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A Mathematical Model for the Interception of a Moving Target

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Abstract

In this paper, a mathematical model for the interception of a target governed by a lod control system is derived. The condition for interception is stated. The interception criterion is the intersection of certain well defined set functions. This condition is equivalent to the controllability of the linear control system and at the same time resolve the optimal control of the system.

1.0 Introduction:

Every life problem has a mathematical dimension associated to it. The necessity to translate a real life problem into a mathematical model therefore cannot be overemphasized. Most of these models are governed by differential systems whose solutions provide clues leading to breakthroughs to real life problems. Population, Economics, disease control and technological models abound in the literature. See [1,2,3].

From simple population models such as

$$\frac{dp}{dt} = kp \tag{1.1}$$

(k constant, describing the rate of increase of population) we have more complex models governed by control systems such as

$$x(t) = A(t)x(t) + B(t)u(t) \tag{1.2}$$

With the realization that most action in life are not instantaneous, that is, causes do not produce their effects immediately, there is therefore need to incorporate time delays in our models giving rise to delay system. These models are found in the study of nuclear reactor dynamics and technological dynamics where the decision in the control function are often shifted or twisted before affecting the evolution. In the study of dynamics of diseases, Yorke [10] obtained the following model for the control of measles.

$$\dot{x}(t) = -B(t)x(t)\,[2\pi + x(t-14) - x(t-12) + z] \tag{1.3}$$

where x(t) denotes the number of susceptible individuals that have not yet been exposed to the disease, $h_1 = 12$, $h_2 = 14$ are delays (incubation periods in days). Quite recently, Chukwu in [1] [2] [3] and Onwuatu and Iheagwam in [7] have provided economic models governed by neutral differential systems for the control of the capital stock of nations.

Models of pursuit games have motivated interest in the study of capture problems and rescue operations. Markus and Sell, reported in Gahl [5], have obtained conditions under which a derelict spaceship drifting in some astronomical system could be saved by a rescue ship. The operation dynamics furnished a nonlinear equation given by.

$$\dot{x}(t) = k(t), x(t), u(t) \tag{1.4}$$

where x(t) is the state of the rescue ship and u(t); the engine thrust. It was found that the conditions for rescue of the derelict ship collapsed to the controllability of system (1.4).

In military quarters where the interception of enemy advances is a common feature (interceptions of energy missiles and menacing aircrafts) the questions that readily come to mind are:

What are the state and energy requirements of the weapon for interception operations? In a situation requiring pursuit, what is the state trajectory?

This study attempts a mathematical response to these and other related questions.

2. Systems Description and Preliminaries

Let E denote the real line.For positive integers n, E^n denotes the space of real n tupples with the usual Euclidean norm $|.|$ and C([a,b], E^n) is the Banach Space. of continuous function from the interval [a,b] into E^n with the topology of

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uniform convergence. The norm of ϕ in $C([a,b]\ E^n)$ is given by

$$\|\phi\| = \sup_{a\le x\le b} |\phi(x)|$$

In this paper, the state space will be E^n or $C([-h,0], E^n)$, the control space will be $L_2\ ([0,\infty), E^n)$. The control set will be a closed and bounded subset U of L_2 with values in

$$C^m = \left\{u : u \in E^m, \left|u_j\right| \le 1, \qquad j = 1,\cdots m\right\}$$

The target G(t) may be a moving point set or a compact set function in the appropriate space.
Consider the system of interest of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1}$$

on $J = [0,\infty)$; $x(t_o) = x_0 \in E^n$ or $\phi(t_o) \in E^n$

A is an nxn constant matrix and B is an nxm constant matrix. Consider the homogeneous part of (2.1)

$$\dot{x}(t) = Ax(t) \tag{2.2}$$

The solution is given by $x(t,\phi,0)$ where $x(t,\phi,0) = X(t,s)\ x(t_o)$. X(t,s) is the fundamental matrix solution of (2.2), that is, X(t,s) satisfies

$$\frac{\partial X(t,s)}{\partial t} = AX(t,s); \qquad t. > 0$$

and $X(t,s) = \begin{cases} 0 & \text{for } t < s \\ I & \text{for } t = s \end{cases}$

