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Successive Over Relaxation Method Which Uses Matrix Norms for Newton Operator.

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Abstract

Succesive Overrelaxation method (S.O.R.) is a well known iterative method which is very sensitive to extensions and modifications as an attempt to obtaining other iterative methods.

An algorithm for S.O.R functional iteration which uses matrix norms for the Jacobi iteration matrices rather than the usual Power method, feasible in Newton Operator for the solution of nonlinear system of equations is proposed. We modified the S.O.R. iterative method known as Multiphase S.O.R. method for Newton operator. Numerical example is given and results from our method are compared with an existing classical S. O.R method. It is shown that our method has superiority over the classical S.O.R. method.

Keywords : Gauss- Siedel method, S.O.R method, Newton method, matrix norm.

Introduction:

The q-step method for Newton-Jacobi and Newton-Gauss-Siedel methods discussed in, [5] is hereby extended to include the Successive Over Relaxation (S.O.R) method for solution of the nonlinear systems of equation

$$F(x) = 0, \tag{1.1}$$

where $F : D \subset R^n \to R^n, F$ is a differentiable function in the sense of Frechet in an open ball $S = S(x,r) = \{x^* \in R^n : \|x - x^*\| < r\} \subset D$, that satisfies $\|F'(x) - F'(x^*)\| \le \|x - x^*\|$ for which F(x*)=0.

We also assume that all sub functions of F are bijective, continuously differentiable with uniformly monotone gradient which, are thus feasible with minimum eigenvalue of the Jacobian matrix $F'(x)$, that are bounded away from zero, since $F'(x)$ is positive definite (see e.g. [4] and [8] for details). Newton's method is attractive because of its almost global convergence under appropriate conditions. Its efficiency strongly depends on the linear system solvers which primarily is the basis of our discussion. [8] is an excellent reference behind this theory and several of its extensions. The simplest well known method which has global convergence for Newton's method is Guass-Siedel method. The procedure goes as follows:

$$x^{(k+1)} = x^{(k)} + d^{(m)}, (k = 0,1,..., m = 0,1,...) \tag{1.2}$$

The $d^{(m)}$ is obtained from solving the linear system

$$F'(x^{(k)})d^{(m)} = -F(x^{(k)}) \quad (k = 0,1,...n, m = 0,1,2,..) \tag{1.3}$$

To update $d^{(m+1)}$ *and* $d^{(m)}$, we successively solve the one dimensional linear equation in (1.3). In what follows, we introduce the relaxation parameters ω_m [see e.g [7] and [8], to obtain an iterative formula

$$d_i^{(m+1)} = d_i^{(m)} + \omega_m \left(d_i - d_i^{(m)}\right), (m = 0,1,...)) \cdot \tag{1.4}$$

The studies of method (1.4) form the basis of our discussions and the subsequent derivation of our constructed algorithm for S.O.R. method. It is well known that for $\omega = 1$, the S.O.R method is the Guass-Siedel method which was historically favored to be globally and monotonically convergent for many linear problems. However, following [8] it is known that the asymptotic convergence rate of Gauss-Siedel iteration matrix is of order magnitude slower than that of S.O.R. method. Thus our objective in this paper is to formulate a multiphase S.O.R. method that will not only include all previously known relaxation methods but will also indicate that the constructed multiphase S.O.R. method converges faster than those previously given in [5]. We paid special emphasis on the computation of the spectral radius of the Jacobi iteration matrix which is very crucial for our studies

1. **THE METHOD.** THE MULTIPHASE S.O.R METHOD:

We set $A = F'(x)$, the Jacobian matrix which has real eigenvalues different from zero. We remember that the matrix A is 2-

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cyclic and consistently weakly ordered in the sense of [8], [9] and thus admits, a regular splitting:

$$A = D - L - U = D(I - B^A) \tag{2.1}$$

where D, L and U are respectively strictly diagonal matrix, strictly lower triangular matrix and strictly upper triangular matrix. If we set $F'(x^{(k)}) = A$ and $F\left(x^{(k)}\right) = b$ in equation (1.1) we will have

$$Ad=b \tag{2.2}$$

where $A \in L\left(R^n\right), d \text{ and } b \in R^n$.

