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Functorial Localization Formula on Mirror Principles.

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Abstract

This paper focuses on function theory or representation theories in a very elegant and substantial way in geometry. It is very interesting to see how special functions enter into geometry. We like to point out that Symmetric, Trigonometric and Theta- functions are representation theoretic formula, equivariant Euler classes, and the geometry of moduli spaces of stable maps. The boundary of these holomorphic discs lie in certain special Lagrangian sub-manifold, which have boundary in some vanishing cycle.

Keywords : Mirror principles, Symmetric series, linear sigma model, Euler data, hypergeometric series.

1.0 Introduction:

Mirror principle is a general method developed in [15] and [16] to compute characteristic classes and characteristic numbers on moduli spaces of stable maps in terms of hypergeometric type series. The counting of the numbers of curves in Calabi- Yau manifolds from mirror symmetry corresponds to the computation of Euler numbers. This principle computes quite general Hirzebruch multiplicative classes such as the total Chern classes.

Recall that a balloon manifold X is a projective manifold with torus action and isolated fixed points. For the purpose of our discussion, we define

$H = (H_1, . . . , H_k)$

as our basis of equivariant Kahler classes. Then X is called a balloon manifold if the following holds:

1. The restriction $H(p) \neq H(q)$ for any two fixed points $p, q \,\varepsilon\, X$.
2. The tangent bundle T_pX has linearly independent weights for any fixed point $p \,\varepsilon\, X$.

The 1-dimensional orbits in X joining every two fixed points in X are called **balloons** which are copies of p^1. The mirror principles are of the following:

- Linear and non-linear sigma model;
- Euler data;
- Ballons and hypergeometric Euler data.

Let X be a projective manifold in the model.

In order to achieve our aims, it is very interesting to see how special functions enter into geometry. Such as Symmetric function, Trigonometric functions and hyper geometric series, we construct the following models

a. Non-Linear sigma model In this model, we define the moduli space of stable maps of degree (1,d) by $M_d^g(X)$ and genus g into P^1 x X:

$$M_d^g(X) = \{(f, C) : f : C \rightarrow P^1 \times X\} \qquad (1.1)$$

with C a genus g (nodal) curve and $f(C) \in H^2(P^1 \times X, Z)$ has bi-degree *(1,d)*, modulo the obvious equivalence. For convenience the degree d will also be used as integers by choosing a basis in $H_2(X, Z)$.

b. Linear sigma model : In this model, we define the moduli space of stable maps of degree (1,d) by W_d for a toric manifold X which was first introduced by [23] and later by [20] for computations. It is a large toric manifold. (see [17])

Example: Let X = P^n, with homogeneous coordinate $[z_0, . . . , z_n]$. Then the linear sigma model is given by the polynomial space W_d with projective coordinate

$$[f_0(w_0, w_1), . . . , f_n(w_0, w_1)] \qquad (1.2)$$

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Where $f_j(w_0, w_1)$ are homogeneous polynomials of degree *d.*
In the genus 0 case, W_d can beviewed as the simplest compactification of the spaces of degree *d* maps from P^1 to X. The following basic Lemma connects this compactification with a stable map moduli space. A proof can be found in [18]
Lemma:
There exists an explicit equivariant map

$$\varphi: M_d^g(P^n) \to W_d. \tag{1.3}$$

Here the equivariance is with respect to the induced actions from the torus actions on X and P^1.
Roughly speaking the computation should be on $M_d^g(X)$. But in general $M_d^g(X)$ is very "singular" and complicated. But W_d is smooth and simple, our main strategy is to push-forward everything to W_d through the map $\varphi!$ The functorial localization formula below is one of the key tricks we used.
Let $M_{g,k}(d,X)$ be the moduli space of stable maps of genus *g* and degree *d* with *k* marked points into *X*. That is

$$M_{g,k}(d,X) = \{(f, C; x_1,...,x_k): \quad f: C \to X\} \tag{1.4}$$

with $x_1, \ldots, x_k$. *k* points on the genus *g* (nodal) curve *C* .

