\mu(x)$ take the forms

$$\mathrm{I}(\mathrm{x})=I_0\left(1+\sin\frac{\pi x}{L}\right)^3 \qquad (2)$$

$$\mu(x)=\mu_0\left(1+\sin\frac{\pi x}{L}\right) \qquad (3)$$

where I_o and μ_o are constants.

The boundary conditions of the above equation (1) are taken to be time dependent, thus at each of the boundary points, there are two boundary conditions written as

$$D_i\,[U(0,t)]=f_i(t) \qquad i=1,2 \qquad (4)$$

and

$$D_i\,[U(L,t)]=f_i(t) \qquad i=3,4 \qquad (5)$$

where D_i are linear homogenous differential operators of order less than or equal to three.

For example, the Rayleigh beam in question is simply supported at both sides then

$$D_1=1,\ D_2=\frac{\partial^2}{\partial x^2},\ D_3=1 \text{ and } D_4=\frac{\partial^2}{\partial x^2}\ ,$$

The initial conditions of the motion at time $t=0$ are specified by two arbitrary functions thus

$$U(x,0)=U_0(x)\ \text{ . and }\ \frac{\partial U(x,0)}{\partial t}=\dot{U}_0(x) \qquad (7)$$

When equations (2) and (3) are substituted into equation (1), the result is a non homogeneous system of partial differential equation with variable coefficients given by

$$\frac{EI_0}{4}\left[2\left(10-6\cos\frac{2\pi x}{L}+15\sin\frac{\pi x}{L}-\sin\frac{3\pi x}{L}\right)\frac{\partial^4 U(x,t)}{\partial x^4}+\left(\frac{24\pi}{L}\sin\frac{2\pi x}{L}+\frac{30\pi}{L}\cos\frac{\pi x}{L}-\frac{6\pi}{L}\cos\frac{3\pi x}{L}\right)\frac{\partial^3 U(x,t)}{\partial x^3}\right.$$

$$\left.+\left(\frac{24\pi^2}{L^2}\cos\frac{2\pi x}{L}-\frac{15\pi^2}{L^2}\sin\frac{\pi x}{L}+\frac{9\pi^2}{L^2}\sin\frac{3\pi x}{L}\right)\frac{\partial^2 U(x,t)}{\partial x^2}\right]+\mu_0\left(1+\sin\frac{\pi x}{L}\right)\frac{\partial^2 U(x,t)}{\partial t^2}$$

$$-\mu_0 R^0\left[\left(1+\sin\frac{\pi x}{L}\right)\frac{\partial^2}{\partial x^2}\frac{\partial^2 U(x,t)}{\partial t^2}+\frac{\pi}{L}\cos\frac{\pi x}{L}\frac{\partial}{\partial x}\frac{\partial^2 U(x,t)}{\partial t^2}\right]+M\delta(x-ct)\left[\frac{\partial^2 U(x,t)}{\partial t^2}+\frac{2c\partial^2 U(x,t)}{\partial x\partial t}+\frac{c^2\partial^2 U(x,t)}{\partial x^2}\right]$$

$$+kU(x,t)=Mg\delta(x-ct). \tag{8}$$

$$\frac{EI_0}{4}\left[\left(10-6\cos\frac{2\pi x}{L}+15\sin\frac{\pi x}{L}-\sin\frac{3\pi x}{L}\right)\frac{\partial^4}{\partial x^4}Z(x,t)+6\frac{\pi}{L}\left(4\sin\frac{2\pi x}{L}+5\cos\frac{\pi x}{L}-\cos\frac{3\pi x}{L}\right)\frac{\partial^3}{\partial x^3}Z(x,t)\right.$$

2.1 Operational Simplifications of Equation

In this paper, the initial-boundary value problem (8) consisting of a non-homogeneous partial differential equation with a non-homogeneous boundary conditions is transformed to a non-homogeneous partial differential equation with homogeneous boundary conditions, using the Midlin-Goodman's method. In order to solve the above initial-boundary value problem, we introduce the auxiliary variable $Z(x,t)$ in the form

$$U(x,t)=Z(x,t)+\sum_{i=1}^{4} f_i(t)g_i(x) \tag{9}$$

Substituting equation (9) into the boundary value problem (8), transforms the latter into a boundary value problem in terms of $Z(x,t)$. The displacement influence functions $g_i(x)$ are chosen so as to render the boundary conditions for the boundary value problem in $Z(x,t)$ homogenous.

Substituting equation (9) into (8) and simplifying yields.

$$+3\frac{\pi^2}{L^2}\left(8\cos\frac{2\pi x}{L}-5\sin\frac{\pi x}{L}+3\sin\frac{3\pi x}{L}\right)\frac{\partial^2}{\partial x^2}Z(x,t)\Bigg]+\mu_0\left(1+\sin\frac{\pi x}{L}\right)Z_{tt}(x,t)-\mu_0 R^0\left[\frac{\partial^2}{\partial x^2}Z_{tt}(x,t)+\sin\frac{\pi x}{L}\frac{\partial^2}{\partial x^2}Z_{tt}(x,t)+\frac{\pi}{L}\cos\frac{\pi x}{L}\frac{\partial^2}{\partial x^2}Z_{tt}(x,t)\right]$$

$$+M\delta(x-ct)\left[Z_{tt}(x,t)+\frac{2c\partial}{\partial x}Z_{tt}(x,t)+\frac{c^2\partial^2}{\partial x^2}Z_{tt}(x,t)\right]+kZ(x,t)=Mg\delta(x-ct)-\sum_{i=1}^{\Psi}\left[\frac{EI_0}{4}f_i(t)\left(10-6\cos\frac{2\pi x}{L}+15\sin\frac{\pi x}{L}-\sin\frac{3\pi x}{L}\right)g_i^{IV}(x)\right.$$

$$+6\frac{\pi}{L}\left(4\sin\frac{2\pi x}{L}+5\cos\frac{\pi x}{L}-\cos\frac{3\pi x}{L}\right)g_i^{III}(x)$$

$$\left.+3\frac{\pi^2}{L^2}\left(8\cos\frac{2\pi x}{L}-5\sin\frac{\pi x}{L}+3\sin\frac{3\pi x}{L}\right)g_i^{II}(x)\right]+\mu_0\ddot{f}_i(t)\left(1+\sin\frac{\pi x}{L}\right)g_i(x)-\mu_0 R^0\ddot{f}_i(t)\left(g_i^{II}(x)+\sin\frac{\pi x}{L}g_i^{II}(x)+\frac{\pi}{L}\cos\frac{\pi x}{L}g_i^{I}(x)\right)$$

$$+M\delta(x-ct)\left[\ddot{f}_i(t)g_i(x)+2c\dot{f}_i(t)g_i^{I}(x)+c^2 f_i(t)g_i^{II}(x)+kf_i(t)g_i(x)\right] \tag{10}$$

Where dot$(\cdot)$ represents the derivative with respect to time, while slash $(')$ represents the derivative with respect to space coordinate.

Now the expression in equation (9) must satisfy the boundary conditions in equations (3) and (4); consequently, one obtains

$$D_i[Z(o,t)]+\sum_{j=1}^{4} f_i(t)D_i[g_i(o)]=f_i(t), \qquad i=1,2. \tag{11}$$

$$D_i[Z(L,t)]+\sum_{j=1}^{4} f_i(t)D_i[g_i(L)]=f_i(t), \qquad i=3,4. \tag{12}$$

Substituting equation (9) into the initial equations (7) yields.

$$Z(x,o)=U(x,o)-\sum_{i=1}^{4} f_i(o)g_i(x) \tag{13}$$

$$\frac{\partial}{\partial t}z(x,o)=\dot{U}_0(x)-\sum_{i=1}^{4}\dot{f}_i(o)g_i(x) \tag{14}$$

Using the Mindlin – Goodman method [16] the boundary conditions (11) and (12) in terms of $Z(x,t)$ can be made homogeneous if the function $g_i(x)$ are chosen such that the sixteen conditions given by

$$D_i[g_i(o)] = \delta_{ij} \qquad (i = 1,2,\ j = 1,2,3,4) \tag{15}$$

And

$$D_i[g_i(L)] = \delta_{ij} \qquad (i = 3,4,\ j = 1,2,3,4) \tag{16}$$

Where

$$\delta_{ij} = \begin{cases} 0 & for \quad i \neq j \\ 1 & for \quad i = j \end{cases} \tag{17}$$

is the Kronecker delta; are satisfied.

Using equations (15) and (16) in the non-homogenous boundary conditions (11) and (12), we obtain the homogenous boundary conditions.

$$D_i[z(o,t)] = 0 \qquad i = 1,2$$

$$D_i[z(L,t)] = 0 \qquad i = 3,4 \tag{18}$$

The original problem now reduces to that of solving the non-homogenous partial differential equation (8) subject to the homogenous boundary conditions in (15) and (16) with the non-homogenous initial conditions (13) and (14).

To the author's best knowledge, a close form solution to equation (8) does not exist. Consequently, an approximate analytical solution is desirable to obtain some vital information about the vibrating system.

3. Analytical Approximate Solution.

It is observed that the initial – boundary – value problem in equation (10) is a fourth order partial differential equation having some coefficients which are not only variable but are also singular. These coefficients are the Dirac delta functions which multiply each term of the convective acceleration operator associated with the inertia of the mass of the moving load. It is remarked at this juncture that this transformed equation is now amenable to a modification of the approximate method commonly called Galerkin's method.

3.1 Galerkin's method

The Galerkin's method is used to solve equations of the form

$$\Gamma[Z(x,t)] - P(x,t) = 0 \tag{19}$$

where , Γ is the differential operator, $Z(x,t)$ is the structural displacement and $P(x,t)$ is the transverse load acting on the structure.

A solution of the form

$$Z_j(x,t) = q_j(t)\phi_j(x) \quad \text{for } j = 1,2,3,...,n\ . \tag{20}$$

is sought when j = 1,2,3, …………..n.