I is the identity matrix.
The variation of constant formula for system (2.1) is thus

$$x(t) = X(t,t_o)x(t_o) + X(t,t_o)\int_{t_o}^{t} X(t_o,s)B(s)u(s)ds \tag{2.3}$$

From equation (2.3), we extract the attainable set, that is, the set of all possible solutions of system (2.1) given as $\{x(t,u);\ u \in U\}$

Definition 2.1:

System (2.1) is Euclidean controllable if there exists a control $u \in U$ which can steer the solution x(t) with $x(t_o) = x_o$, to $x(t_1) = x_1$ for $x_1 \in E^n$ in finite time interval $[t_o,t_1]$, $t_1>t_o$.

Theorem 2.1:

The attainable set Ậ(t) is convex and compact

Proof:

The convexity of Ậ(t) follows trivially from the convexity of the control set U. To show that Ậ(t) is bounded, we let the set S to be a convex and compact subset of the space C of continuous functions. Since $x(t,\phi,0)$ is continuous, $x(t,\phi,0)$ is bounded. Also since $X(t,t_o)$, and B(t) are integrable and $u(t) \in U$, Ậ(t) is bounded. From the weak compactness argument in [3] and the compactness of S. it is clear that Ậ(t) is closed in E^n. Thus, the boundedness and closedness of Ậ(t) establishes its compactness.

3. Results

3.1 Formulation of Model

In this section, we shall formulate a mathematical model for the interception of a moving target. Models of this type are expected to represent games of pursuit. They are even more relevant in military adventures in the interception of bombs, missiles and menacing aircrafts.

Let G(t) be a continuously moving target and x(t), the pursuer's position at any time t, which is often referred to as the state. Let the distance between the target and the state be D(t) for any time t. Assume D(t) is decreasing with increasing time, that is if

$t_o \le t_1 \le \ldots \le t_n$, then $D(t_o) > D(t_1) > \ldots \ge D(t_n)$

Assume further that the rate of change of state, $\dot{x}(t)$ varies as the distance D(t)., that is

$$\dot{x}(t) \propto D(t)$$

This implies

$$\dot{x}(t) = KD(t) \tag{3.1}$$

K is the constant of variation.
Define

$$D(t) = \max_{1 \le i \le n} |x_i(t) - G_i(t)| \quad (3.2)$$

where $x(t) = \sum_{i=1}^{n} x_i(t)$ and $G(t) = \sum_{i=1}^{n} G_i(t)$

From (3.1) and (3.2) we get

$$\dot{x}(t) = K(x(t) - G(t)) \quad (3.3)$$

Let u(t) be the control energy requirement for the pursuit. Let the amount of control energy needed to increase the speed of the state vary between 0 and 1, ie $0 \le u(t) \le 1$, u = 0 when no energy is applied. Of course there may be times when there will be the need for the application of brakes in the pursuit. This reverse operation places u(t) between –1 and 0. That is u(t) lies in the interval $-1 \le u(t) \le 0$.
Evidently, u(t) defined on $-1 \le u(t) \le 1$ and so is an admissible control.
Incorporating the control energy into system (3.3), we have

$$\dot{x}(t) = K(x(t) - b(t))\, u(t) \quad (3.4)$$

Describing the configuration of the distance between the state and the target by a family of curves f(t, x(t), u(t)), system (3.4) becomes

$$\dot{x}(t) = f(t, x(t), u(t)) \quad (3.5)$$

which is a nonlinear dynamic. However using the method in [9] where f is such that

$$\frac{\partial f}{\partial x} = Ax(t) \text{ where A is nxn matrix}$$

$$\frac{\partial f}{\partial u} = Bu(t) \text{ where B is nxm matrix.}$$

equation (3.5) now becomes

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (3.6)$$

which is the contemporary model for a linear ordinary differential equation.

