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On certain isotopic maps of central loops *†

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Abstract

It is shown that the Holomorph of a C-loop is a C-loop if each element of the automorphism group of the loops is left nuclear. Condition under which an element of the Bryant-Schneider group of a C-loop will form an automorphism is established. It is proved that elements of the Bryant-Schneider group of a C-loop can be expressed a product of pseudo-automorphisms and right translations of elements of the nucleus of the loop. The Bryant-Schneider group of a C-loop is also shown to be a kind of generalized Holomorph of the loop.

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1.0 Introduction:

Central loops(C-loops) are loops which satisfy one of the identities called 'Central identity' as named by F. Fenyves [9], [10]. Closely related to the central identity are left central (LC) and right central (RC) identities. The expressions for the mentioned identities are as follows;

$(yx \cdot x)z = y(x \cdot xz)$ central identity (1)

$i. xx \cdot yz = (x \cdot xy)z \equiv ii. (x \cdot xy)z = x(x \cdot yz) \equiv iii. (xx \cdot y)z = x(x \cdot yz)$ LC-identities (2)

$i. yz \cdot xx = y(zx \cdot x) \equiv ii. (yz \cdot x)x = y(zx \cdot x) \equiv iii. (yz.x)x = y(z \cdot xx)$ RC-identities (3)

Recently Phillips and Vojtechovsky [20], found out that in addition to the identities above, LC and RC identities can also be defined respectively by,

$(y \cdot xx)z = y(x \cdot xz)$ and $(yx \cdot x)z = y(xx \cdot z)$ (4)

$C - loops$ are one of the least studied loops. Few publications that have considered C-loops include Fenyves [9], [10], Phillips and Vojtechovsky [18] [20] [19], Chein [5]. The difficulty in studying them is as a result of the nature of their identities when compared with other Bol-Moufang identities (the element occurring twice on both sides has no other element separating it from itself).

1 Preliminaries

Theorem 2.1 ([10], [20]) Let $(L,\cdot)$ be an LC-loop(RC-loop). Then :

1. $(L,\cdot)$ is a left (right) alternative loop,
2. $(L,\cdot)$ is a left (right) inverse property loop,
3. $(L,\cdot)$ is a left (right) nuclear square loop,
4. $(L,\cdot)$ is a left (right) power alternative loop,
5. $(L,\cdot)$ is a middle square loop,
6. $(L,\cdot)$ is power associative loop.

Definition 2.1 A triple (α, β, γ) of bijections is called an isotopism of loop $(L,\cdot)$ onto a loop $(H,\circ)$ provided $x\alpha \circ y\beta = (x \cdot y)\gamma \; \forall \; x, y \in L. (H,\circ)$ is called an isotope of $(L,\cdot)$. The loops $(L,\cdot)$ and $(H,\circ)$ are said to be isotope to each other.

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Definition 2.2 Let α and β be a permutation of L and let i denote identity map on L. Then (α,β,ι) is a principal isotopism of a loop $(L,\cdot)$ onto a loop $(L,\circ)$ which imply that (α,β,ι) is an isotopism of $(L,\cdot)$ onto $(L,\circ)$.
Definition 2.3 An isotopism of $(L,\cdot)$ onto $(L,\cdot)$ is called an autotopism of $(L,\cdot)$. The group of autotopism of L is denoted by $A(L)$.
Remark 2.1 The components of isotopism are usually denoted by lower case Greek letters, thus if $T=(U,V,W)$ is an autotopism of a loop $(L,\cdot)$, then

$$xU\cdot yV=(xy)W, \forall\, x,y\in L.$$

The set of all autotopism of a loop is a group with the inverse of $TT^{-1}=(U,V,W)^{-1}=(U^{-1},V^{-1},W^{-1})$. The identity element of the group being (I,I,I) where I is the identity map of L. If $T=(U,U,U)$, then T is called the automorphism of $(L,\cdot)$.
Definition 2.4 If $\langle U,V,W\rangle$ is autotopism of an inverse property loop $(L,\cdot)$, then $\langle W,JVJ,U\rangle$ and $\langle JUJ,W,V\rangle$ are autotopisms of L. Moreover if $\langle U,V,W\rangle=\langle S,SR_{c,}SR_c\rangle$ the S is called a pseudoautomorphism of L with companion c. The set of all pseudoautomorphisms of L is denoted by $PS(L,\cdot)$.
Definition 2.5 Let $(L,\cdot)$ be an inverse property loop with the nucleus denoted by N. Then an automorphism α of $(L,\cdot)$ is left nuclear iff $a\alpha\cdot a^{-1}\in N$ for all $a\in L$.
Definition 2.6 Let $(L,\cdot)$ be a loop and $BS(L,\cdot)$ be the set of all permutations θ of Q such that