We hereby introduce the generalized class of linear stationary iterative methods of first degree in the form

$$d^{(m+1)} = Gd^{(m)} + c \quad (m = 0,1,..) \tag{2.3}$$

Here $d^{(o)}$ is an arbitrary vector and, for some nonsingular matrix H, we have

$$G = I - H^{-1}A, c = H^{-1}b \tag{2.4}$$

The splitting matrix H is called a preconditioner for the coefficient matrix A and satisfies the relationship [9] in the form:

$$A=H-(H-A) \tag{2.5}$$

From an iterative convergent sequence of type

$$Hd^{(m+1)} = (H - A)d^{(m)} + b, (m = 0,1,...,); \tag{2.6}$$

under certain conditions for values of H in (2.6), the following preconditioners are always used [see [3], [4], [8] and [9] in the form:

H=I- The Null preconditioner (Richardson method),

$H = (D - \omega L)$ - The S.O.R. preconditioner ,

$H = \frac{1}{2-\omega}\left(\frac{1}{\omega}D - L\right)\left(\frac{1}{\omega}D\right)^{-1}\left(\frac{1}{\omega}D - U\right)$ - The symmetric successive over relaxation preconditioner,

$H = \left(\bar{D} - \bar{L}\right)\left(\bar{D}\right)\left(\bar{D} + \bar{U}\right)$ - The incomplete LU factorization.

The term $\bar{D}$ is usually taken as block triangular matrix. Let us note that the closer the product $H^{-1}A$ approximates the identity matrix I, the faster will be the convergence of these iterative methods.

We will be more interested in the manipulation of S.O.R. method since S.O.R. formula is very sensitive to extensions and modifications due to the nature of Jacobi iteration matrix arising there from. We aim to achieve this feat with huge success.

The one step Newton S.O.R. method [4] needs the evaluation of $(D_m - \omega L)$ with $\frac{n(n-1)}{2}$ partial derivatives as well as the solution of the triangular systems of equations.

Thus this paper further exposes the variants of these traditional Newton – S.O.R. method wherein, we propose a new brand multiphase S.O.R. methods by a craft full combinations of S.O.R. method with itself.

To steer the course of our discussions in the right perspective, first we present the method of [5] in the form:

(q-step Newton-Jacobi method)

$$x_i^{(k+1)} = x_i^{(k)} + \hat{d}^{\left(m+\frac{\lambda+1}{q}\right)} \tag{2.7}$$

where

$$\hat{d}^{\left(m+\frac{\lambda+1}{q}\right)} = \frac{1}{a_{ii}}\left(-F\left(x^{(k)}\right) - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} d_j^{\left(m+\frac{\lambda}{q}\right)}\right) \tag{2.8}$$

(m=0,1,…, k=0, 1….q=2,…t, λ=0,1,…q-1, and $t \in N$).

and

(q-step Newton-Gauss-Siedel method)

$$x^{(k+1)} = x^{(k)} + d^{\left(m+\frac{\lambda+1}{q}\right)} \tag{2.9}$$

where

$$d^{\left(m+\frac{\lambda+1}{q}\right)} = \frac{1}{a_{ii}}\left(-F\left(x^{(k)}\right) - \sum_{j=1}^{i-1} a_{ij} d_j^{\left(m+\frac{\lambda+1}{q}\right)} - \sum_{j=i+1}^{n} a_{ij} d_j^{\left(m+\frac{\lambda}{q}\right)}\right) \tag{2.10}$$

(m=0,1,…,k=0,1,…q=2,…t,λ=0,1…q-1 and $t \in N$).

Usually we take q=2 since higher values of q>2 call for higher computational costs which in essence, may not really lead to enhancement of convergence of our computation in terms of speed, robustness and stability with reference to inner iterations.
Method (2.7) and (2.10) may be seen as a convolution of Jacobi and Gauss-Siedel methods respectively, using one Newton step length. We note that analysis of S.O.R method is dependent on the spectral radius of Block Jacobi iteration matrix. Therefore, following [1], the convergence analysis for the Jacobi iteration matrix B holds if

$$\rho(D^{-1}(L+U)) < 1 \tag{2.11}$$

Here ρ denotes the spectral radius of a matrix B. We must note that the convergence of Jacobi iteration method is linear with a rate that is at least as fast as ρ. Practically, we assumed that given any vector norm $\|\cdot\|$, there exists a corresponding constant K such that the error,

$$e^{(n)} \in R^n \text{ with } e^{(m)} = d^{(m+1)} - d^{(m)} \text{ which satisfies } \left\|e^{(n)}\right\| \le K\rho^{(n)}.$$

The explicit one point S.O.R. iteration matrix in the sense of [8] can then be written as

$$\ell_\omega^A = (D-\omega L)^{-1}\{(1-\omega)D + \omega U\}. \tag{2.12}$$

It follows that solution to the linear system (2.2) will be written in the form:

$$d_i^{(m+1)} = \ell_\omega^A d^{(m)} + \omega(D-\omega L)^{-1} b \quad (m = 0,1,....). \tag{2.13}$$