3.2 State Trajectories

Equation (3.6) provides a model describing the state of the act. To obtain the configuration of the trajectories of the state of system (3.6), we consider the homogeneous part of system (3.6) given by

$$\dot{x}(t) = A\,x(t) \quad (3.7)$$

as in (2.2).
Thus the matrix function X(t) such that

$$\frac{\partial X(t,s)}{\partial t} = AX(t,x) \quad \text{and } X(0) = I$$

is called the fundamental matrix solution of system (3.6) and has the following exponential form : $X(t) = e^{At}$ such that the solution of the homogeneous system has the representation

$$x(t) = X(t)x_0 = e^{At}x_o \quad (3.8)$$

where $x(t_o) = x_o$ is a given initial vector.

The origin (0, 0) is the only critical point under the assumption that matrix A is non singular. The graph of the vector equation (3.8) is the trajectory that starts at $x_o = (y_o, z_o)$. Because e^{At} can sometimes be complicated, it may not be easy to picture such a trajectory. However, by similarity transformation of A, we obtain the diagonal matrix N. That is, we can find a non singular matrix P such that $P^{-1}AP = N$. N is simpler to handle than A but still preserves the basic properties of A. By the same similarity transformation, we have that

$e^{At} = Pe^{At}P^{-1} = Pe^{Nt}P^{-1}$. The trajectory equation thus becomes

$P^{-1}x = e^{N}P^{-1}$

If we set $M = P^{-1}x$, we get

$$M = e^{N}M_o \tag{3.9}$$

The equation $M = P^{-1}x$ can be viewed as representing a fairly simple transformation of the XY plane to MX plane. This transformation maps the trajectory

$$x = e^{At}x_o \text{ onto } M = P^{Nt}M_o$$

The similarity transformation does not only preserve the basic properties of A but that of the system trajectory, thereby enabling us to obtain basic facts about the phase portrait of equation (3.6) through studying the phase portrait of the equation

$$\dot{x} = Nx \tag{3.10}$$

where $N = P^{-1}AP$

Note that the diagonal elements of N are the eigenvalues of A. For a 2 x 2 matrix let these eigenvalues be λ_1, λ_2. They are non zero since $|A| \neq 0$ and so we can write e^{Nt} as

$$e^{Nt} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \tag{3.11}$$

From this, we obtain in component form equation (3.9) as

$$x = e^{x_1 t}x_o \quad , \quad y = e^{\lambda_2 t}y_o \tag{3.12}$$

With these as parametric equations, λ being the parameter, we can obtain possible trajectories of equation (3.6) vis-à-vis equation (3.9).

As an illustration, consider the system

$$\dot{x} = Ax(t)$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} ; \quad x = (x_1, x_2)$$

Rewriting the equation, we have

$$\dot{x} = x_1 \tag{3.13a}$$

$$\dot{x} = x_1 - x_2 \tag{3.13b}$$

Let $x_1 = \alpha e^{\lambda t}$ and $x_2 = \beta e^{\lambda t}$ where α and β are constants. Evidently, $\lambda = 1$ (twice repeated) and so we have the parametric equation for the state trajectory as

$$x_1 = 0, \quad x_1 = e^{t}$$

where $\alpha = 0$ and $\beta = 1$ (say)

Sometimes certain equations yield themselves to direct integration, like equation (3.13a). Integrating (3.13a) we have

$$\ln x_1 = t + c \Rightarrow x_1 = Ae^{t}.$$

Assuming the solution of system (3.13a) passes through the point (x_o, y_o) at time t_o, we have

$$x_1 = x_o e^{t}.$$

Substituting this result in integration of equation (3.13b) we have the problem of finding the solution of the resulting linear equation

$$\dot{x}_2 - x_2 = x_o e^{t}$$

whose solution is given as

$$e^{t}x_2 = \int x_o e^{2t}dx + B$$

where B is a constant.