$$\langle\theta R_g^{-1},\theta L_f^{-1},\theta\rangle$$

is an autotopism of $(L,\cdot)$ for some $f,g\in L$, then $BS(L,\cdot)$ is called the Bryant-Schneider group of the loop.
Definition 2.7 Let $(L,\cdot)$ be a loop, $A(L)$ a group of automorphisms of loop $(L,\cdot)$ and let $HH=A(L)\times L$ and define

$$(\alpha,x)\circ(\beta,y)=(\alpha\beta,x\beta\cdot y)$$

$V(\alpha,x),(\beta,y)\in H$. Then the loop $(H,\circ)$ is called the $A(L)-holomorph$ of $(L,\cdot)$ or simply holomorph of $(L,\cdot)$.

3 Holomorphy

Theorem 3.1 Let $(L,\cdot)$ be an LC-loop and $A(L)$ be a group of automorphism of $(L,\cdot)$. Then the $A(L)-holomorph$ $(H,\circ)$ of $(L,\cdot)$ is an LC-loop if and only if

$$(x\alpha\cdot xy)z=x\alpha(x\cdot yz) \quad (5)$$

$\forall\, x,y,z\in L$ and $\forall\,\alpha\in A(L)$.

Proof.
Suppose $A(L)-holomorph$ $(H,\circ)$ of $(L,\cdot)$ is an LC-loop we have

$$\{(\alpha,x)\circ[(\alpha,x)\circ(\beta,y)]\}\circ(\gamma,z)=(\alpha,x)\circ\{(\alpha,x)\circ[(\beta,y)\circ(\gamma,z)]\} \quad (6)$$

$\forall\, x,y,z\in L$ and $\forall\,\alpha,\beta,\gamma\in A(L)$. Thus

$$\{(\alpha,x)\circ(\alpha\beta,x\beta\cdot y)\}\circ(\gamma,z)=(\alpha,x)\circ\{(\alpha,x)\circ(\beta\gamma,y\gamma\cdot z)\}$$
$$\{\alpha\cdot\alpha\beta,x\alpha\beta\cdot(x\beta\cdot y)\}\circ(\gamma,z)=(\alpha,x)\circ\{(\alpha\cdot\beta\gamma,x\beta\gamma\cdot(y\gamma\cdot z)$$
$$\{(\alpha\cdot\alpha\beta)\gamma,[x\alpha\beta\cdot(x\beta\cdot y)]\gamma\cdot z\}=\{\alpha(\alpha\cdot\beta\gamma),x\alpha\cdot\beta\gamma\cdot x\beta\gamma(y\gamma\cdot z)\}$$

$\forall\, x,y,z\in L$ and $\forall\alpha,\beta,\gamma\in A(L)$. Therefore

$$\{x\alpha\beta\cdot(x\beta\cdot y)\}\gamma\cdot z=x\alpha\cdot\beta\gamma\cdot x\beta\gamma(y\gamma\cdot z)$$

$\forall\, x,y,z\in L$ and $\forall\,\alpha,\beta,\gamma\in A(L)$. Therefore,

$$\{x\alpha\cdot\beta\gamma\cdot(x\beta\gamma\cdot y\gamma)\}\cdot z=x\alpha\cdot\beta\gamma\cdot x\beta\gamma\cdot(y\gamma\cdot z)$$

putting $\emptyset=\beta\gamma$, gives

$$\{x\alpha\emptyset\cdot(x\emptyset\cdot y\gamma)\}z=x\alpha\emptyset\cdot x\emptyset(y\gamma\cdot z)$$

Hence $\{x\alpha\cdot(x\cdot y\gamma\emptyset^{-1})\}\cdot z\emptyset^{-1}=\{x\alpha\cdot x(y\gamma\emptyset^{-1}\cdot z\emptyset^{-1})\}$

