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Solvability of linear Interval System of Equations via Oettli-Prager Theorem and Rohn's bound.

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Abstract

In general, numerical results computed by interval methods tend to grow in diameters as a result of data dependencies and cluster effects which may be traced to error from one source that can affect every other source and thereby drastically lower the efficiency of the interval inclusion methods. We describe in this paper how this can be reduced and an attempt is made to address the above problems subject to tolerable solution sets. Basic computational tools at our disposal are the Oettli-Prager's theorem and Rohn's method which combine floating point operation with an interval method.

Keywords : linear interval systems of equations, hull of solution set of linear interval systems, Oettli-Prager theorem, Bauer-Skeel bounds

Introduction:

We consider the problem of enclosing solution of linear interval systems of equations

$$Ax=b \tag{1.1}$$

where

$$\left\{A = [A_c-\Delta, A_c+\Delta], b=[b-\delta, b+\delta]\right\}, A_c=\frac{1}{2}(\underline{A}+\bar{A}), \Delta=\frac{1}{2}(\bar{A}-\underline{A}), A\in IR^{n\times n},$$

$$x_c=\frac{1}{2}\left(\underline{x}-\bar{x}\right), \delta=\frac{1}{2}\left(\bar{b}-\underline{b}\right), b\in IR^n .$$

Finding inclusion bounds for system (1.1) is proved to be NP-hard [8]. We hereby pay special attention to Oettli-Prager theorem [4]. Further results in [2], [7] and [9] will play some crucial roles as basic tools to achieve our purpose.
Computing solution to system (1.1) requires some computational skills such as being able to handle efficiently the interval arithmetic operations of $\{+,-,\times,/\}$. Sufficient conditions for regularity or singularity of interval matrix as regards linear interval system (1.1) are well detailed in the works of [5] and [6].
Writing

$$G=\left|I-RA_c\right|+\left|R\right|\Delta \ , \tag{1.2}$$

it is easy to obtain [9] a vector d

$$d=\left(I-\left|G\right|\right)^{-1}\left(\left(\left|I-G\right)x-g\right|+\delta\right), \tag{1.3}$$

for which the inequality

$$\left|\left(I-G\right)x-g\right|<\left(I-\left|G\right|\right)d \ , \tag{1.4}$$

holds, that solves the linear interval system (1.1) with high yield of mathematical certainty .
After d has been computed sufficiently well, the solution x^* as an enclosure to system (1.1) is bounded by the inequality

$$x-d<x^*<x+d \ . \tag{1.5}$$

Using the fact that $I-G\approx I-R\Delta$, it follows that $(I-\left|G\right|)^{-1}\approx\left(I-R\Delta\right)^{-1}$, and that $\left|R\right|\Delta\le\left(I-\left|I-RA_c\right|\right)^{-1}\left|R\right|\Delta$. Letting $M_0=\left(I-\left|A_c^{-1}\right|\Delta\right)^{-1}\ge 0, R_0=A_c^{-1}$ the $[A_c-\Delta, A_c+\Delta]^{-1}$ enclosure is given by the equation

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$$[A_c-\Delta, A_c+\Delta]^{-1}=[R-(M-I)|R|, R+(M-I)|R|], \quad (1.6)$$

which is valid for any regular matrix.
[3] used an identity matrix as a preconditioner for the linear interval system (1.1) and was later studied and modified by [7]. In what follows we take $M=(I-\Delta)^{-1}$ with the property that $M\Delta=\Delta M=M-I$ where $m_{ii}\geq 1$. This implies that $2m_{ii}-1\geq 1$.
We note that $\rho(G)<1$ and for $A_c=I$ it is easy to see that $|RA-I|=|RA_c-I+R(A-A_c)|\leq G$, which implies the inclusion of $RA\in[I-G, I+G]$. It follows that the inverse preconditioned interval matrix with midpoint matrix is finitely bounded in the form $\left|A^{-1}R^{-1}-M\right|=\left|A^{-1}R-\frac{1}{2}(\underline{M}+\bar{M})\right|\leq\frac{1}{2}(\bar{M}-\underline{M})=\Delta M$, (1.7)

where ΔM signifies the deviation of the radius of inverse interval matrix A from the preconditioned interval matrix A.
As a result, inclusion for the upper bound for the solution set is given in the form

$$\bar{x}_i=x_i^*+m_{ii}\left(b_c-|b_c|\right)_i=\left(M\bar{b}\right)_i , \quad (1.8)$$

where from $A\in[A-\Delta, A+\Delta], b\in[b_c-\delta, b+\delta]$ and $M\left(x'-|x|\right)+|x|\leq Mb$,

and that $x'=\left(|x_1|,|x_2|,...,|x_{i-1}|,x_i,|x_{i+1}|,...,|x_n|\right)^T$.