With the initial condition $x_2(o) = y_o$, the solution becomes

$$x_2(t) = x_o e^{t} + (y_o - x_o t)e^{t}$$

The state trajectory's now becomes the pair of equations

$$x_1 = x_o e^{t}$$

$$x_2 = \frac{x_o}{2} e^{t}(y_o - x_o t)e^{-t}.$$

3.3 The Controllability Question

When we want the trajectories to follow a desired pattern or to reach a certain target, system (3.6) becomes the focus of attention.

In this case, we commence a search for a control function capable of transforming the initial state $x(t_o)$ of system (3.6) to some desired final state x_t in finite time. This of course raises the question of the controllability of system (3.6).

Consider our system of interest

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where A is an nxn constant matrix and B is also constant nxm matrix for any nonsingular matrix P of order nxn, let x = Pz where z is also a state vector. Equation (3.6) can be rewritten as

$$P\dot{z} = APz + Bu$$

or

$$\dot{z} = P^{-1}APz + P^{-1}Bu \qquad (3.14)$$

set $N = P^{-1}AP$ and $M = P^{-1}B$ so that (3.14) becomes

$$\dot{z} = Nz + Mu$$

where N is a diagonal matrix and can be written as $N = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. where λ_i , i = 1…,m, are eigenvalues of A.

Let us consider the simple case where n = m = 2, then equation (3.14) takes the form

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Simplifying, we have

$$\left.\begin{aligned} \dot{z}_1 &= \lambda_1 z_1 + b_1' u \\ \dot{z}_2 &= \lambda_2 z_2 + b_2' u \end{aligned}\right\} \qquad (3.15)$$

where b_i' = max(b_{i1}, b_{i2}) for each i. b_{ij} are components of B.

It is seen from system (3.15) that if b_i, the ith now of M has all zero components, then $\dot{z}_1 = \lambda_1 z_i + 0$ and the control function u(t) has no influence on the ith mode of the system in which case the mode is said to be uncontrollable. On the other hand, where all the modes are controllable, the system is said to be completely state controllable.

To obtain controllability criterion for the system under study, we make B in equation (3.6) a column matrix, b. Of course the result obtained using the column vector b holds for a more general case.

Equations (3.6) and (3.14) then become

$$\dot{x}(t) = Ax(t) + bu$$

$$\dot{z}(t) = Nz + b_1 u \qquad \text{respectively.}$$

where $b_1 = P^{-1}b$.

Define $b_1 = \{\beta_1, \beta_2, \ldots, \beta_n)^T$; T is transpose

and $Q = [b_1, Nb_1, N^2b_1, \ldots, N^{n-1}b_1]$

$$= \begin{bmatrix} \beta_1 & \lambda_1\beta_1 & \cdots & \lambda_1^{n-1}\beta_1 \\ \beta_2 & \lambda_2\beta_2 & \cdots & \lambda_2^{n-1}\beta_2 \\ \vdots & \vdots & & \vdots \\ \beta_n & \lambda_n\beta_n & & \lambda_n^{n-1}\beta_n \end{bmatrix}$$

Q_1 being a Vandermonde matrix has all the columns linearly independent and is non-singular. Recall that

$$b_1 = P^{-1}b$$
$$Nb_1 = P^{-1}Ab$$
$$N^2b_1 = P^{-1}A^2b$$
$$\vdots$$
$$N^{n-1}b_1 = P^{-1}A^{n-1}b.$$

So that

$$Q_1 = P^{-1}[b, Ab, \ldots, A^{n-1}b] = P^{-1}Q$$

where

$$Q = [b, Ab, \ldots, A^{n-1}b] \qquad (3.16)$$

From the sequel, system (3.6) is controllable if the components of Q are not zero. This means that Q is non singular and so has n^n linearly independent columns. The rank of Q is therefore n. clearly, system (3.6) is controllable if

$$\text{Rank } [b, Ab, A^2b, \ldots, A^{n-1}B] = n.$$

In general where the system is multivariate (ie B has many columns),

$$\text{Rank } Q = \text{rank } [B, AB, \ldots, A^{n-1}B] = n.$$

provides a computable criterion for the controllability of system (3.6), credited to R.E. Kalman as in [7].