The functions $\phi_j(x)$ are chosen to satisfy the approximate boundary conditions. The Galerkin's method requires that the expression (20) be orthogonal to the function $\phi_i(x)$ for $i = 1,2,3,...,n$.

Thus

$$\int_0^l \left[\Gamma \sum_{j=1}^{n} q_j(t)\phi_j(x) - P \right] \phi_i(x) dx = 0 \ , \quad \text{for i =1, 2,…….. n} \tag{21}$$

This gives us a set of ordinary differential equations in $q_j(t)$ to be solved. These differential equations are called Galerkin's equations.

The Galerkin's method requires that the solution of equation (10) takes the form

$$Z_n(x,t) = \sum_{m=1}^{n} Y_m(t) V_m(x) \tag{22}$$

where $V_m(x)$ is chosen such that the desired boundary conditions were satisfied**.**

Thus, substituting equation (22) into equation (10) to obtain

$$\sum_{m=1}^{\eta} \left[\frac{EI_o}{4} \left(\left(10 - 6\cos\frac{2\pi x}{L} + 15\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L} \right) V_m^{IV}(x) + 6\frac{\pi}{L}\left(4\sin\frac{2\pi x}{L} + 5\cos\frac{\pi x}{L} - \cos\frac{3\pi x}{L} \right) V_m^{IIIi}(x) \right.\right.$$
$$\left. + 3\frac{\pi^2}{L^2}\left(8\cos\frac{2\pi x}{L} - 5\sin\frac{\pi x}{L} + 3\sin\frac{3\pi x}{L} \right) V_m^{II}(x) \right) Y_m(t) + \mu_o \left(V_m(x) + \sin\frac{\pi x}{L} V_m(x) \right) \ddot{Y}_m(t)$$

$$-\mu_o R^0\left(V_m^{II}(x)+\sin\frac{\pi x}{L}V_m^{II}(x)+\frac{\pi}{L}\cos\frac{\pi x}{L}V_m^{I}(x)\right)\ddot{Y}_m(t)+M\delta(x-ct)\left(V_m(x)\ddot{Y}_m(t)+2CV_m^{I}(x)\dot{Y}_m(t)+C^2V_m^{II}(x)Y_m(t)\right)$$

$$+KV_m(x)Y_m(t)\,]-Mg\delta(x-ct)+\sum_{i=1}^{4}\left[\frac{EI_o}{4}f_i(t)\left(\left(10-6\cos\frac{2\pi x}{L}+15\sin\frac{\pi x}{L}-\sin\frac{3\pi x}{L}\right)g_i^{IV}(x)\right]\right.$$

$$+6\frac{\pi}{L}\left(4\sin\frac{2\pi x}{L}+5\cos\frac{\pi x}{L}-\cos\frac{3\pi x}{L}\right)g_i^{III}(x)+3\frac{\pi^2}{L^2}\left(8\cos\frac{2\pi x}{L}-5\sin\frac{\pi x}{L}+3\sin\frac{3\pi x}{L}\right)g_i^{II}(x)+\mu_o\ddot{f}_i(t)\left(1+\sin\frac{\pi x}{L}\right)g_i(x)$$

$$-\mu_o R^o\ddot{f}_i(t)\left(g_i^{II}(x)+\sin\frac{\pi x}{L}g_i^{II}(x)+\frac{\pi}{L}\cos\frac{\pi x}{L}g_i^{I}(x)\right)+M\delta(x-ct)\left(\ddot{f}_i(t)g_i(x)+2C\dot{f}_i(t)g_i(x)+C^2f_i(t)g_i^{II}(x)\right)\quad+Kf_i(t)g_i(x)\,]=0 \tag{23}$$

In order to determine $Y_m(t)$, it required that the expression on the left hand side of equation (23) be orthogonal to the function $V_k(x)$
.
Thus,

$$\sum_{m=1}^{n}\left[\left[H_i(m,k)+H_2(m,k)-R^0\left(H_3(m,k)+H_4(m,k)+\frac{\pi}{L}H_5(m,k)\right)\right]\ddot{Y}_m(t)\right.$$
$$+\left\{\frac{EI_o}{4}\left([10H_6(m,k)+15H_7(m,k)-6H_8(m,k)-H_9(m,k)]+6\frac{\pi}{L}[4H_{10}(m,k)+15H_{11}(m,k)-H_{12}(m,k)]\right.\right.$$
$$+3\frac{\pi^2}{L^2}[8H_{13}(m,k)+15H_4(m,k)+3H_{14}(m,k)]+\frac{K}{\mu_0}H_i(m,k)\,\}Y_m(t)$$
$$+\frac{M}{\mu_o}\left[H_{15}(m,k)\ddot{Y}_m(t)+2CH_{16}(m,k)\dot{Y}_m(t)+C^2H_{17}(m,k)Y_m(t)\right]\Bigg]-\frac{Mg}{\mu_0}V_k(ct)+[G_a(t)-G_b(t)+G_c(t)$$
$$-G_d(t)+G_e(t)+G_f(t)-G_g(t)\;-G_d(t)+G_e(t)+G_f(t)-G_g(t)+G_h(t)-G_i(t)$$
$$+G_j(t)+G_k(t)+G_l(t)-G_m(t)-G_N(t)-G_o(t)+G_p(t)+G_q(t)+G_R(t)+G_s(t)\,]=0 \tag{24}$$

where

$H_i(m,k)$ .Where ($i=1,2,........16$) are the resulting integrals and

$$G_a(t)=10\frac{EI_o}{4\mu_o}\sum_{i=1}^{4}f_i\int_0^L g_i^{IV}(x)V_k(x)dx\quad,\quad G_b(t)=\frac{6EI_o}{4\mu_o}\sum_{i=1}^{4}f_i(t)\int_0^L\cos\frac{2\pi x}{L}g_i^{IV}(x)V_k(x)dx$$

$$G_c(t)=\frac{15EI_o}{4\mu_o}\sum_{i=1}^{4}f_i(t)\int_0^L\sin\frac{\pi x}{L}g_i^{IV}(x)V_k(x)dx\,,\quad G_d(t)=\frac{EI_o}{4\mu_o}\sum_{i=1}^{4}f_i(t)\int_0^L\sin\frac{3\pi x}{L}g_i^{IV}(x)V_k(x)dx$$

$$G_e(t)=\frac{24EI_o}{4\mu_o}\frac{\pi}{L}\sum_{i=1}^{4}f_i(t)\int_0^L\sin\frac{2\pi x}{L}g_i^{III}(x)V_k(x)dx\,,\;G_f(t)=\frac{30EI_o}{4\mu_o}\frac{\pi}{L}\sum_{i=1}^{4}f_i(t)\int_0^L\cos\frac{\pi x}{L}g_i^{III}(x)V_k(x)dx\quad,$$

$$G_g(t)=\frac{6EI_o}{4\mu_o}\frac{\pi}{L}\sum_{i=1}^{4}f_i(t)\int_0^L\cos\frac{3\pi x}{L}g_i^{III}(x)V_k(x)dx,.,.G_h(t)=\frac{24EI_o}{4\mu_o}\frac{\pi^2}{L^2}\sum_{i=1}^{4}f_i(t)\int_0^L\cos\frac{2\pi x}{L}g_i^{II}(x)V_k(x)d\,,\;G_h(t)=\frac{24EI_o}{4\mu_o}\frac{\pi^2}{L^2}\sum_{i=1}^{4}f_i(t)\int_0^L\cos\frac{2\pi x}{L}g_i^{II}(x)V_k(x)dx$$

$$G_i(t)=\frac{15EI_o}{4\mu_o}\frac{\pi^2}{L^2}\sum_{i=1}^{4}f_i(t)\int_0^L\sin\frac{\pi x}{L}g_i^{II}(x)V_k(x)dx,\quad ,G_j(t)=\frac{9EI_o}{4\mu_o}\frac{\pi^2}{L^2}\sum_{i=1}^{4}f_i(t)\int_0^L\sin\frac{3\pi x}{L}g_i^{II}(x)V_k(x)dx\,,\quad G_k(t)=\sum_{i=1}^{4}\ddot{f}_i(t)\int_0^L g_i(x)V_k(x)dx\quad,$$

$$G_l(t)=\sum_{i=1}^{4}\ddot{f}_i(t)\int_0^L\sin\frac{\pi x}{L}g_i(x)V_k(x)dx$$

$$G_m(t)=R^o\sum_{i=1}^{4}\ddot{f}_i(t)\int_0^L g_i^{II}(x)V_k(x)dx\quad,\quad G_N(t)=R^o\sum_{i=1}^{4}\ddot{f}_i(t)\int_0^L\sin\frac{\pi x}{L}g_i^{II}(x)V_k(x)dx$$

$$G_O(t)=R^o\frac{\pi}{L}\sum_{i=1}^{4}\ddot{f}_i(t)\int_0^L\cos\frac{\pi x}{L}g_i^I(x)V_k(x)dx\ ,\quad G_p(t)=\frac{M}{\mu_o}\sum_{i=1}^{4}\ddot{f}_i(t)\int_0^L\delta(x-ct)g_i(x)V_k(x)dx$$

$$G_R(t)=\frac{C^2M}{\mu_o}\sum_{i=1}^{4}f_i(x)\int_0^L\delta(x-ct)g_i^{II}(x)V_k(x)dx\ ,\quad G_s(t)=\frac{K}{\mu_o}\sum_{i=1}^{4}f_i(t)\int_0^L g_i(x)V_k(x)dx \qquad (25)$$

Since our beam has simple supports at both ends x=0 and x=L, we therefore choose the function; $V_m(x)=\sin\frac{\lambda_m x}{L}$ which implies

$$V_k(x)=\sin\frac{\lambda_k x}{L}\ \text{and}\ V_m(ct)=\sin\frac{\lambda_m(ct)}{L} \qquad (26)$$

The frequency equation $\sin\lambda_m x=\sin\lambda_k x=0$, hence, $\lambda_m x=\lambda_k x$ (27)

The property of the Dirac-Delta function as an even function to express it in series form thus,

$$\delta(x-ct)=\left(\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}\right) \qquad (28)$$