2 Characterization of strong Regularity of interval matrix.

One of the surest ways to check if an interval matrix A in the linear interval system is strongly regular is by computing the spectral value of $\rho(|R|\Delta)$ <1. It is necessary that $(I-|R|\Delta)^{-1}\geq 0$
holds. Since regularity of interval matrix A implies the existence of its inverse , as a result we provide inverse inclusion also for interval matrix A.
In what follows we introduce the Bauer-Skeel bounds for enclosing solution set to system (1.1) as follows:
Theorem 2.1. [11], Bauer-Skeel bounds
if
$\rho\left(|A_c^{-1}|\Delta\right)<1$, for each A, b such that $|A-A_c|\leq\Delta$ and $|b_c-b|<\delta$, then A is non-singular and the solution of the system (1.1) satisfies the inequality

$$-x^*+x_c+|x_c|\leq x\leq x^*+x_c-|x_c| , \quad (2.1)$$

$$M=\left(I-|A_c^{-1}|\Delta\right)^{-1}, \quad (2.2)$$

$$x^*=M\left(|x_c|+|A_c^{-1}|\delta\right) \quad (2.3)$$

A careful analysis will reveal that method (2.1) has similar representation as that of method (1.5).
Theorem 2.2 [4]. The set of all admissible solutions of system (1.1) is a polytope:

$$X=\left\{x:\|Ax-b\|_\infty\leq\varepsilon\|x\|_1+\delta\right\} \quad (2.4)$$

where $\|x\|_1=\sum|x_k|$, and X is a non-convex set, ε is the precision to which the interval data of the matrix are rounded to.
We define nonsingularity radius for the interval perturbations as the reciprocal to the
$(\infty,1)$ -norm of the inverse matrix A in the form
$\rho(A)=\dfrac{1}{\|A^{-1}\|_{\infty,1}}$. Let us note that calculation of such norm is NP-hard [9].

Definition 2.1. Let Q be a diagonal matrix such that $\forall i,(1\leq i\leq n)|Q_{ii}|=1$. The set $\{x|Qx\geq 0\}$ is called an orthant of R^n.
Theorem 2.3, [9]. Let A be a non-singular interval matrix .Then the matrix equations

$$QA_cT_z+|Q|\Delta=E \quad (2.5)$$

$$Q'A_cT_z-|Q'|\Delta=E \quad (2.6)$$

have unique solution Q, Q'. E is the canonical set of $(1,1,...,1)^T$.

$$T_z = diag(z_1, z_2, .., z_n) = \begin{pmatrix} z_1 & 0 \dots & 0 \\ 0 & z_2 \dots & 0 \\ .. & \dots & \dots \\ 0 & 0\dots & z_n \end{pmatrix}$$ is the matrix prescribed by the orthant.

As a result, using equations 2.5 and 2.6, and letting y be the sign vector of the i-th row of the matrix Q, it is easily deduced that $|Q|_i = Q_i T_y$ so that $QA_c T_z + |Q|\Delta = E$, one then obtains $Q\left(A_c + T_y \Delta T_z\right) = \left(T_z\right)_i$. In this case we have the expressions for the values of Q_i and $-Q_i$ in the form:

$$Q_i = \left(A_c + T_y \Delta T_z\right)_i^{-1} \quad \text{for } z_i = 1,$$

$$-Q_i = \left(A_c + T_y \Delta T_z\right)_i^{-1} \quad \text{for } z_i = -1.$$

With these we are able to solve the system

$$\left(A_c + T_y \Delta T_z\right) x = b_c - T_y \delta \,. \tag{2.7}$$

Computing the hull of the solution set is an NP.hard problem [8]. In order to formulate the exact bounds for the interval solution x we define the interval vector x_z by the equations

$$(\underline{x}_z)_i = \begin{cases} \left(Qb_c - |Q|\delta\right)_i, \; for\; z_i = 1 & (2.8) \\ -\left(Q^/ b_c + |Q^/|\delta\right)_i, \; for \;\; z = -1 & (2.9) \end{cases}$$

$$(\bar{x}_z)_i = \begin{cases} \left(Q^/ b_c + |Q^/|\delta\right)_i, \; for \;\; z = 1 & (2.10) \\ -\left(Qb_c - |Q^/|\delta\right)_i, \; for \;\; z = -1 & (2.11) \end{cases}$$