This modulli space may have higher dimension than expected, even worse, its different components may have different dimensions .To compute integrals on such space, we need to first define the integral. For this purpose, we have the notion of virtual cycles: the virtual fundamental class which is first given by [14] and later by [3]. Let us denote the modulli space by

$$LT_d^g(X) \in A_*(M_d^g)_T, \tag{1.5}$$

The equivariant analogue of the virtual fundamental cycle which is a class in the equivariant Chow group of cycles of $M_d^g(X)$. Another virtual cycle will also be used:

$$LT_{g,k}(d,X) \in A_*(M_{g,k}(d,X))_T. \tag{1.6}$$

Now let us introduce the starting data of the argument. We let V→ X be an equivalent concavex bundle. The notion of concave bundles was introduced in [4], it represents a direct sum of appositive and a negative bundles on X. From a concavex bundle V, we can induce vector bundles $V_d^g(X)$ on $M_{g,k}(d,X)$ by taking either $H^0(C, f^*V)$ *or* $H^1(C, f^*V)$, or their direct sum. Let b be a multiplicative characteristic class.

Problem: The main problem of mirror principle is to compute the integral

$$K_d^g = \int_{LT_{g,k}(d,X)} b(V_d^g). \tag{1.7}$$

More precisely, let λ, q be two formal variables. We would like to compute the generating series,

$$F(q,\lambda) = \sum_{d,g} K_d^g \lambda^g q^d \tag{1.8}$$

in terms of a certain natural explicit hyper geometric series. So far we have rather complete success for the case of balloon manifolds and genus *g = 0.*

2.0 Rational curves:

The mirror principle for the genus 0 case has been more or less fully developed, which implies almost all of the genus 0 conjectural formulas from string theory [16], [17], [18]. The most famous corollary is possibly the Candelas formula (Cd). In this note we will briefly review our approach to the higher genus mirror principle, which is still under progress with partial successes as discussed in [17]. Roughly speaking we now have the following general theorem:

Theorem: Assume g = 0. Mirror principle holds for balloon manifolds and any concavex bundles.
Remarks:
1. For toric manifolds, the above mirror principle implies almost all mirror conjectural formulas derived from string theory.
2. In the above statement of mirror principle, we need to require *splitting type* on V when restricted onto each balloon P^1, and certain condition on the first Chern class $c_1(V)$. (see [15,16]).

There are many non-split bundle V with given splitting type, such as TP^n; and many equivalent bundles over toric manifolds[19]. Such bundles will give non-complete intersection Calabi-Yau manifolds, such as Pfaffian variety; moduli space of rank 2 bundles over Riemann surface.

3. The special case of P^n., with V the sum of positive line bundles, b the Euler class, a second approach can be found in [4],[21] following [8]; for V direct sum of positive and negative line bundles, (see [5]). A mirror formula was proved by using relative stable maps.

Recently, Lee, the functorial localization formula [15] and deformations of normal cone were used to derived a mirror formula with V the sum of positive or negative line bundles,, and b the Euler class. However, it requires strong restrictions on the first Chern class of V, and it yields no information when V is the trivial bundle.

One of the most interesting corollary of the mirror principle is when we take V to be the sum of negative bundles. This gives the so-called local mirror symmetry, which is called geometric engineering in [10]. The examples include:

a). Take X to be the de Pezzo surface, P^1 x P^1 or P^2. Take V = K_X, the canonical line bundle and b the Euler class. In this case , the corresponding hypergeometric series are periods of elliptic curves, which are called the Seiberg-Witten curves[10]. Indeed the total space of *Kx* is an open CY, its mirror is the elliptic curve, the Seiberg-Witten curve.

b) The simplest but very interesting example is when X = P^1, $V = O(-1) \oplus O(-1)$ and *b* the Euler class. In this case we have the multiple cover formula of Candelas et al: $K_d = d^{-3}$. When X = $P^{1,}$ $V = O(-2)$ and b the total Chern class, we get a similar multiple cover formula $K_d = d^{-3.}$ Another very interesting example is when X = $P^{1,}$ $V = O(-3)$ and b the total Chern class [15].

3.0 Higher genus

As one may notice that, almost all of the techniques for genus 0 case work well for higher genus, except the last step of finding the hypergeometric type series which is more difficult in higher genus due to the complicated fixed point moduli spaces.

Functorial localization formula is one of the simple techniques used in the approach.
Let X and Y be two manifolds with torus action.