This converges to a unique solution if and only if the spectral radius of $\ell_\omega < 1$. We in the same spirit as those works wish to construct multiphase algorithms for S.O.R. method using Newton step length bearing in mind those matrix norms to estimate the dominant eigenvalue for Jacobi iteration matrix is very crucial in our methodology in the form:

$$x^{(k+1)} = x^{(x)} + d_{jbsor}^{\left(m+\frac{\lambda+1}{2}\right)}, \tag{2.14}$$

where

$$d_{jbsor}^{m+\frac{\lambda+1}{2}} = \frac{\omega}{a_{ii}}\left(-F\left(x^{(k)}\right) - \sum_{j=1}^{i-1} a_{ij} d^{\left(m+\frac{\lambda+1}{2}\right)} - \sum_{j=i+1}^{n} a_{ij} d^{\left(m+\frac{\lambda}{2}\right)}\right) + (1-\omega) d^{\left(m+\frac{\lambda}{2}\right)} \tag{2.15}$$

(i=1,2,…,n).
Method (2.14) has very high speed of convergence the classical S.O.R. method. Because of space, analysis of the convergence order will form another work which will be reported elsewhere.
The optimum relaxation parameter can be obtained in line with [8], depends greatly on the associated block Jacobi matrix B obtained from the splitting matrix A=D-L-U.
Let ℓ_ω be the block successive over relaxation matrix that is convergent for $\omega = 1$ which by continuity, is also convergent for some interval in ω containing unity be defined. Let $R\left(\ell_\omega^m\right)$ be defined for all sufficiently large positive integer m, which symbolizes the average rate of convergence for which

$$\min_{\omega\, real} \rho(\ell_\omega) = \rho(\ell_{\omega_b}). \tag{2.16}$$

Assuming the $\rho(B)$ is real and let p be a positive integer, it can be derived, [3] and [8] that

$$\rho(B)\omega_b)^p = [p^p(p-1)^{1-p}].(\omega_b - 1). \tag{2.17}$$

The relaxation parameter is optimal when p=2 from which we obtain

$$\omega_b = \frac{2}{1+\sqrt{1-\rho^2(B)}} = 1 + \left(\frac{2}{1+\sqrt{1-\rho^2(B)}}\right)^2. \tag{2.18}$$

We define the asymptotic rate of convergence $R(\ell_\omega)$ to be $R_\omega(\ell_{\omega b}) = -In(\omega_b - 1)(\rho - 1)$

and $R_\omega(\ell_1) = -p.\ln \rho(B)$, so that the ratio $\dfrac{R_\omega(\ell_{\omega b})}{R_\omega(\ell_1)}$ is practicable.

2. CONVERGENCE

Theorem 1 [4].

Suppose that $F : D \subset R^n \to R^n$ is F-differentiable on a convex set $D_o \subset D$ and that for each $x \in D_o$, $F'(x)$ is nonsingular and satisfies

$$\|F'(x) - F'(y)\| \le \eta \|x - y\|,$$
$$\|F'(x)^{-1}\| \le \beta, \forall x, y \in D_o$$

If $x^{(o)} \in D$ and r is the radius such that

$$\left\| F'\left(x^{(o)}\right)^{-1} F\left(x^{(o)}\right) \right\| \le \eta$$ and $\alpha = \frac{1}{2}\beta\eta\gamma < 1$ as well as $\bar{S}\left(x^{(o)}, r_o\right) \subset D_o$ where

$$r_o = \eta \sum_{i=0}^{\omega} \alpha^{2i-1} \le \frac{\eta}{1-\alpha},$$

then the Newton iterates given by

$$x^{(k+1)} = x^{(k)} - F'(x^{(k)})^{-1} F(x^{(k)}), k = 0,1,2,....$$

remain in $S\left(x^{(0)}, r_o\right)$ and converges to solution $x^* \in S\left(x^{(0)}, r_o\right)$ of F(x)=0

moreover

$$\left\| x^* - x^{(k)} \right\| < \varepsilon_k \left\| x^{(k)} - x^{(k-1)} \right\|^2, i = 1,2,..$$

Where

$$\varepsilon_k = \left(\frac{\alpha}{\eta}\right) \sum_{j=0}^{\infty} (\alpha^{2k})^{2i-1} \le \alpha(\eta(1-\alpha^{2k}))^{-1}$$

The proof of this theorem can be found in [4], [see theorem 12.46, pp412].