Thus, in view of equations (26) and evaluating the integrals in equation (24), after some simplifications and rearrangement, one obtains

$$\sum_{m=1}^{n}\left[\alpha_0(m,k)\ddot{Y}_m(t)+\alpha_1(m,k)Y_m(t)+\in\left(\left[H_1(m,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{1A}(m,n,k)\right]\ddot{Y}_m(t)\right.\right.$$

$$\left.\left.+2C\left[H_{18}(m,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{18A}(m,n,k)\right]\dot{Y}_m(t)+C^2\left[H_3(m,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}H_{3A}(m,n,k)\right]Y_m(t)\right)\right]$$

$$=\frac{Mg}{\mu_0}\left[\sin\lambda_k\frac{ct}{L}+A_k\cos\lambda_k\frac{ct}{L}+B_k\sinh\frac{\lambda kct}{L}+C_k\cosh\lambda_k\frac{ct}{L}\right]-\left[G_a(t)-G_b(t)+G_c(t)-G_d(t)+G_e(t)+G_f(t)-G_g(t)\right.$$

$$\left.+G_h(t)-G_i(t)+G_j(t)+G_k(t)+G_l(t)-G_m(t)-G_n(t)-G_o(t)+G_p(t)+G_q(t)+G_r(t)+G_s(t)\right] \qquad (29)$$

Where

$$\in=\frac{mL}{\mu_0} \qquad (30)$$

$$\alpha_0(m,k)=\left[H_1(m,k)+H_2(m,k)-R^0\left(H_3(m,k)+H_4(m,k)+\frac{\pi}{L}H_5(m,k)\right)\right] \qquad (31)$$

and

$$\alpha_1(m,k)=\frac{EI_0}{4\mu_0}\left[\left[10H_6(m,k)+15H_7(m,k)-6H_8(m,k)-H_9(m,k)\right]+6\frac{\pi}{L}\left[4H_{10}(m,k)+5H_{11}(m,k)-H_{12}(m,k)\right]\right.$$

$$\left.+\frac{3\pi^2}{L^2}\left[8H_{13}(m,k)-5H_4(m,k)+3H_{14}(m,k)\right]\right]+\frac{K}{\mu_0}H_1(m,k) \qquad (32)$$

Equation (29) is the transformed equation governing the problem of non-uniform Rayleigh beam resting on a constant Winkler elastic foundation and transverse by a moving load. This second order differential equation is valid for all variants of the classical boundary conditions. In what follows, we shall consider boundary conditions such as simply supported boundary conditions as illustrative examples.

3.2 Simply-Supported Boundary Conditions.

The deflection and bending moment at $x=0$ and $x=L$ vanish for a non-uniform Rayleigh beam having simple supports at both ends.

$$Z(0,t)=0=Z(L,t)\ ,\quad \frac{\partial^2 Z(0,t)}{\partial x^2}=0=\frac{\partial^2 Z(L,t)}{\partial x^2} \qquad (33)$$

also, for normal modes

$$V_m(0)=0=V_m(L)\ ,\quad \frac{d^2V_m(0)}{dx^2}=0=\frac{d^2V_m(L)}{dx^2} \qquad (34)$$

Similarly

$$V_k(0) = 0 = V_k(L) \quad , \quad \frac{d^2V_k(0)}{dx^2} = 0 = \frac{d^2V_k(L)}{dx^2} \tag{35}$$

Substituting equations (26) and (27) and evaluating the resulting integrals after simplifications and rearrangements equation (29) yields.

$$\sum_{m=1}^{\infty} \alpha_0^*(m,k)\ddot{Y}_m(t) + \alpha_1^*(m,k)Y_m(t) + \in \left[Q_1(t)\ddot{Y}_m(t) + Q_2(t)\dot{Y}_m(t) + Q_3(t)Y_m(t)\right]$$

$$= \frac{Mg}{\mu_0}\sin\frac{k\pi ct}{L} - \left[G_a(t) - G_b(t) + G_c(t) - G_d(t) + G_e(t) + G_f(t) - G_g(t) + G_h(t) - G_i(t)\right.$$

$$\left. + G_j(t) + G_k(t) + G_l(t) - G_m(t) - G_n(t) - G_o(t) + G_p(t) + G_q(t) + G_r(t) + G_s(t)\right] = 0 \tag{36}$$

where

$$\alpha_0^*(m,k) = \left(I_1 + I_{17} + R^0\frac{m^2\pi^2}{L^2}\left(I_1 + I_{17} - \frac{1}{m}I_{33}\right)\right) \tag{37}$$

$$\alpha_1^*(m,k) = \frac{EI_0}{4\mu_0}\left[\frac{m^4\pi^2}{L^4}\left(5L + \frac{3}{2}mL\frac{-60mL}{\pi\left[1-(m-k)^2\right]\left[1-(m+k)^2\right]} + \frac{12mkL}{\pi\left[9-(m-k)^2\right]\left[9-(m+k)^2\right]}\right)\right.$$

$$\frac{-m^3\pi^4}{L^4}\left(6kL + \frac{12kL\left(9+m^2-l^2\right)}{\pi\left[9-(m-k)^2\right]\left[9-(m+k)^2\right]} + \frac{60kL\left(1+m^2-k^2\right)}{\pi\left[1-(m-k)^2\right]\left[1-(m+k)^2\right]}\right)$$

$$\left. + \frac{m^2\pi^4}{L^4}\left(6mL - \frac{60mkl}{\pi\left[1-(m-k)^2\right]\left[1-(m+k)^2\right]} + \frac{108mkl}{\pi\left[9-(m-k)^2\right]\left[9-(m+k)^2\right]}\right)\right] \tag{38}$$

$$Q_1(k,m,t) = \frac{L}{2}\left(1 + 4\sum_{m-1}^{\infty}\sin\frac{k\pi ct}{L}\sin\frac{m\pi ct}{L}\right) \tag{39}$$

$$Q_2(k,m,n,t) = -4cmk\left(\frac{1}{m^2-k^2} + \frac{2\sum_{n=1}^{\infty}\left(n^2+m^2-k^2\right)\cos\frac{n\pi ct}{L}}{\pi\left[n^2-(m-k)^2\right]\left[n^2-(m+k)^2\right]}\right) \tag{40}$$

and

$$Q_3(k,m,t) = -\frac{c^2m^2\pi^2}{L^2}Q_1(k,m,t) \tag{41}$$

At this juncture, it is pertinent to obtain the particular functions $g_i(x)$ that ensure zeros of the right hand sides of the boundary conditions for simply supported beam. In view of equations (15) and (16), the $g_i(x)$ are obtained for simply supported non-uniform Rayleigh beam with time dependent boundary conditions thus,

$$g_1(x) = 1 - \frac{x}{L},\; g_2(x) = -\frac{L}{3}x + \frac{x^2}{L} - \frac{1}{6L}x^3,\; g_3(x) = \frac{x}{L} \text{ and } g_4(x) = -\frac{L}{6}x \tag{42}$$

It is only necessary to compute those of the $g_i(x)$ for which the corresponding $f_i(t)$ do not vanish. Thus, we need only $g_1(x)$ and $g_3(x)$ for our boundary displacement influence functions $f_1(t)$ and $f_3(t)$

In view of equations (42),

$$G_a(t) = G_b(t) = G_c(t) = G_d(t) = G_e(t) = G_f(t) = G_g(t) = G_h(t) = G_I(t) = G_j(t) = G_m(t) = G_n(t) = G_r(t) = 0 \tag{43}$$

While,

$$G_k(t) = \ddot{f}_1(t)N_1 + \left(\ddot{f}_3(t) - \ddot{f}_1(t)\right)\frac{1}{L}N_2 \tag{44}$$

$$G_l(t) = \ddot{f}_1(t)N_3 + \left(\ddot{f}_3(t) - \ddot{f}_1(t)\right)\frac{1}{L}N_4 \tag{45}$$

$$G_0(t) = \frac{R^0N_5}{L}\left(\ddot{f}_3(t) - \ddot{f}_1(t)\right), \tag{46}$$

$$G_p(t)=\left[\frac{M}{\mu_0}\ddot{f}_1(t)\sin\frac{k\pi ct}{L}+\frac{M}{L^2\mu_0}\left(N_2+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}N_6\right)\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right)\right] \tag{47}$$

$$G_q(t)=\frac{2cm}{L\mu_0}\sin\frac{k\pi ct}{L}\left(\dot{f}_3(t)-\dot{f}_1(t)\right) \tag{48}$$

$$G_5(t)=\left[\frac{k}{\mu_0}f_1(t)N_1+\frac{k}{L\mu_0}\left(f_3(t)-f_1(t)\right)\right] \tag{49}$$

where

$$N_1=\int_o^L \sin\frac{K\pi x}{L}dx,\quad N_2=\int_o^L x\sin\frac{K\pi x}{L}dx.,\quad N_3=\int_o^L \sin\frac{\pi x}{L}\sin\frac{K\pi x}{L}dx,\quad N_4=\int_o^L x\sin\frac{\pi x}{L}\sin\frac{K\pi x}{L}dx,$$

$$N_5=\int_o^L \cos\frac{\pi x}{L}\sin\frac{K\pi x}{L}dx \text{ and } N_6=\int_o^L \cos\frac{n\pi x}{L}\sin\frac{K\pi x}{L}dx. \tag{50}$$

Substituting equations (43)-(49) and the evaluation of the integrals in equation (50) into equation (36) after some rearrangements and simplifications, one obtains

$$\sum_{m=1}^{n}\left[\alpha_0^*(m,k)\ddot{Y}_m(t)+\alpha_1^*(m,k)Y_m(t)+\in\left(Q_1(t)\ddot{Y}_m(t)+Q_2(t)\dot{Y}_m(t)+Q_3(t)Y_m(t)\right)\right]$$

$$=\frac{Mg}{\mu_0}\sin\frac{k\pi ct}{L}-\left[F_1(t)+F_2(t)+F_3(t)+F_4(t)\right]-\in L\left[F_5(t)+F_6(t)+F_7(t)+F_8(t)\right] \tag{51}$$