3.4 Conditions for Interception of a moving target

We shall now state conditions for the possibility to intercept a moving target where the target is either a moving point function or a compact set function.

Theorem 3.1:

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where A is an nxn and B is nxm metrics.

Suppose:

(i) system (3.6) is controllable

(ii) the set functions: reachable set $R(t_o,t_1)$, attainable set, $\hat{A}(t_1,t_o)$, and target set G, are compact, then

$$\hat{A}(t) \cap G(t) \neq 0$$

This is the interception of the moving target G(t).

Proof:

Let G(t) be the target, we need to show that, there exists a control u such that the state $x(t) \in \hat{A}(t)$ can be steered to G(t) in finite time.

Let u_n, be a sequence in U.

Since U is compact,

$$\lim_{n\to\infty} u_n = u.$$

Now $x(t,\phi,u_n) \in \hat{A}(t)$ and

$$x(t,\phi,u_n) = x(t,\phi,0) + X(t,t_o) \int_{t_o}^{t} X(t_o,s)B(s)u_n(s)ds$$

Taking limits on both sides, we have

$$\lim_{n\to\infty} x(t,\phi,u_n) = x(t,\phi,0) + X(t,t_o)\int_{t_o}^{t} X(t_o s)B(s)\lim_{n\to\infty} u_n ds$$

$$\Rightarrow \quad x(t,\phi,u_n) = x(t,\phi,0) + X(t,s)\int_{t_o}^{t} X(t_o s)B(s)\lim_{n\to\infty} u_n ds$$

Since $\hat{A}(t)$ is assumed compact

$$x(t,\phi,u) = G(t) \in \hat{A}(t)$$

This shows that

$$\hat{A}(t) \cap G(t) \neq \varnothing \qquad (3.17)$$

It is evident that the required control function must be able to steer the state into the attainable set as well as the target set. This is the condition that $\hat{A}(t_o,t_1) \cap G(t) \neq \varnothing$ and this completes the proof.

4. Discussion of Results/Applications

This study has not only provided a model for the pursuit and interception of a moving target, but has also established conditions for its interception. It raises the following questions:

i) What is the state trajectory to the target?

ii) What is the choice of appropriate control function to steer the state of the system to the target?

iii) What is the duration of the journey of meeting the target?

For a quick and vivid understanding of the analysis, we illustrate using the following optimal control problem. Consider the system

$$\dot{x} = Ax(t) + Bu(t);$$

$$x(o) = (x_o, y_o) \qquad (4.1)$$

$$\{u \in C^m : \|u\| \leq 1\}$$

C^m is a unit cube in E^m, the m-dimensional Euclidean space. A and B are given respectively by

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

First of all we test the system for controllability using Kalman's criterion as in [7].

$$\text{rank } [B, AB] = \text{rank} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2$$

Since the computed rank is the same as the dimension of the state space, E^2, we conclude that the system (4.1) is controllable on a finite time interval $\{t_o,t_1\}$ this shows that any target can be intercepted in a context described by the control system. The next issue to be discussed is obtaining the control capable of steering the state to the target at the shortest possible time, that is, the optimal control

$$u^* = \text{sgn } \eta^T (X^{-1}B) \ ; \ \eta \in E^2$$

where X is the fundamental matrix of the homogeneous part of system (4.1) and X^{-1} its inverse. The exponential characterization of X(t) is e^{At}.