$\forall\, x,y,z\in L$ and $\forall\,\alpha,\beta,\gamma\in A(L)$. If we put $\bar{y}=y\gamma\emptyset^{-1}$ and $\bar{z}=z\emptyset^{-1}$, we obtain $(x\alpha\cdot x\bar{y})\bar{z}=x\alpha\cdot(x\cdot\bar{y}\bar{z})$
And replacing $\bar{y}$ and $\bar{z}$ by y and z respectively we have

$$(x\alpha\cdot xy)z=x\alpha(x\cdot yz)$$

$\forall\, x,y,z\in L$ and $\forall\,\alpha\in A(L)$, which is equation (5).
The converse is obtained by reversing the process.
Corollary 3.1 Let $(L\,\cdot)$ be a loop and $A(L)$ be the group of all automorphism of L, then L is an LC-loop if

$$B=\langle L_xL_{x\alpha},I,L_xL_{x\alpha}\rangle \quad (7)$$

is an autotopism of L, $\forall\, x,y,z\in L$ and $\forall\,\alpha\in A(L)$
Proof. This is a consequence of (5)
Theorem 3.2 Let $(L\,\cdot)$ be a loop and $A(L)$ be a group of automorphism of $(L\,\cdot)$. Then the A(L)-holomorph $(H,\circ)$ of $(L\,\cdot)$ is an RC-loop if and only if

$$y\big((z\cdot x\alpha)x\big)=(yz\cdot x\alpha)x \quad (8)$$

$\forall\, x,y,z\in L$ and $\forall\,\alpha\in A(L)$.

Proof.
The procedure for the proof is like that of Theorem 3.1 above hence it is omitted.
Corollary 3.2 Let $(L\,\cdot)$ be any loop and $A(L)$ be the group of all automorphisms of L, then L is an RC-loop if and only if

$$B=\langle I,R_{x\alpha}R_x,R_{x\alpha}R_x\rangle \quad (9)$$

is an autotopism of L, $\forall\, x,y,z\in L$ and $\forall\,\alpha\in A(L)$

Proof.
From (8)

$y\big((z\cdot x\alpha)x\big)=(yz\cdot x\alpha)x$

$\Rightarrow y\cdot zR_{x\alpha}R_x=yzR_{x\alpha}R_x$

$\forall\, x,y,z\in L$ and $\forall\, \alpha\in A(L)$.

$\Longrightarrow \langle I,R_{x\alpha}R_x,R_{x\alpha}R_x\rangle$

is an autotopism of $(L,\cdot)$ $\forall\, x,y,z\in L$ and $\forall\, \alpha\in A(L)$

Conversely, suppose (9) hold, then $\forall\, y,z\in L$ we have

$yI\cdot zR_{x\alpha}R_x=yzR_{x\alpha}R_x$

$y\big((z\cdot x\alpha)x\big)=y(x\alpha\cdot xz)$

$\forall\, x,y,z\in L$ and $\forall\, \alpha\in A(L)$.

Theorem 3.3 Let $(L,\cdot)$ be a loop and $A(L)$ be the group of automorphism of $(L\ \cdot)$. Then the A(L)-holomorph $(H,\circ)$ of $(L\ \cdot)$ is a C-loop if and only if

$$(y\cdot x\alpha)x\cdot z=y(x\alpha\cdot xz) \qquad (10)$$

$\forall\, x,y,z\in L$ and $\forall\, \alpha\in A(L)$.

Proof.
The procedure for the proof is like that of theorem 3.1 hence it is omitted.

Corollary 3.3 Let $(L,\cdot)$ be a loop and $A(L)$ be the group of automorphism of $(L\ \cdot)$. Then L is a C-loop if and only if

$$B=\langle R_{x\alpha}R_x,L_{x\alpha}^{-1}L_{x^{-1}},I\rangle \qquad (11)$$

is an autotopism of $(L,\cdot)$ $\forall\, x,y,z\in L$ and $\forall\, \alpha\in A(L)$.

Proof. From (10)

$(y\cdot x\alpha)x\cdot z=y(x\alpha\cdot xz)$

$\Longrightarrow yR_{x\alpha}R_x\cdot z=y\cdot zR_xR_{x\alpha}$

$\forall\, x,y,z\in L$ and $\forall\, \alpha\in A(L)$.

substituting $\bar z=zL_xL_{x\alpha}$ we have

$yR_{x\alpha}R_x\cdot \bar zL_{(x\alpha)^{-1}}L_{x^{-1}}=y\bar z$

$\forall\, x,y,z\in L$ and $\forall\, \alpha\in A(L)$.

$\Longrightarrow \langle R_{x\alpha}R_x,L_{(x\alpha)^{-1}}L_{x^{-1}},I\rangle$

is an autotopism of $(L,\cdot)$ $\forall\, x,y,z\in L$ and $\forall\, \alpha\in A(L)$.