Lemma: Let $f : X \to Y$ be an equivariant map. Let $f \subset Y$ be a fixed component, and s be the fixed components in X. Let f_0 = $f|_E$, then for an equivariant cohomology class $\omega \in H_T^*(X)$, we have the identity on F:

$$f_{0*}[\frac{i_E^* \omega}{e_T(E/X)}] = \frac{i_F^*(f_* \omega)}{e_T(F/Y)}. \qquad (3.1)$$

It is interesting to note that this functorial localization formula is very much in the spirit of the Riemann- Roch formula. Functorial localization is one of the key ideas in [15] and [16]. This same idea was later used in [2] and [14]. Which apply this formula to φ, the collapsing map. Before that, let us first work out the fixed points in the nonlinear and linear sigma models with respect to the induced S^1-action from P^1, as well as some of its key properties.

The fixed points in $M_d^g(X)$ induced by the S^1- action on P^1 are given by the components:

$$F_r^{g_1,g_2} = M_{g_1,1}(r, X) \times_X M_{g_2,1}(d - r, X) \qquad (3.2)$$

With $g_1+g_2 = g$ and $r = 0, \ldots d$. By considering the pull-back of $b_T(V_d^g)$ through the projection:

$$\pi : M_d^g(X) \to M_{g,0}(d, X) \qquad (3.3)$$

And its restriction to $F_r^{g_1,g_2}$, we have the important

4.0 Gluing identity:

$$b_T(V) b_T(V_d^g) = b_T(V_d^{g_1}) b_T(V_{d-r}^{g_2}). \qquad (4.1)$$

The collapsing map φ, when restricted to $F_r^{g_1,g_2}$ is just the evaluation map *ev* into X at the gluing point. The next step is to get the so-called **Euler data** from the above gluing identity. Let us write

$$A_d^g = ev_*[\frac{i^* \pi^* b_T(V_d^g) \cap LT_{g,1}(d, X)}{e_T(F_d^{g,0} / M_d^g(X))^v}] \qquad (4.2)$$

which comes from the left hand side of the *Functorial Localization Formula*. Here we have actually used a virtual version of the functorial localization formula, which is proved by using the virtual Atiyah. Bott formula as generalized in[7]. The denominator $e_T(F_d^{g,0} / M_d^g(X))^v$ denotes the virtual equivariant Euler class. (See [16] and [17]).

Let us form the generating series:

$$A_d = \sum_g A_d^g \lambda^g, \quad A = \sum_d A_d e^{dt} \tag{4.3}$$

From gluing identity and the functorial localization formula we can derive the following identity:

$$b_T(V) \cdot i_r^* A_d^{g_1,g_2} = \bar{A}_r^{g_1} \cdot A_{d-r}^{g_2} \tag{4.4}$$

Where $i_r^* A_d^{g_1,g_2}$ is the local term from the localization of $b_T(V_d^g)$ onto $F_r^{g_1,g_2}$, and $\bar{A}_r$ denotes the switch of sign: α → -α. Here α is the weight of the S^1-action induced from the action on P^1. This then gives us quadratic relations among the A_d's. (see [17]).

The right hand side of the functorial localization formula is the localization of the push-forward class by φ:

$$\varphi_*[b_T(V_d^g) \cap LT_d^g(X)] \in A_*(W_d)_T \tag{4.5}$$

This is a polynomial class in α, Note that A_d^g is actually a rational class in α. Through functorial localization formula and localization on W_d, we derive, from the gluing identity, that $\{A_d\}$is an Euler data.

Here Euler data, roughly speaking, are the sequences of classes like A_d with properties like (4.4) and (4.5). The connection between (4.4) and (4.5) is the functorial localization formula. From the above discussion, we see that any triple *(X,V,b)* induces an Euler data through the functorial localization formula.

On the other hand, we know that knowing A_d^g is equivalent to knowing K_d^g, as given by the following:

Lemma: We have the following

$$\alpha^{g-3}(2-2g-d\cdot t)K_d^g = \int_X e^{-t\cdot H/\alpha} A_d^g. \tag{4.6}$$

So the problem is reducing to the computation of the Euler data A_d. The next step in our approach is to approximate A_d by restricting to "smooth part" or "generic part" of $M_{g,k}(d,X)$.