Remark:

The question now is why $F\left(x^*\right) = 0$?

That F(x*) =0 can be inferred from the inequalities

$$\left\| F'(x)^{-1}(F'(x^{(k)})(x^{(k+1)} - x^{(k)}) \right\| \le \left\| F'(x^*)^{-1}(F'(x^{(k)})(x^{(k+1)} - x^{(k)}) \right\| + \left\| x^{(k+1)} - x^{(k)} \right\|$$
$$\le \{\eta \left\| x^* - x^{(k)} \right\| + 1\} \left\| x^{(k+1)} - x^{(k)} \right\|, k = 0,1,..,$$

Where from,

$$\left\| F'(x^*)^{-1} F(x^*) \right\| = \lim \left\| F'(x^{(k)}) \right\|$$
$$= \lim_{k \to \infty} \left\| F'(x^*)^{-1} F'(x^{(k)})(x^{(k+1)} - x^{(k)}) \right\| = 0$$

In the limit as $k \to \infty$, it is easily seen that $\eta_k = 0$ which shows that F(x)=0. Thus signifying that the Filter condition is satisfied in the set $D_o \subset D$.

5. Numerical Experiment/Discussion

Consider the nonlinear system of equation taken from [2] and [8] as the scalar test problem

$$F(x) = \begin{cases} 20x_1 - \cos^2 x_2 + x_3 - \sin x_3 = 37 \\ \cos 2x_1 + 20x_2 + \log_e(1 + x_4^2) = -5 \\ \sin(x_1 + x_2) - x_2 + 19x_3 + \arctan x_3 = 12 \\ 2\tanh x_2 + e^{-2x_3^2 + 0.5} + 21x_4 = 0 \end{cases}$$

$$X^{(0)} = (2.0154195, -0.3182241, 0.6364483, -0.0874438)^T.$$

We present the numerical result for the non stationary S.O.R. method in Table 1.

TABLE 1. RESULTS FOR OUR METHOD (2.14).

No of iterations	x_1	x_2	x_3	x_4
0	2.0154195	-0.3182241	0.6364483	-0.0874438
1	1.891312879	-0.227954795	0.534767008	-0.02241936
2	1.895738905	-0.20960129	0.542783284	-0.02391607
3	1.896537738	-0.210269186	0.542045354	-0.023888312
4	1.896513584	-0.210267774	0.542088833	-0.023884314

For $\lambda = 1$ the asymptotic rate of convergence of the q-step for non stationary S.O.R method is precisely the same as that of the original S.O.R method applied on a linear system (2.2).
It is then to be expected that the q-step for non stationary S.O.R method is q faster than the original S.O.R method since for the corresponding linear problem, one complete iteration cycle of the k-step process amounts to q-S.O.R functional iterations.
To obtain the relaxation factor ω, we set $\omega = \frac{2}{1+\sqrt{1-\rho^2(B)}}$ where $\rho(B)$ is the spectral radius of Jacobi iteration matrix.

Practically in our work, the replacement of the spectral radius of the matrix B by the maximum column-sum or maximum row-sum norm instead of the traditional power method has been found very satisfactory for our purpose.

Table 2: Results for classical S.O.R. Method

No of Iterations	x_1	x_2	x_3	x_4
0	2.0154195	-0.3182241	0.6364483	-0.0874438
1	1.897248711	-0.209174571	0.535969139	-0.023944194
2 3	1.872279338 1.896500462	-0.208788915 -0.210219191	0.542381896 0.542057717	-0.023920517 -0.023891724
4	1.896514524	-0.210266422	0.542060190	-0.023887175

From results presented in both Tables 1 and 2, it can be seen that we halt each inner successive iteration after four complete cycles. We have found out that the use of S.O.R. with 2 steps gives better results than that of one step with respect to inner iterations. It is suggested therefore that our method should be of good computational utility for those researchers in the Numerical Linear algebra or in the areas of Partial differential equations.

6. Conclusion

The paper presented a modification for Successive Over Relaxation (S.O.R) method. The method uses matrix norms rather than the well known Power method for finding the eigenvalues of the resulting Jacobi Iteration matrices applied on Newton operator for finding zeros of systems of nonlinear equations. Of special interest at the peak of our findings was that, the modified S.OR. method which makes use of two steps iteration is iteratively faster than the Classical S.O.R method for obtaining inner iterations in each Newton step and that our presented method includes all the previously known class of S.O.R methods.

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