Evidently

$$X(t) = e^{At} = 1 + At + \frac{A^2t^2}{2!} + \ldots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ and } X^{-1}(t) = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$$

Clearly, system (4.1) is normal since B is a column matrix (see [6]). Hence there exists optimal control that is unique and Bang - Bang (maximum control power) as given below:

$$u^* = \begin{cases} 1 & if \ \eta^T X^{-1} B(x) > 0 \\ -1 & if \ \eta^T X^{-1} B(t) < 0 \end{cases}$$

For some non-zero vector $\eta = (\eta_1, \eta_2)^T$ we can easily calculate

$$\text{sgn} \left[(\eta_1, \quad \eta_2) \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

$$= \text{sgn}(-\eta_1 t + \eta_2)$$

Hence

$$u^* = \begin{cases} 1 & for \, \text{sgn}[(-\eta_1 t + \eta_2)) > 0 \\ -1 & for \, \text{sgn}(-\eta_1 t + \eta_2) < 0 \end{cases}$$

and has only one switch between -1 and 1. The switch time is obtained by equating $-\eta_1 t + \eta_2$ to zero to have

$$t = \frac{\eta_2}{\eta_1}$$

Evidently, the optimal control exists and is unique and Bang – Bang (maximum control power). There is no loss of generality in assuming that the target is the origin.

To obtain the optical trajectory therefore, we re-state system (4.1) and with u* = 1, we have

$$\frac{dx_1}{dt} = x_2 \qquad \text{and} \qquad \frac{dx_2}{dt} = 1$$

$$\Rightarrow \quad \frac{dx_2}{dx_1} = \frac{1}{x_2} \tag{4.2}$$

By the method of separation of variables, we have

$$\frac{x_2^2}{2} = x_1 + C$$

Since the solution passes through (x_o, y_o) we have

$$x_1 = \frac{x_2^2}{2} - \frac{y_o^2}{2} + x_o \qquad (4.3)$$

which is a parabola with x_2 increasing,

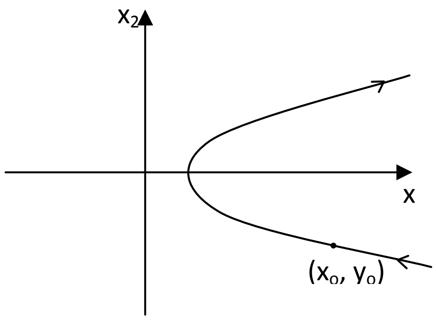

Fig. 4.1: Arc C_1 (Increasing Optimal trajectory)

Taking the other value of the optimal control, that is u* = –1, we have

$$\frac{dx_1}{dt} = x_2 \ ; \quad \frac{dx_2}{dt} = -1$$

$$\Rightarrow \quad \frac{dx_2}{dx_1} = \frac{-1}{x_2} \qquad (4.4)$$

By direct integration, we have

$$\frac{x_2^2}{2} = x_1 + C$$

Since this solution passes through the initial point (x_o, y_o), we have

$$x_1 = \frac{-x_2^2}{2} + \left(x_o + \frac{y_o^2}{2}\right) \qquad (4.5)$$

In this case, x_2 is decreasing

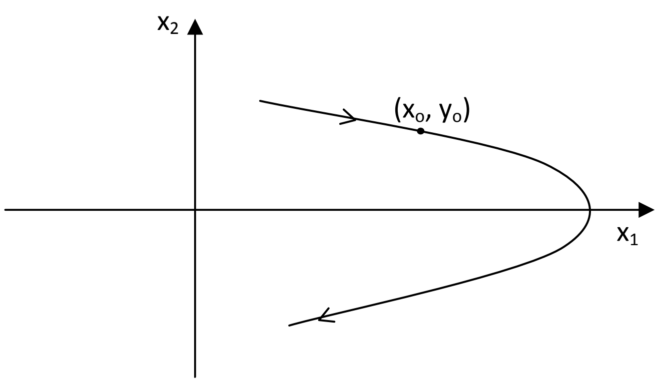

Fig. 4.3: Arc C_2 (Decreasing Optimal Control)