3.1 Nuclear Automorphism

Theorem 3.4 Let $(L,\cdot)$ be a loop and $A(L)$ be the group of automorphism of $(L\ \cdot)$. Then the A(L)-holomorph $(H,\circ)$ of $(L\ \cdot)$ is a C-loop if and only if $(L,\cdot)$ is a C-loop and each $\alpha\in A(L)$ is a left nuclear automorphism of $(L,\cdot)$.

Proof. Suppose $(H,\circ)$ is a C-loop. Since $(L,\cdot)$ is isomorphic to a subloop of $(H,\circ)$, it follows that $(L,\cdot)$ must be a C-loop. From Theorem (3.1), equation (5) holds $\forall\, x,y,z\in L$ and $\forall\, \alpha\in A(L)$. Furthermore, by Theorem (3.1) and Corollary (3.3),

$A(x)=\langle R_x^2,L_x^{-2},I\rangle$ and $B(x)=\langle R_xR_{x\alpha},L_x^{-1}L_{x\alpha}^{-1},I\rangle$

are autotopisms of $(L,\cdot)$, $\forall\, x\in L$ and $\forall\, \alpha\in A(L)$. Therefore by Theorem (3.1) and we have

$A_\lambda(x)=\langle L_x^{-2},I,L_x^{-2}\rangle$, $A_\mu^{-1}(x)=\langle I,R_x^{-2},R_x^{-2}\rangle$, $B_\lambda^{-1}(x)=\langle L_{x\alpha}L_x,I,L_{x\alpha}L_x\rangle$ and $B_\mu(x)=\langle I,R_xR_{x\alpha},R_xR_{x\alpha}\rangle$ are autotopisms of $(L,\cdot)$, $\forall\, x\in L$ and $\forall\, \alpha\in A(L)$. If these are combined we have

$A_\lambda(x)B_\lambda^{-1}(x)=\langle L_x^{-2},I,L_x^{-2}\rangle\langle L_{x\alpha}L_x,I,L_{x\alpha}L_x\rangle$

$$A_\lambda(x)B_\lambda^{-1}(x)=\langle L_x^{-1}L_{x\alpha},I,L_x^{-1}L_{x\alpha}\rangle \qquad (12)$$

And $B_\mu(x)A_\mu^{-1}(x)=\langle I,R_xR_{x\alpha},R_xR_{x\alpha}\rangle\langle I,R_x^{-2},R_x^{-2}\rangle$

$$B_\mu(x)A_\mu^{-1}(x)=\langle I,R_{x\alpha}R_x^{-1},R_{x\alpha}R_x^{-1}\rangle \qquad (13)$$

as autotopisms of $(L,\cdot)$, $\forall\, x\in L$ and $\forall\, \alpha\in A(L)$. Now if we apply (12) and (13) to $1\cdot b$ and $a\cdot 1$ respectively, we have

$1L_x^{-1}L_{x\alpha}\cdot b=(1\cdot b)L_x^{-1}L_{x\alpha}$

$(x\alpha\cdot x^{-1})b=bL(x)^{-1}L_{x\alpha}$

$bL_{x\alpha\cdot x^{-1}}=bL_x^{-1}L_{x\alpha}$

and $a\cdot 1R_{x\alpha}R_x^{-1}=(a\cdot 1)R_{x\alpha}R_x^{-1}$

$a(x\alpha\cdot x^{-1})=aR_{x\alpha}R_x^{-1}$

$aR_{x\alpha\cdot x^{-1}}=aR_{x\alpha}R_x^{-1}$

And respectively we have

$$L_{x\alpha\cdot x^{-1}}=L_x^{-1}L_{x\alpha} \qquad (14)$$

$$R_{x\alpha\cdot x^{-1}}=R_{x\alpha}R_x^{-1} \qquad (15)$$

$\forall\, x\in L$ and $\forall\, \alpha\in A(L)$. If we put equations (14) and (15) into equations (12) and (13) respectively, we have

$A_\lambda(x)B_\lambda^{-1}(x)=\langle L_{x\alpha\cdot x^{-1}},I,L_{x\alpha\cdot x^{-1}}\rangle$

and $B_\mu(x)A_\mu^{-1}(x)=\langle I,R_{x\alpha\cdot x^{-1}},R_{x\alpha\cdot x^{-1}}\rangle$

$\forall\, x\in L$ and $\forall\, \alpha\in A(L)$. These therefore imply that $x\alpha\cdot x^{-1}\in N_\lambda(L)$ and $x\alpha\cdot x^{-1}\in N_\rho(L)$.