When the genus g = 0, by localization to smooth fixed points, the multiple covers of the balloons, which are those complex 1- dimensional orbits in X. When restricting the A_d to those smooth fixed points in $M_{0,1}(d,X)$, we get another class B_d which is an explicit hypergeometric type and cohomology class. Here we just illustrate by a typical example:

Example: Let X = P^n, $V = O(l)$ and b = Euler class. Then we have

$$B_d = \frac{\prod_{m=0}^{ld}(lH - m\alpha)}{\prod_{m=1}^{d}(H - m\alpha)^{n+1}}. \tag{4.7}$$

The general toric case is very similar, and B_d is also read out from localization on the balloons. Here for general vector bundle V, the splitting type comes in.

By applying localization formula on the sigma model W_d of X, we find that B_d is also an Euler data. And we know that A_d = B_d at the smooth points, which we called them linked. Together with a Lagrange interpolation type argument, we derive the following uniqueness lemma by using localization again:

Lemma: If $\deg_\alpha (A_d = B_d) \le -2$, then $A_d = B_d$.

That is to say that B_d determines A_d up to degree -2. But in general B_d has higher degree is zero. Then we can always find a so-called mirror transformation to decrease its degree to -2. Here is one typical example of the mirror formula as a corollary of mirror principle:

Example: Let X be a toric manifold; consider the case of g = 0. Let $D_1, \ldots, D_N$ be the T-invariant divisors, and V the direct sum of positive line bundles: $V = \oplus_i L_i, \quad c_1(L_i) \ge 0$ and $c_i(X) = c_i(V)$.

Let $b(V) = e(V)$, $\Phi(T) = \Sigma K_d e^{d.T}$, and

$$B(t) = e^{-H\cdot t}\sum_d \prod_i \prod_{k=0}^{\langle c_i(L_i),d\rangle}(c_i(L_i)-k) \times \frac{\prod_{\langle D_\alpha,d\rangle<0}\prod_{k=0}^{-\langle D_\alpha,d\rangle-1}(D_\alpha+k)}{\prod_{\langle D_\alpha,d\rangle\ge 0}\prod_{k=1}^{\langle D_\alpha,d\rangle}(D_\alpha-k)} e^{d\cdot t}.$$

Then the mirror principle implies that there are explicitly computable functions *f(t), g(t),* such that

$$\int_X (e^f B(t) - e^{-H\cdot T} e(V)) = 2\Phi - \sum T_i \tfrac{\partial\Phi}{\partial T_i} \tag{4.8}$$

where $T = t + g(t)$. From this formula we can determine Φ uniquely.

The whole argument is actually genus independent except finding B_d. The problem is that for g > 0, the good fixed points are given by the Deligne-Mumford moduli space of stable curves $M_{g,1}$. And when localizing A_d to such fixed points, we get explicit Hodge integrals on $M_{g,1}$, which are all explicitly computable. Our approach works well until the last step: we can not figure out a simple B_d from the integral on $M_{g,1}$, which is again an Euler data, to approximate A_d.

But the fact that A_d is an Euler data already puts very strong restriction on such sequences, and this determines it up to certain degree. Such restrictions are all quadratic and compatible with mirror symmetry for higher genus by [10]. Even if we take X a single point in our non-linear sigma model, we already get strong information on the Hodge integrals on $M_{g,1}$.[6,22] At this point we are trying to design more refined localization involving M_g to find some more refined relations among the A_d's.

Euler date is a general notion that can include general Gromov-Witten invariants. We can consider marked points to the moduli spaces and add the pull-back classes to the A_d's. More precisely we can try to compute integrals of the form:

$$K^g_{d,k} = \int_{LT_{g,k}(d,X)} \prod_j ev_j^* \omega_j \cdot b(V_d^g) \qquad (4.9)$$

where $\omega_j \in H^*(X)$.

By introducing the generating series with summation over k, we can still get Euler data. The *Ultimate Mirror* we are searching for is the following statement: *Compute this series by explicit hypergeometric series!* Our discussion above has reduced this to the problem of finding the hypergeometric Euler data B_d*'s.*

Conclusion

Counting holomorphic discs: the boundary of these holomorphic discs lie in certain special Lagrangian sub-manifold, which is some vanishing cycle, in X. We hope to extend mirror principle to deal with such problems. Nonlinear sigma model has been studied by Fukaya et al, and linear sigma model has been worked out in string theory. In this situation both sigma models have boundaries. The string theorists [10] have made several interesting conjectures. Some progresses have been made in [11] and [13].