To lend credence to our discussion here, let the initial point be (5, -1), that is, $x_o = 5$, $y_o = -1$. Starting with the choice of control u* = -1 from equation (4.4), we have

$$x_1 = \frac{1}{2}\left(11 - x_2^2\right)$$

$$\text{or} \qquad x_2 = -\sqrt{11 - 2x_1} \qquad \text{since } x_2 < 0$$

At the point of intersection, we change control to u* = 1 and there

$$x_2 = -\sqrt{2x_1} \quad \text{since} \quad \left(\frac{dx_1}{dx_2} = x_2\right)$$

and so

$$-\sqrt{11-2x1} = -\sqrt{2x_1}$$

$$\Rightarrow \quad 11 - 2x_1 = 2x_1 \quad \Rightarrow \quad x_1 = {11}/{4}$$

Then

$$x_2 = \sqrt{11-2\left({11}/{4}\right)} = -\frac{1}{2}\sqrt{22}$$

From the point of intersection, $\left({11}/{4}\ ,\ -\frac{\sqrt{22}}{2}\right)$ of the two arcs, a straight line to the origin completes the pursuit.

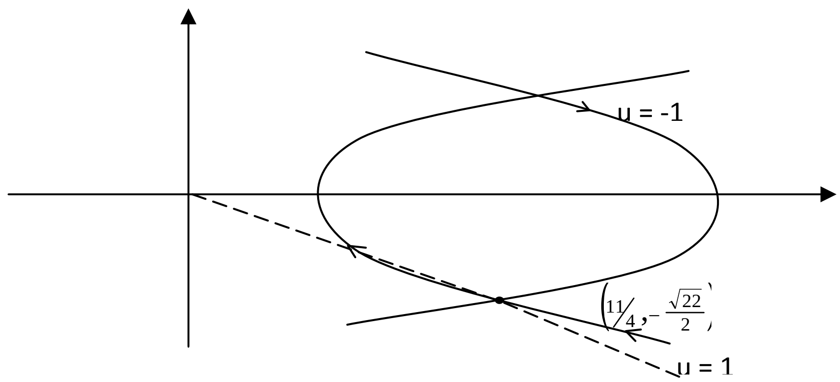

Fig. 4.3: Arc C_3 (Optimal Trajectory)

The next and final question is what is the total time taken for this pursuit to intercept the target. Solving the state equations (4.4) we have

$$x_2(t) = -t_1 + y_o = \frac{\sqrt{22}}{2}$$

so

$$t_1 = -\frac{\sqrt{22}}{2} - 1 \qquad \text{(since } y_o = -1\text{)}$$

$$= \frac{\sqrt{22}}{2} + 1 \qquad \text{(since time is positive)}$$

Similarly, after switching to u = 1 traversing the curve C_1

$$\dot{x}_1 = x_2\ , \quad \dot{x}_2 = 1 \quad \text{from} \left(\frac{11}{4}, -\frac{\sqrt{22}}{2}\right)$$

to the origin (0, 0) and solving for x_2, we have

$$x_2(t) = t_2 - \frac{\sqrt{22}}{2}$$

with $x_2(t)$ now zero, we have

$$t_2 - \frac{\sqrt{22}}{2} = 0 \quad \Rightarrow t_2 = \frac{\sqrt{22}}{2}$$

The total time taken for the pursuit is $t_1 + t_2 = \dfrac{\sqrt{22}+2}{2} + \dfrac{\sqrt{22}}{2} = \sqrt{22} + 1$

From the foregoing, the capture problem is an illuminating application of the theory of optimal control.

Conclusion:

This study furnishes a model governed by an ordinary control system for the interception of a moving target. It is evident from the model that the primary concern in the pursuit for the interception of a moving target is the state of the weapon and the control energy requirement. The control energy should be such that, it has the potential of steering the state from its initial position through the phase portrait described by the moving target to reach it. This is communicated by the assumption that $\hat{A}(t) \cap G(t) \neq \phi$. This condition in other words is the optimal control of the system. Clearly (3.17) provides a computable interval for the interception of a moving target in the context described by an ordinary differential autonomous equation..

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