Where

$$F_1(t)=\left(\ddot{f}_1(t)+(-1)^{k+1}\ddot{f}_3(t)\right)\frac{L}{k\pi} \tag{52}$$

$$F_2(t)=\frac{L}{2}\ddot{f}_1(t) \tag{53}$$

$$F_3(t)=\left(f_1(t)+(-1)^{k+1}f_3(t)\right)\frac{l}{\mu_0\pi} \tag{54}$$

$$F_4(t)=\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right)\left(\frac{2R^0k}{\pi\left(1-k^2\right)}-\frac{4kl}{\pi^2\left(1-k^2\right)^2}\right) \tag{55}$$

$$F_5(t)=\ddot{f}_1(t)\sin\frac{k\pi ct}{L} \tag{56}$$

$$F_6(t)=\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right)\frac{(-1)^{k+1}}{k\pi} \tag{57}$$

$$F_7(t)=\left(\ddot{f}_3(t)-\ddot{f}_1(t)\right)\sum_{n=1}^{\infty}\left\{\frac{(k-n)(-1)^{k+n}+(k+n)(-1)^{k-n}}{\pi\left(k^2-n^2\right)}\right\}\cos\frac{n\pi ct}{L} \tag{58}$$

$$F_8(t)=\left(\dot{f}_3(t)-\dot{f}_1(t)\right)\frac{2c}{L}\sin\frac{k\pi ct}{L} \tag{59}$$

Equation (51) represents the transformed equation of the non-uniform Rayleigh beam simply-supported at both ends and having boundary and initial conditions which are time dependent. To solve equation (51), two cases are considered :(a)Moving force case and (b)Moving mass case respectively.

3.2.1 Simply Supported Rayleigh Beam Traversed By Moving Force.

This model neglects the inertia effect of the moving mass M. Thus, in equation (51), ε is set to zero. On this consideration, the transformed equation (51) reduces to

$$\ddot{Y}_m(t)+\gamma_{mf}^2Y_m(t)=A_0\sin\frac{k\pi ct}{L}-\frac{1}{\alpha_0^*(m,k)}\left[F_1(t)+F_2(t)+F_3(t)+F_4(t)\right] \tag{60}$$

where

$$\gamma_{mf}^2 = \frac{\alpha_1^*(m,k)}{\alpha_0^*(m,k)} \quad (61)$$

and

$$A_0 = \frac{mg}{\mu_0 \alpha_0^*(m,k)} \quad (62)$$

As in the previous seminar we consider a simply–supported beam, one of whose end $x = 0$, (say) is subjected to a sine-wave (undamped) transient displacement, starting from rest and the other end, $x = l$ is subjected to a damped sine wave transient displacement starting from rest. Consequently, we have

$$f_1(t) = BSin\Omega t \quad \text{and} \quad f_3(t) = Ae^{-\beta t} Sin\Omega t \quad (63)$$

Where A, B are amplitudes, Ω is frequency and β is parameter.
Substituting $f_1(t)$, $f_3(t)$, $g_1(x)$ and $g_3(x)$ in the initial conditions (7), one obtains

$$Z(x,0) = 0 \quad \text{and} \quad \frac{\partial Z(x,0)}{\partial t} = -\Omega \quad (64)$$

Substituting equations (63), (64) into (60) after simplifications and rearrangements, yields

$$\ddot{Y}_m(t) + \gamma_{mf}^2 Y_m(t) = A_0 Sin\frac{k\pi c}{L}t \; + C\,F_9 Sin\Omega t + C\,F_{10} Cos\Omega t + C\,F_{11} e^{-\beta t} Sin\Omega t \quad (65)$$

where

$$C_{F9} = \frac{1}{\alpha_0^0(m,k)}\left(\frac{\Omega^2 L}{k\pi} - \frac{\Omega^2 L}{2} - \frac{L}{\mu_0 \pi} - \frac{2\Omega^2 k}{\pi(1-k^2)}\left(R^0 - \frac{2L}{\pi(1-k^2)}\right)\right) \quad (66)$$

$$C_{F10} = \frac{1}{\alpha_0^x(m,k)}\left(\frac{2\beta\Omega L}{k\pi c} + \frac{4\beta\Omega k}{\pi(1-k^2)}\left(R^0 - \frac{2L}{\pi(1-k^2)}\right)\right) \quad (67)$$

$$C_{F11} = \frac{-1}{\alpha_0^x(m,k)}\left((-1)^{k+1}\frac{(\beta^2-\Omega^2)L}{k\pi} + (-1)^{k+1}\frac{L}{\mu_0\pi} + \frac{2k(\beta^2-\Omega^2)}{\pi(1-k^2)}\left(R^0 - \frac{2L}{\pi(1-k^2)}\right)\right) \quad (68)$$

while when equation (65) is solved by Laplace Transform and Convolution theory yield expression in $Y_m(t)$ thus,

$$Y_m(t) = \frac{1}{\gamma_{mf}}\left[\frac{A_0\gamma_{mf}}{\gamma_{mf}^2 - \left(\frac{K\pi c}{L}\right)^2}\left(Sin\frac{K\pi c}{L}t - \frac{K\pi c}{L\gamma_{mf}}Sin\gamma_{mf}t\right) + \frac{C_{F9}\gamma_{mf}}{2(\gamma_{mf}^2-\Omega^2)}\left(Sin\Omega t - \frac{\Omega}{\gamma_{mf}}Sin\gamma_{mf}t\right) + \frac{C_{F10}\gamma_{mf}}{(\gamma_{mf}^2-\Omega^2)}(Cos\Omega t - Cos\gamma_{mf}t)\right.$$
$$+ \frac{2C_{F11}\gamma_{mf}(\gamma_{mf}^2-\Omega^2+\beta^2)}{2\left[(\gamma_{mf}^2+\Omega^2+\beta^2)^2 - 4\gamma_{mf}^2\Omega^2\right]}e^{-\beta t}Sin\Omega t + \frac{4C_{F11}\gamma_{mf}\Omega\beta}{2\left[(\gamma_{mf}^2+\Omega^2+\beta^2)^2 - 4\gamma_{mf}^2\Omega^2\right]}e^{-\beta t}\cos\Omega t - \frac{2C_{F11}\Omega(\gamma_{mf}^2-\Omega^2-\beta^2)}{2\left[(\gamma_{mf}^2+\Omega^2+\beta^2)^2 - 4\gamma_{mf}^2\Omega^2\right]}Sin\gamma_{mf}t$$
$$\left.+ \frac{-4C_{F11}\gamma_{mf}\Omega\beta}{2\left[(\gamma_{mf}^2+\Omega^2+\beta^2)^2 - 4\gamma_{mf}^2\Omega^2\right]}Cos\gamma_{mf} + \frac{\Omega L}{k\pi}\left(1+(-1)^{k+1}\right)Sin\gamma_{mf}t\right] \quad (69)$$

and on inversion yields

$$Z(x,t) = \sum_{m=1}^{n}\left[\frac{A_0}{(\gamma_{mf}^2 - Z_0^2)}\left(SinZ_0 t - \frac{Z_0}{\gamma_{mf}}Sin\gamma_{mf}t\right) + \frac{C_{F9}}{2(\gamma_{mf}^2-\Omega^2)}\left(Cos\Omega t - \frac{\Omega}{\gamma_{mf}}Sin\gamma_{mf}t\right) + \frac{C_{F10}}{(\gamma_{mf}^2-\Omega^2)}(Cos\Omega t - Cos\gamma_{mf}t)\right.$$
$$\left.+ \frac{C_{F11}Q_2}{Q_0}e^{-\beta t}Sin\Omega t + \frac{2C_{F11}\Omega\beta}{Q_0}e^{-\beta t}Cos\gamma_{mf}t + \left(\frac{\Omega L}{K\pi}\left(1+(-1)^{k+1}\right) - \frac{C_{F11}\Omega Q_3}{Q_0}\right)\frac{1}{\gamma_{mf}}Sin\gamma_{mf}t - \frac{2C_{F11}\Omega\beta}{Q_0}Cos\gamma_{mf}t\right]Sin\frac{m\pi x}{L} \quad (70)$$

Where

$$Q_0 = \left[\left(\gamma_{mf}^2 + \Omega^2 + \beta^2\right)^2 - 4\gamma_{mf}^2\Omega^2\right], \quad Q_1 = \left(\gamma_{mf}^2 + \Omega^2 + \beta^2\right),$$
$$Q_2 = \left(\gamma_{mf}^2 - \Omega^2 + \beta^2\right), \quad Q_3 = \left(\gamma_{mf}^2 - \Omega^2 - \beta^2\right) \quad (71)$$

$$Z_0 = \frac{k\pi c}{L} \tag{72}$$

But From equation (9),

$$U(x,t) = Z(x,t) + \sum_{i=1}^{4} f_1(t) g_i(x)$$

Consequently,

$$U(x,t) = Z(x,t) + Sin\Omega t + \left(e^{-\beta t} - 1\right)\frac{x}{L} Sin\Omega t. \tag{73}$$

where $Z(x,t)$ is as given in equation (70).

Equation (73) is the dynamic response of the non- uniform Rayleigh beam to moving force whose two simply–supported edges undergo displacements which vary with time.