The Gopakunar-Vafa formula: This formula [9] reinterpretes the rational number K_d^g in terms of certain integer valued instanton numbers n_d^g, generalizing the multiple cover formula for rational curves. In particular, in the genus zero case, this gives rise to integer series expansions for Yukawa couplings. The question of integrality and divisibility of these series expansions were first studied in[19], as a special of integrality conjecture. Using the formula of [9] and this conjecture as a guide, we hope to construct hypergeometric Euler data, which is linked to A_d.

Lastly, this paper shows the extent at which the special functions enter into geometry through the symmetric, Trigonometric functions and Hypergeometric series of modulli spaces of stable maps. The boundary of these holomorphic discs lie in certain special Lagrangian sub-manifold, which have boundary in some vanishing cycle.

References

[1] Atiyah M and Bott R(1984). *The moment map and equivariant cohomology*, Topology Bull. Amer. Math. Soc. **23** pp 1-28.

[2] Bertram, A.(1999) *Another way to enumerate rational curves with tours action*,. Proc. Amer. Math. Soc. **85**, 140-145.

[3] Behrend K and Fentachi B(1997). *The intrinsic normal cone*,Physica Scripta, vol. 128 pp 45-88.

[4] Bini G., De Concini C., Polito M. and Procesi C.,(1998) *Givental's work relative to mirror symmetry* math. Bull. London Math. Soc. **15**, 233-242.

[5] Elezi A., (2000)*Mirror symmetry for concave bundles on projective spaces*, math. Duxbury Press, Belmont.

[6] Faber C., and Pandharipande R.(1998) *Hodge Integrals and Gromov-Witten Theory*,. Acta Sci. Math. Szeged **12** 75-81

[7] Graber T. and Pandharipanda R.(1997) *Localization of virtual classes*, al-geom. Math. Ann. 79, 150-156

[8] Givental A.,(1996)Equivariant Gromov-Witten invariants, Proc. Lond. Math. Soc., II. **70**, 234-245.

[9] Gopakumar, R. and Vafa, C.(1998) *M-theory and topological strings* II, Proc. Lond. Math. Soc., III. **55**, 347-356.

[10] Kaatz S, Klemm A and Vafa C (1997). *Geometric engineering of quantum field theories*, Nuclear Physics B497 pp 173-195.

[11] Katz, S, Liu, C. (2001)*Enumerative geometry of stable maps with Lagrangian boundary conditions and the multiple cover formula for the disc*, math. Stoch. Proc. Appl. 64, 285-299.

[12] Kontsevich, M. (2001)*Enumeration of rational curves via torus actions*. In: The Moduli Space of Curves, edited by Dijkgraaf, R., Faber, G., Geer van der, progress in Math. Vol. 129 Birkhauser, 195, pp. 333 – 368.

[13] Li, J, Song, Y.(2001) *Open string instantons and relative stable morphisms,* J. of American mathematics Society 8, no 2, pp. 301 – 312.

[14] Li J and Tian G (1998). *Virtual moduli cycle and Gromov-Witten invariants of algebraic varieties*, J. of American mathematics Society 11, no 1 pp. 119-174.
[15] Lian B, Liu K and Yau S. T(1997). *Mirror principle*, I Asian J. Math 1 pp. 729-763.
[16] Lian B, Liu K and Yau S. T(1999). *Mirror principle*, II Asian J. Math 3. pp.154-177
[17] Lian B, Liu K and Yau S. T(2000). *Mirror principle*, III Asian J. Math 3. Pp.362 -392
[18] Lian B, Liu K and Yau S. T(2000). *Mirror principle*, IV Asian J. Math. Survey in Differential Geometry. Pp. 567- 603
[19] Lian, B, Yau, S.T,(1994) *Arithmetic properties of mirror map and quantum Coupling*. Proc. Camb. Phil. Soc. 102, 287-296.
[20] Morrison, D. and Plesser R.,(1994) *Summing the instantons: quantum cohomology and mirror symmetry in toric varieties*, Commun. Math. Phys. 176, 321-343
[21] Pandharipande, R., (1998) *Rational curves on hypersurfaces (after givental)* math.3rd edition. McGraw-Hill..
[22] Tian G, Zhou J (2001). *Quadratic recursion relations of Hodge integrals via localization*. J. of American mathematics Society 8, no2, Pp. 504 – 562.
[23] Witten, E.(1997) *Phases of N=2 theories in two dimension*, Cambridge University Press