The dual linear programming problem

$$\min(-\left(b_c - \delta\right)^T y_1 + \left(b_c + \delta\right)^T y_2, -\left(A_c T_z + \Delta\right)^T y_1 + \left(A_c T_z - \Delta\right)^T y_2 \geq e^i, y_1 \geq 0, y_2 \geq 0)$$

which can be obtained from the linear programming problem : $\max(x_i^/ : -\left(A_c T_z + \Delta\right)x^/ \leq -\left(b_c - \delta\right), \left(A_c T_z - \Delta\right)x^/ \leq \left(b_c + \delta\right), x^/ \geq 0)$ has feasible solution (y_1, y_2) and $x \in X \cap IR_z^n$. The Oettli-Prager theorem implies that

$$[\min_z \left\{(\underline{x}_z)_i, \max_z (\bar{x}_z)_i\right\}] \subseteq X^I \,.$$

Theorem 2.4, [1]:

Assume that A_c =I, and let Q be an orthant . Then the hull of solution set to linear interval system (1.1) is in the form:

$$x \in \sum \cap Q \Leftrightarrow \begin{cases} (I - \Delta)Qx \leq Qb + \delta, \\ (I + \Delta)Qx \geq Qb \_ \delta, \\ \qquad Qx \geq 0. \end{cases} \tag{2.12}$$

It is shown in [1] that for a given orthant Q,

$x_Q = QM\left(Qb + \delta\right)$, and that $(I - \Delta)Qx_Q = \left(Qb + \delta\right)$. Theoretical results [1] reveals that $|x_Q|$ maximizes $|x|$ in $\sum \cap Q$ where $\sum(A, b)$ is the interval hull of solution set of the linear interval system.

3. Numerical Experiment.

EXAMPLE 1. Consider the following example given as problem 1.

AX=**b**

with $\mathbf{A} = I \pm \Delta$ and:

$$\Delta = \begin{pmatrix} 0.1, & 0.1, & 0.1, & 0.1, & 0.1 \\ 0.1, & 0.2, & 0.1, & 0.1, & 0.1 \\ 0.2, & 0.3, & 0.1, & 0.2, & 0.2 \\ 0.1, & 0.4, & 0.1, & 0.1, & 0.1 \\ 0.1, & 0.5, & 0.1, & 0.1, & 0.1 \end{pmatrix}, \qquad b = \left([1,7],[-10,-4],[-6,8],[8,9],[-10,2]\right)^T$$

The following results are given in Table 1.

Table 1. Our results computed from the given problem using Matlab 7.0 version

k	Oettli- Prager theorem with tolerable solution set x_k	Rohn's method (1.6) with tolerable solution set x_k	Results from original Oettli-Prager theorem (2.11) without tolerable solution set. x_k
1	[3.7500, 4.2500]	[2.5839, 5.4161]	[-0.7000, 4.8000]
2	[-7.4500, -6.5500]	[-9.3513, -4.6487]	[-12.7000, -5.8000]
3	[0.7000, 1.3000]	[-2.5158, 4.5158]	[-9.8000, 4.8000]
4	[6.6500, 10.3500]	[4.2785, 12.7215]	[3.3000, 8.0000]
5	[-6.5500, -1.4500]	[-9.1566, 1.1566]	[-5.7000, 2.6000]

4. Conclusion

We studied the effects of tolerable solution sets for the linear interval system with Oettli-Prager theorem and compared such results with a formular derived by (Rohn,2010) where by the inverse midpoint matrix happened to be a unit matrix. It was shown that tolerable solution sets for the Oettli-Prager theorem and a formular due to (Rohn,2010) could be the better alternatives due to their ability to narrow down the interval widths in the obtained results when compared with the results obtained from the original Oettli-Prager theorem of equation (2.11).

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