3.2.2 Simply Supported Non-Uniform Rayleigh Beam Traversed by Moving Mass

In this case, the moving load has mass commensurable with that of the beam. Consequently, $\varepsilon \neq 0$. As mentioned earlier, there is no exact analytical solution to this problem. Thus, we resort to the modified Struble's asymptotic technique. In order to solve equation (51), it is rearranged to take the form

$$\sum_{m=1}^{n}\left[\ddot{Y}_m(t) + \frac{\varepsilon Q_2(t)}{\left[\alpha_o^x(m,k) + \varepsilon Q_1(t)\right]}\dot{Y}_m(t)\right] + \frac{\alpha_1^x(m,k) + \varepsilon Q_3(t)}{\left[\alpha_o^x(m,k) + \varepsilon Q_1(t)\right]} Y_m(t)$$

$$= \frac{\varepsilon L}{\left[\alpha_o^x(m,k) + \varepsilon Q_1(t)\right]}\left[g \sin k\frac{\pi c t}{L} - F_1^0(t) + F_2^0(t) + F_3^0(t) + F_4^0(t) + F_5(t) + F_6(t) + F_7(t) + F_8(t)\right] \tag{74}$$

where

$$F_1^0(t) = \frac{L\mu_o}{k\pi m}\left(\overset{\infty}{f}_1(t) + (-1)\right)^{k+1}\overset{\infty}{f}_3(t), \qquad F_2^0(t) = \frac{L\mu_o}{2m}\overset{\infty}{f}_1(t),$$

$$F_3^0(t) = \frac{L}{m\pi}\left(f_1(t) + (-1)^{k+1} f_3(t)\right), \qquad F_4^0(t) = \frac{\mu_o}{m}\left(\frac{2R^o K}{\pi(1-k^2)} - \frac{4LK}{\pi^2(1-k^2)^2}\right)\left(\overset{\infty}{f}_3(t) - \overset{\infty}{f}_1(t)\right) \tag{75}$$

and

As in the previous section, the homogeneous part of equation (74) is first considered as a modified frequency corresponding to the frequency of the free system due to the presence of the moving mass is sought. An equivalent free system operator defined by the modified frequency then replaces equation (74). To do this, consider a parameter $\lambda < 1$ for any arbitrary mass ratio ε defined as.

$$\lambda = \frac{\varepsilon}{1+\varepsilon} \tag{76}$$

It can be shown that $\varepsilon = \lambda + 0(\lambda^2)$ (77)

All the various time dependent coefficients of the differential operator which acts on $Y_m(t)$ in equation (74) can be written in terms of λ when one considers that to $0(\lambda)$.

$$\varepsilon = \lambda \tag{78}$$

Thus,

$$\frac{1}{\left[\alpha_o^x(m,k) + \lambda\frac{L}{2}\left(1 + 4\sum_{m-1}^{\infty}\sin\frac{k\pi ct}{L}\sin\frac{m\pi ct}{L}\right)\right]} = \frac{1}{\alpha_o^x(m,k)}\left[1 - \frac{1}{\alpha_o^x(m,k)}\lambda\frac{L}{2}\left(1 + 4\sum_{m-1}^{\infty}\sin\frac{k\pi ct}{L}\sin\frac{m\pi ct}{L}\right) + 0(\lambda^2) + \ldots\ldots\ldots\ldots\right] \tag{79}$$

where

$$\left|\frac{1}{\alpha_o^x(m,k)}\lambda\frac{L}{2}\left(1 + 4\sum_{m-1}^{\infty}\sin\frac{k\pi ct}{L}\sin\frac{m\pi ct}{L}\right)\right| < 1 \tag{80}$$

Now, using (78) and (79) in (74), one obtains.

$$\sum_{m=1}^{n}\left[\ddot{Y}_m(t) - \frac{\lambda Q_2(t)}{\alpha_0^*(m,k)}\dot{Y}_m(t) + \gamma_{mf}^2\left(1 - \frac{\lambda}{\alpha_0^*(m,k)}\left(Q_1(t) - \frac{Q_3(t)}{\gamma_{mf}^2}\right)\right)Y_m(t)\right]$$

$$= \frac{\lambda L}{\alpha_0^x(m,k)}\left[g \sin\frac{k\pi ct}{L} - \left(F_1^o(t) + F_2^0(t) + F_3^o(t) + F_4^o(t) + F_5(t) + F_6^o(t) - F_7(t) - F_8(t)\right)\right] \tag{81}$$

when $\lambda = 0$, a case corresponding to the case when the inertial effect of the mass of the systems is neglected is obtained. In such a case, the solution is of the form.

$$\mathrm{Y}_m(t) = C_1 \cos(\gamma_m(t) - \phi_m) \tag{82}$$

where C_1 and ϕ_m are constants (83)

Since $\lambda < 1$ for any arbitrary mass ratio $\in$, Strubble's techniques requires that the solution of the homogenous part of equation (74) takes the form,

$$\mathrm{Y}_m(t) = \phi(m,t)\cos[\gamma_{mf}(t) - \psi(m,t)] + \lambda \mathrm{Y}_1(t) + 0(\lambda^2) \tag{84}$$

where $\phi(m,t)$ and $\psi(m,t)$ are slowly varying function of time. The modified frequency is obtained by substituting equation (84) and its derivatives into the homogenous part of equation (81). The resulting variational equations describing the behavior of $\phi(m,t)$ and $\psi(m,t)$ during the motion of the mass determine the modified frequency.

To this end, substituting (84) and it derivative into (81) in conjunction with the expanded expressions in equations (78) and (79) yields

$$-2\gamma_{mf}\,\dot\phi(m,t)\sin[\gamma_{mf}t - \psi(m,t)] + 2\gamma_{mf}\,\dot\psi(m,t)\phi(m,t)\cos[\gamma_{mf}\,t - \psi(m,t)] - \frac{4\lambda c m k \gamma_{mf}\,\phi(m,t)}{\alpha_0^*(m,k)(m^2 - k^2)}\sin[\gamma_{mf}t - \psi(m,t)]$$
$$-\frac{8\lambda c m k \gamma_{mf}\sum_{n=1}^{\infty}(n^2 + m^2 - k^2)\phi(m,t)\cos\frac{n\pi\,\pi ct}{L}\sin[\gamma_{mf}t - \psi(m,t)]}{\pi\alpha_o^*(m,k)[n^2 - (m-k)^2][n^2 - (m+k)^2]} - \frac{\lambda\,\gamma_{mf}^2}{\alpha_o^*(m,k)}\left[\left(\frac{c^2 m^2\pi^2}{L^2\gamma_{mf}^2} + 1\right)\frac{L}{2}\phi(m,t)\cos[\gamma_{mf}t - \psi(m,t)]\right.$$
$$+\left(\frac{c^2 m^2 \pi^2}{L^2\gamma_{mf}^2} + 1\right)\frac{L}{2}\phi(m,t)4\sum_{n=1}^{\infty}\sin\frac{k\pi\,ct}{L}\sin\frac{m\pi\,ct}{L}\cos[\gamma_{mf}t - \psi(m,t)] + \lambda_1\,y_1(t) = 0 \tag{85}$$

retaining terms to order $0(\lambda)$ only.

At this juncture, the following trigonometric identities are noted, namely

$$\cos\frac{n\pi ct}{L}\sin[\gamma_{mf}t - \psi(m,t)] = \frac{1}{2}\left[\sin\left(\frac{n\pi ct}{L} + \gamma_{mf}t - \psi(m,t)\right) - \sin\left(\frac{n\pi ct}{L} - \gamma_{mf}t + \psi(m,t)\right)\right] \tag{86}$$

and

$$\sin\frac{k\pi ct}{L}\sin\frac{m\pi ct}{L}\cos[\gamma_{mf}t - \psi(m,t)] = \frac{1}{4}\left[\cos\left\{(m-k)\frac{\pi ct}{L} + \gamma_{mf}t + \psi(m,t)\right\}\right.$$
$$+\cos\left\{(m-k)\frac{\pi ct}{L} - \gamma_{mf}t + \psi(m,t)\right\} - \cos\left\{(m+k)\frac{\pi ct}{L} - \gamma_{mf}t - \psi(m,t)\right\}$$
$$\left.-\cos\left\{(m+k)\frac{\pi ct}{L} - \gamma_{mf}t + \psi(m,t)\right\}\right] \tag{87}$$

Since only the terms involving $\sin\{\gamma_{mf}t + \psi(m,t)\}$ and $\cos\{\gamma_{mf}t - \psi(m,t)\}$ contribute to the variational equations describing the behavior of $\phi(m,t)$ and $\psi(m,t)$, in view of the identities (86) and (87) equation (85) reduces to

$$-2\gamma_{mf}\dot\phi(m,t)\sin[\gamma_{mf}t - \psi(m,t)] - \frac{4\lambda c m k \gamma_{mf}\phi(m,t)}{\alpha_o^*(m,k)(m^2 - k^2)}\sin[\gamma_{mf}t - \psi(m,t)]$$
$$+2\gamma_{mf}\dot\psi(m,t)\phi(m,t)\cos[\gamma_{mf}t - \psi(m,t)] - \frac{\lambda\gamma_{mf}^2}{\alpha_o^*(m,k)}\left(\frac{c^2 m^2\pi^2}{L^2\gamma_{mf}^2} + 1\right)\frac{L}{2}\phi(m,t)\cos[\gamma_{mf}t - \psi(m,t)] = 0 \tag{88}$$

The variational equations of our problem are obtained by setting the coefficients of $\sin\{\gamma_{mf}t - \psi(m,t)\}$ and $\cos\{\gamma_{mf}t - \psi(m,t)\}$ to zero and solved to obtain

$$\phi(m,t) = e^{\left(\frac{-2\lambda c m k}{\alpha_o^*(m,k)(m^2 - k^2)}\right)tk_1} \quad \text{and} \quad \Psi(m,t) = \lambda\gamma_{mf}\left[\left(\frac{c^2 m^2\pi^2}{L\gamma^2{}_{mf}} + L\right)\frac{1}{4\alpha_0^x(m,k)}\right]t + \Psi_m \tag{89}$$

respectively, where k_1, k_2 and Ψ_m are constants of integration.

Therefore, when the effect of the mass of the particle is considered, the first approximation to the homogenous system is

$$Y_m(t) = \phi(m,t)\cos[\gamma_m t - \Psi_m] \quad \text{where} \quad \therefore \gamma_m = \gamma_{mf}\left[1 - \lambda\left(\frac{c^2 m^2\pi^2}{L\gamma_{mf}{}^2} + L\right)\frac{1}{4\alpha_0{}^*(m,k)}\right] \tag{90}$$

is called the modified natural frequency representing the frequency of the free system due to the presence of the moving mass. Thus, the homogeneous part of (81) can be written as

$$\ddot{Y}(t)+\gamma_m^2 Y_m(t)=0 \tag{91}$$

Hence, the entire equation (74) taking into account (90) takes the form

$$\ddot{Y}(t)+\gamma_m^2 Y_m(t) = \frac{\lambda L}{\alpha_0^x(m,k)}\left[g\sin\frac{k\pi c t}{L}-\left(F_1^o(t)+F_2^0(t)+F_3^o(t)+F_4^o(t)+F_5(t)+F_6^o(t)-F_7(t)+\_F_8(t)\right)\right] \tag{92}$$

In order to solve the non-homogeneous equation (92), it is first simplified and re arranged, using trigonometric identities and the standard formulae, some 36 integrals are evolved and evaluated to obtain

$$\begin{aligned} Y_m(t) &= A_{10}\sin z_o t + A_{20}\sin\gamma_m t + A_{30}\sin\Omega t + A_{40}\cos\gamma_m t + A_{50}e^{-\beta t}\cos\Omega t \\ &+ A_{60}e^{-\beta t}\sin\Omega t + A_{70}\cos(z_o-\Omega)t + A_{80}(z_o+\Omega)t + A_{90}e^{-\beta t}\cos\gamma_m t + A_{91}e^{-\beta t}\sin\gamma_m t \\ &+ A_{92}e^{-\beta t}\sin(z_o-\Omega)t + A_{93}e^{-\beta t}\sin(z_o+\Omega)t + A_{94}e^{-\beta t}\cos(z_o+\Omega)t + A_{95}\sin(z_o+\Omega)t \\ &+ A_{96}\sin(z_o-\Omega)t + A_{97}e^{-\beta t}\cos(z_o+\Omega)t + A_{98}e^{-\beta t}\sin(z_o+\Omega)t + A_{99}e^{-\beta t}\cos(z_1-\Omega)t \\ &+ A_{991}e^{-\beta t}\sin(z_1-\Omega)t + A_{992}\sin(z_1+\Omega)t + A_{993}\sin(z_1-\Omega)t \end{aligned} \tag{93}$$

where $A_{10} = \dfrac{-T_1}{\gamma_m^2 - z_o^2}$,

$$\begin{aligned} A_{20} = \frac{1}{\gamma_m}\Bigg[& \frac{-T_1 z_o}{\gamma_m^2-z_o^2}+\frac{-T_2\Omega}{\gamma_m^2-\Omega}+\frac{1}{Q_o}\left(T_3\Omega\left(\beta^2+\Omega^2-\gamma_m^2\right)+T_4\beta\left(\beta^2+\gamma_m^2+\Omega^2\right)\right)-T_6\left(\frac{(z_o+\Omega)}{Q_1}\left(\beta^2+(z_o+\Omega)^2-\gamma_m^2\right)-\frac{(z_o-\Omega)}{Q_2}\left(\beta^2+(z_o-\Omega)^2-\gamma_m^2\right)\right) \\ & -T_7\beta\left(\frac{\left(\beta^2+\gamma_m^2+(z_o+\Omega)^2\right)}{Q_2}-\frac{\left(\beta^2+\gamma_m^2(z_o+\Omega)^2\right)}{Q_1}\right)+T_6\left(\frac{z_o+\Omega}{\gamma_m^2+(z_o+\Omega)^2}+\frac{z_o-\Omega}{\gamma_m^2-(z_o-\Omega)^2}\right)-T_9\left(\frac{(z_1+\Omega)\left(\beta^2+(z_1+\Omega)^2-\gamma_m^2\right)}{Q_3}-\frac{(z_1-\Omega)\left(\beta^2+(z_1-\Omega)^2-\gamma_m^2\right)}{Q_4}\right) \\ & +T_{10}\beta\left(\frac{\left(\beta^2+\gamma_m^2+(z_1+\Omega)^2\right)}{Q_3}+\frac{\left(\beta^2+\gamma_m^2+(z_1-\Omega)^2\right)}{Q_4}\right)+T_{11}\left(\frac{z_1+\Omega}{\gamma_m^2-(z_o+\Omega)^2}-\frac{(z_1-\Omega)}{\gamma_m^2-(z_1-\Omega)^2}+C^o\right)\Bigg] \end{aligned}$$

$$A_{30} = \frac{T_2}{\gamma_m^2-\Omega^2}$$

$$\begin{aligned} A_{40} = \Bigg[& \frac{1}{Q_o}\left(2\beta\Omega T_3-\left(\beta^2+\gamma_m^2-\Omega^2\right)\right)+T_s\left(\frac{1}{\gamma_m^2-(z_o-\Omega)^2}+\frac{1}{\gamma_m^2-(z_o+\Omega)^2}\right)+2\beta T_6\left(\frac{z_o+\Omega}{Q_1}-\frac{z_o-\Omega}{Q_2}\right) \\ & +T_7\left(\frac{\left(\beta^2+\gamma_m^2-(z_o-\Omega)^2\right)}{Q_2^!}-\frac{\left(\beta^2+\gamma_m^2-(z_o+\Omega)^2\right)}{Q_1^!}\right)+2T_9\beta\left(\frac{z_1+\Omega}{Q_3^!}-\frac{z_1-\Omega}{Q_4^!}\right)+T_{10}\left(\frac{\left(\beta^2+\gamma_m^2-(z_1+\Omega)^2\right)}{Q_3^!}-\frac{\left(\beta^2+\gamma_m^2-(z_1-\Omega)^2\right)}{Q_4^!}\right)\Bigg] \end{aligned}$$

$$A_{50} = \frac{1}{Q_0^!}\left(T_3+T_4\left(\beta^2+{}^2-\gamma_m^2-\Omega^2\right)\right) \qquad A_{60} = \frac{1}{Q_0^!}\left(T_3+\left(\beta^2+{}^2-\gamma_m^2-\Omega^2\right)-2\beta\Omega T_4\right), \qquad A_{70} = \frac{-T_5}{\gamma_m^2-(z_o-\Omega)^2}, \qquad A_{80} = \frac{-T_5}{\gamma_m^2-(z_o+\Omega)^2}$$

$$A_{90} = \left[-2\beta T_6\left(\frac{z_o+\Omega}{Q_1^!}+\frac{z_o-\Omega}{Q_2^!}\right)-2\beta T_9\left(\frac{z_1+\Omega}{Q_3^!}+\frac{z_1-\Omega}{Q_4^!}\right)\right]$$

$$A_{91} = \frac{1}{\alpha_m}\left[T_6\left(\frac{(z_o+\Omega)\left(\beta^2+(z_o+\Omega)^2-\gamma_m^2\right)}{Q_1^!}-\frac{(z_o-\Omega)\left(\beta^2+(z_o+\Omega)^2-\gamma_m^2\right)}{Q_2^!}\right)+T_9\left(\frac{(z_o+\Omega)\left(\beta^2+(z_1+\Omega)^2-\gamma_m^2\right)}{Q_3^!}-\frac{(z_o-\Omega)\left(\beta^2+(z_1+\Omega)^2-\gamma_m^2\right)}{Q_4^!}\right)\right]$$

$$A_{92} = \frac{2\beta T_7(z_o-\Omega)}{Q_2^1}, \qquad A_{93} = \frac{-2\beta T_7(z_o+\Omega)}{Q_1^1} \qquad A_{94} = -T_7\left[\frac{\left(\beta^2+\gamma_m^2-(z_o-\Omega)^2\right)}{Q_2^1}+\frac{\left(\beta^2+\gamma_m^2-(z_o+\Omega)^2\right)}{Q_1^1}\right] \quad A_{95} = \frac{-T_6}{\gamma_m^2-(z_o+\Omega)^2}, \qquad A_{96} = \frac{-T_6}{\gamma_m^2-(z_o-\Omega)^2}$$

$$A_{97} = \frac{T_{10}\left(\beta^2+\gamma_m^2-\left(z_1+\Omega^2\right)\right)}{Q_3}, \qquad A_{98} = \frac{-2\beta\left(z_1+\Omega^2\right)T_{10}}{Q_3} \qquad A_{99} = \frac{T_{10}\left(\beta^2+\gamma_m^2-\left(z_1-\Omega^2\right)\right)}{Q_4^!}, \quad A_{991} = \frac{-2\beta\left(z_1-\Omega^2\right)T_{10}}{Q_4^!} \quad A_{992} = \frac{-T_{11}}{\gamma_m^2-(z_1+\Omega)^2}$$

and

$$A_{993} = \frac{T_{11}}{\gamma_m^2-(z_1-\Omega)^2} \tag{94}$$

where

$$\begin{aligned} Q_o^* &= \beta^4+\gamma_m^4+\Omega^4+2\left[\beta^2\gamma_m^2+\beta^2\Omega^2-\gamma_m^2\Omega^2\right] \\ Q_1^* &= \beta^4+\gamma_m^4+(z_o+\Omega)^4+2\left[\beta^2\gamma_m^2+\beta^2(z_o+\Omega)^2-2\gamma_m^2(z_o+\Omega)^2\right] \\ Q_2^* &= \beta^4+\gamma_m^4+(z_o-\Omega)^4+2\left[\beta^2\gamma_m^2+\beta^2(z_o-\Omega)^2-2\gamma_m^2(z_o-\Omega)^2\right] \\ Q_3^* &= \beta^4+\gamma_m^4+(z_1+\Omega)^4+2\left[\beta^2\gamma_m^2+\beta^2(z_1+\Omega)^2-2\gamma_m^2(z_1+\Omega)^2\right] \end{aligned}$$

$$Q_4^* = \beta^4 + \gamma_m^4 + (z_1 - \Omega)^4 + 2\left[\beta^2\gamma_m^2 + \beta^2(z_1-\Omega)^2 - 2\gamma_m^2(z_1-\Omega)^2\right] \quad (95)$$

Therefore,

$$Z_n(x,t) = \sum_{m=1}^{n} Y_m(t)\sin\frac{m\pi x}{L} \quad (96)$$

Consequently in view of equation (9),

$$U(x,t) = Z(x,t) + \sum_{l=1}^{t} f_i(t)g_i(t)$$

$$= Z(x,t) + \sin\Omega t + \left(e^{-\beta t} - 1\right)\left(x/L\right)\sin\Omega t \quad (97)$$

Equation (97) is the dynamic response of a non-uniform Rayleigh beam to Moving Mass whose two simply supported edges undergo displacements which vary with time.

4.0 Discussion of the Analytical Solution

If the undamped system such as this is studied, it is desirable to examine the response amplitude of the dynamical system which may grow without bound. This is termed resonance when it occurs. Equation (61) clearly shows that the simply supported elastic Rayleigh beams transverse by a moving force will be in state of resonance whenever

$$\gamma_{mf} = \frac{m\pi u}{L} \quad (98)$$

While equation (90) shows that the same beam under the action of moving mass experiences resonance effect whenever

$$\gamma_m = \frac{m\pi u}{L} \quad (99)$$

From equation (90), it implies,

$$\gamma_{mf} = \frac{\frac{m\pi u}{L}}{1 - \lambda\left(\frac{u^2 m^2 \pi^2}{L\gamma_{mf}^2} + L\right)\frac{1}{4\alpha_0^*(m,k)}} \quad (100)$$

From equations (90) and (100), we deduced that for the same natural frequency, the critical speed for the system of a simply supported elastic beam on an elastic foundation and traversed by a moving force is greater than that traversed by moving mass. Thus, resonance is reached earlier in the moving mass system than in the moving force system.

4.0.1 Numerical Calculation and Discussion of the Results for Simply-Supported Non-Uniform Rayleigh **Beam.**

To illustrate the foregoing analysis, the Non-uniform Rayleigh beam of length L=12.192m, is considered. Furthermore, the load velocity u = 8.123, $E = 2.109x10^9 kg/m$, $\frac{EI}{\mu} = 2200m^4/s^2$ and the ratio of the mass of the load to mass of the beam is 0.25. The values of the foundation modulli K are varied between 0 and 400000units while the values of axial force N are varied between 0 and 40000.The traverse deflections of the non-uniform Rayleigh beam are calculated and plotted against time for various values of rotatory inertia r^o, axial force N and foundation stiffness K.

Fig.1, displays the transverse displacement response to a moving force of simply supported non- uniform Rayleigh beam for various values of foundation modulli K and for fixed value of rotatory inertia r^o, and axial force N. The graph shows that the response amplitude of the beam decreases as the values of the foundation modulli K increases. Fig.2 also shows the deflection profile due to moving force of a simply supported non-uniform Rayleigh beam for fixed value of foundation modulli K and axial force N and for various values of rotatory inertia r^o.

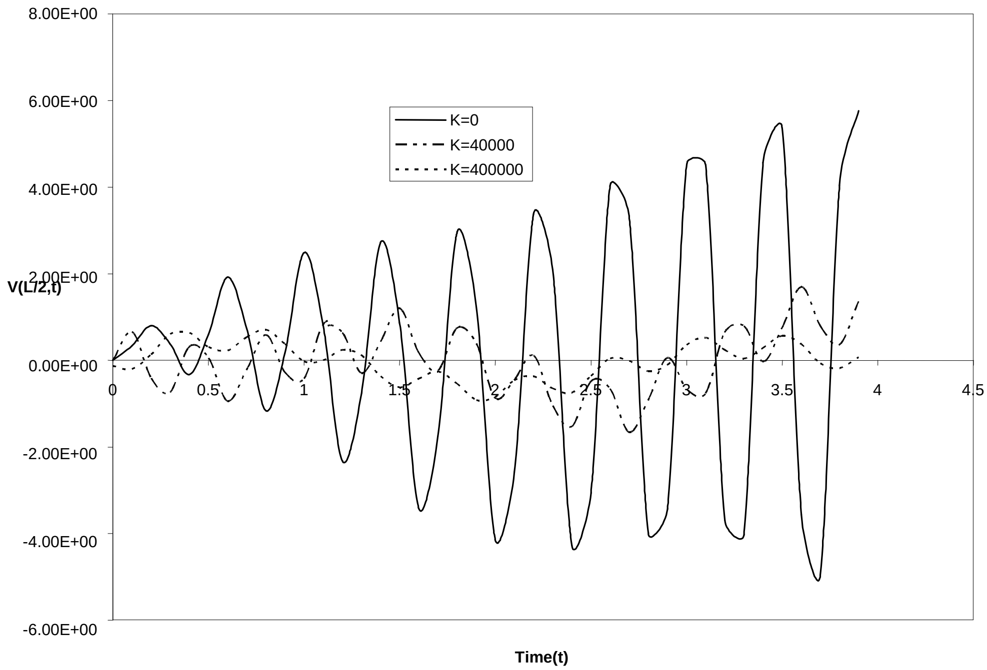

Fig.1: Deflection profile of a simply supported non-uniform Rayleigh beam under moving force for various values of foundation modulli K and for fixed value of axial force N and rotator inertia r(1)

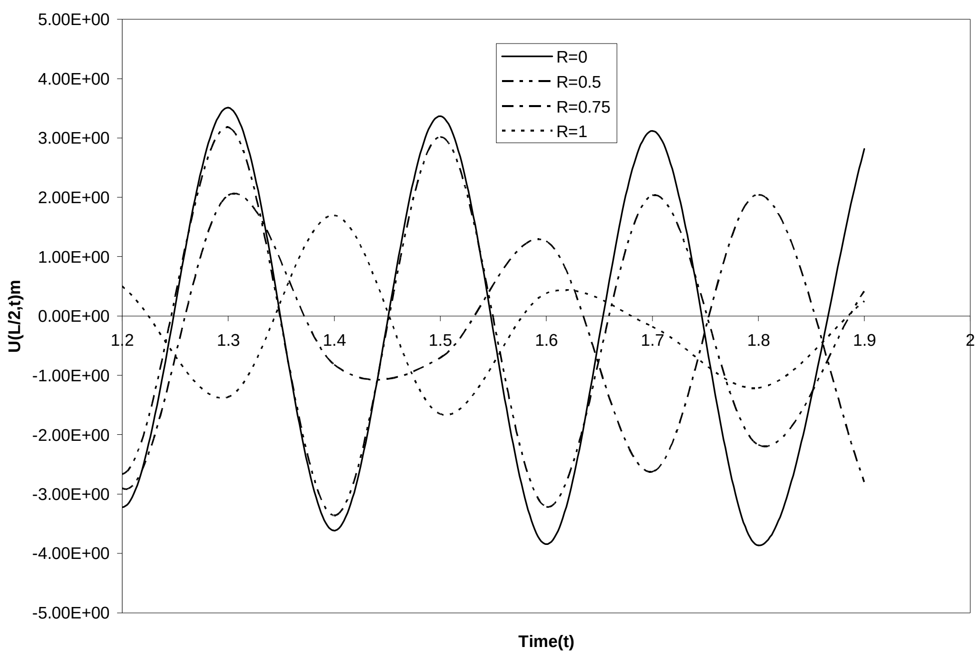

Fig 2: Deflection profile of the simply surported non-uniform Rayleigh beam under a moving force for various values of rotatory inertia and for fixed value of foundation K = 40000

The graph shows that the response amplitude of the beam decreases as the values of the rotatory inertia correction factor r^o are increased. Furthermore, fig.3 shows the deflection profile of simply supported Non-Uniform Rayleigh beam under the action of moving force for various values of axial force N and fixed value of rotatory inertia r^o and foundation modulus K. The graph shows that as the value of axial force N increases the displacement response of the beam decreases. In fig.4, the deflection profile of simply supported Non-Uniform Rayleigh beam under the action of moving mass for various values of axial force and fixed values of rotatory inertia r^o and foundation modulus K is shown. Evidently, as the axial force increases

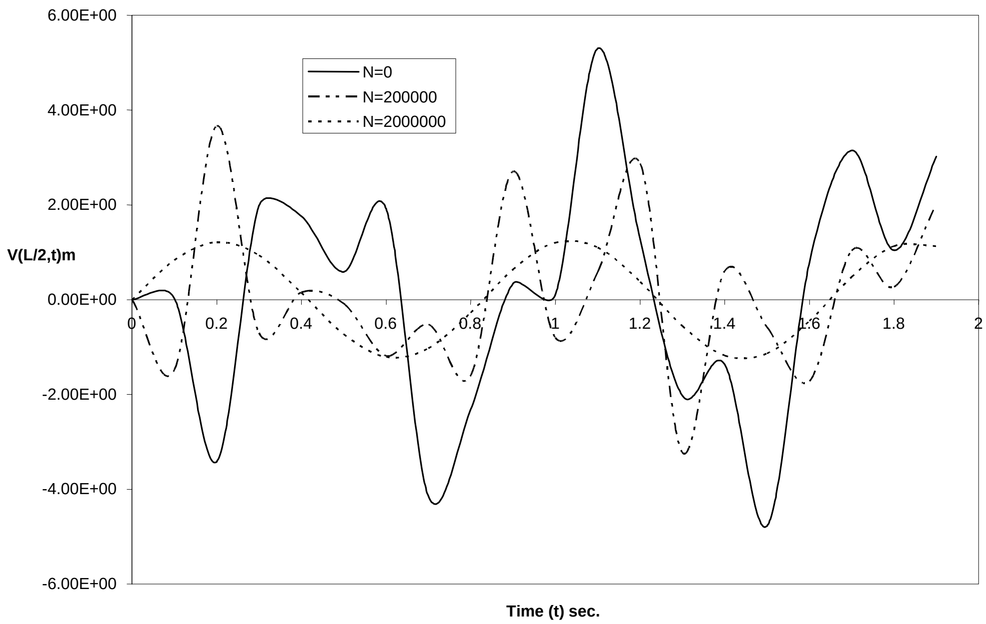

Fig3.Deflection profile of simply supported Non-Uniform Rayleigh beam under the action of moving force for various values of axial force N and for fixed value of rotatory inertia r(1) and for fixed value of foundation modulus K(40000).

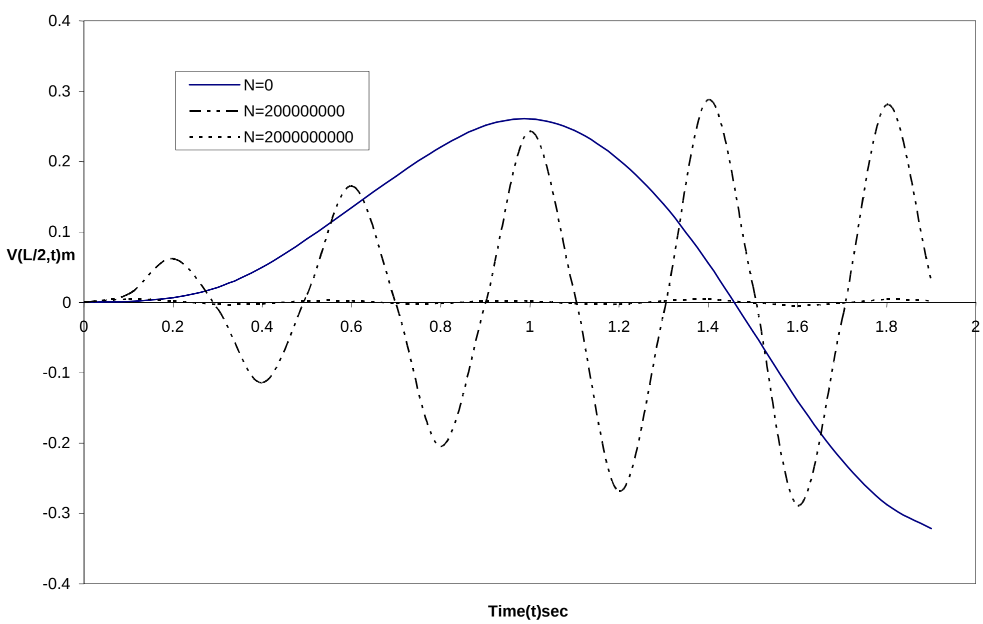

Fig4: Deflection Profile of a simply supported Rayleigh beam under the actions of moving mass for various values of axial force N and for fixed values of Rotatory inertia r(1) and foundation modulus K(40000)

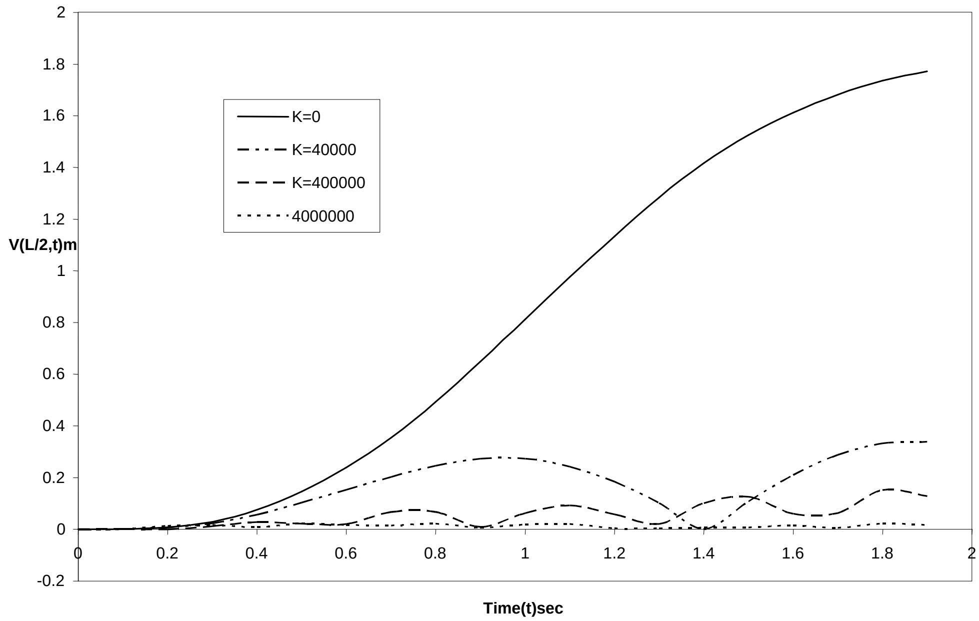

Fig.5: Deflection profile of a simply supported non-uniform Rayleigh beam under the action of moving mass for various values of foundation modulus K and for fixed values of axial force N(20000) and rotatory inertia r(1)

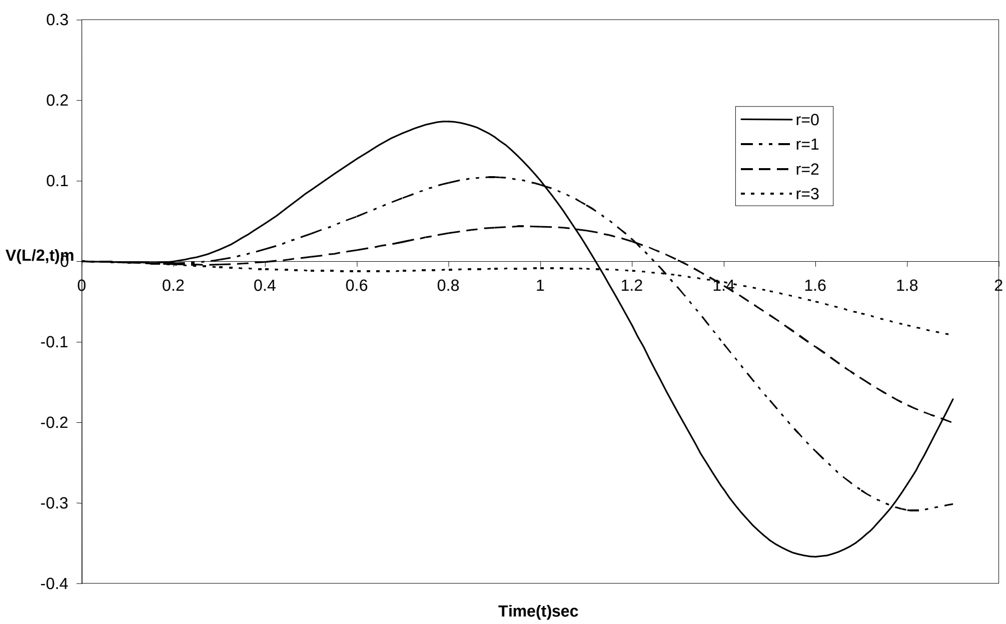

Fig .6: Deflection profile for simply supported non-uniform Raylegih beam under the action of moving mass for various values of rotatory inertia and for fixed values of axial force N(20000) and foundation modulus K(40000)

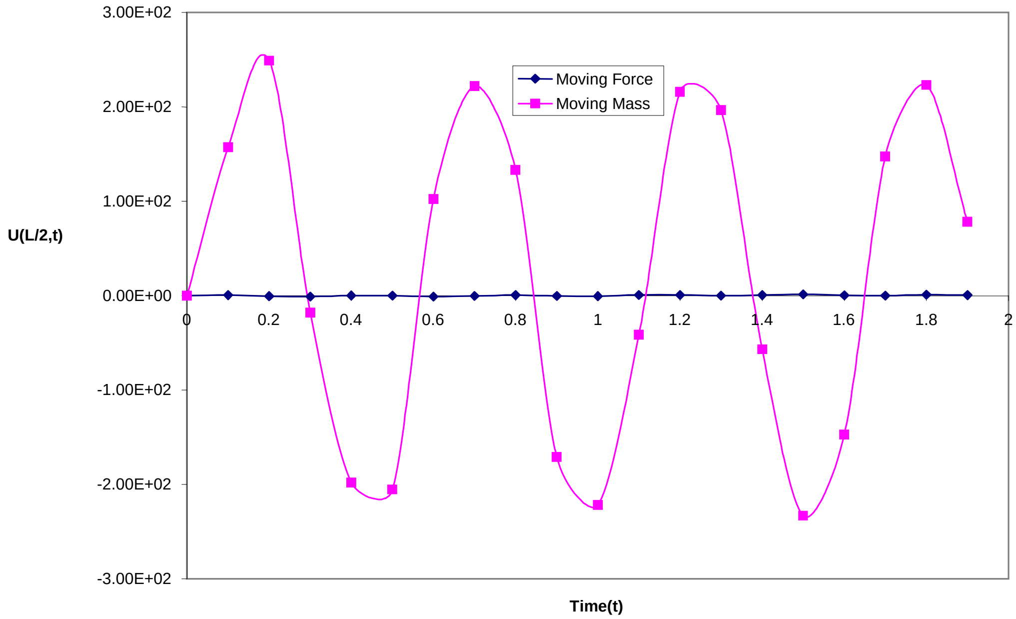

Fig. 7: Comparison of the displacement response of moving force and moving mass cases for Simply supported non-uniform beam for fixed axial force N(40000), foundation

modulus K(40000) and rotatory inertia r(1)

the response amplitude of the beam decreases. In fig.5, the transverse displacement response to a moving mass of simply supported non-uniform Rayleigh beam for fixed values of rotatory inertia r^o and axial force N and for various values of foundation moduli K is displayed. From the graph it is shown that the response amplitude of the beam decreases as the values of the foundation moduli K increases. Furthermore, fig.6,the depicts deflection profile of simply supported Non-Uniform Rayleigh beam under the action of moving mass for various values of rotatory inertia and for fixed values of axial force N and foundation modulus K. The graph shows that the response amplitude of the beam decreases as the values of rotatory inertia r^o increases.

Finally, fig.7 shows the comparison of the transverse displacement of the moving force and moving mass cases of simply supported Non-uniform Rayleigh beams with time dependent boundary conditions under the action of moving loads for fixed values of foundation moduli K, axial force N and rotatory inertia r^o. It is also evidently confirmed that the moving force solution is not an upper bound for the accurate solution of the moving mass problem.

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