Consequently, $x\alpha \cdot x^{-1} \in N(L)$ since $(L,\cdot)$ is an inverse property loop. Hence $\alpha \in A(L)$, is left nuclear.
Conversely, suppose $(L,\cdot)$ is a C-loop and each $\alpha \in A(L)$ is left nuclear. Then for each $\alpha \in A(L)$ and each $x \in L$ the element $x\alpha \cdot x^{-1} \in N_\mu(L)$, thus

$x\alpha \cdot y = ((x\alpha \cdot x^{-1})x)y$

$x\alpha \cdot y = (x\alpha \cdot x^{-1})xy$

$\forall\, y \in L$.

$yL_{x\alpha} = yL_xL_{x\alpha\cdot x^{-1}} \Rightarrow L_x^{-1}L_{x\alpha} = L_{x\alpha\cdot x^{-1}}$

$\forall\, x \in L$ and $\forall\, \alpha \in A(L)$. But for $\forall\, x \in L$ and $\forall\, \alpha \in A(L)$, we know that $x\alpha \cdot x^{-1} \in N_\lambda(L)$. Hence,

$C = \langle L_{\alpha\cdot x^{-1}}, I, L_{x\alpha\cdot x^{-1}}\rangle = \langle L_x^{-1}L_{x\alpha}, I, L_x^{-1}L_{x\alpha}\rangle$

is an autotopism of $(L,\cdot), \forall\, x \in L$ and $\alpha \in A(L)$. But again, $A = \langle L_x^2, I, L_x^2\rangle$ is an autotopism of $(L,\cdot), \forall\, x \in L$. Therefore, $AC = \langle L_xL_{x\alpha}, I, L_xL_{x\alpha}\rangle$
is an autotopism of $(L,\cdot)$, $\forall\, x \in L$ and $\forall\, \alpha \in A(L)$. So also is $(AC)_\lambda^{-1} = \langle R_{x\alpha}R_x, L_{x\alpha}^{-1}L_x^{-1}, I\rangle$. Therefore of yz, $\forall\, y,z \in L$, we have $yR_{x\alpha}R_x \cdot zL_{x\alpha}^{-1}L_x^{-1} = yz$
if we put $\bar{z} = zL_{x\alpha}^{-1}L_x^{-1}$, in this we have

$yR_{x\alpha}R_x \cdot \bar{z} = y \cdot \bar{z}L_xL_{x\alpha}$

$\big((y \cdot x\alpha)x\big)\bar{z} = y(x\alpha \cdot x\bar{z})$

$\forall\, x, y, \bar{z} \in L$ and $\forall\, \alpha \in A(L)$. Replacing $\bar{z}$ by $z, \forall\, x, y, z \in L$ and $\alpha \in A(L)$ and we a central identity. Hence, $(H,\circ)$ is a C-loop.

Theorem 3.5 The set $S(L)$ of all left nuclear automorphism of a C-loop $(L,\cdot)$, is a normal subgroup of the automorphism group of $(L,\cdot)$.

Proof. $S(L) \neq \emptyset$, from the Theorem 3.4 it was shown that

$L_{u\alpha\cdot u^{-1}} = L_u^{-1}L_{u\alpha}$

$\forall\, u \in L$ and $\forall\, \alpha \in S(L)$ (since for an inverse property loop L, $L_{u^{-1}} = L_u^{-1}\ \forall\, u \in L$). Then $u\alpha \cdot u^{-1} \in N_\lambda(L,\cdot), \forall\, u \in L$ and $\forall\, \alpha \in S(L)$. It follows then that

$A(\alpha, u) = \langle L_{u\alpha\cdot u^{-1}}, I, L_{u\alpha\cdot u^{-1}}\rangle = \langle L_u^{-1}L_{u\alpha}, I, L_u^{-1}L_{u\alpha}\rangle$

$\forall\, u \in L$ and for all $\alpha \in L$. Hence if $\alpha, \beta \in S(L)$, we have

$A(\alpha, u)A(\beta, u\alpha) = \langle L_u^{-1}L_{u\alpha}, I, L_u^{-1}L_{u\alpha}\rangle\langle L_{u\alpha}^{-1}L_{u\alpha\beta}, I, L_{u\alpha}^{-1}L_{u\alpha\beta}\rangle$

$$A(\alpha, u)A(\beta, u\alpha) = \langle L_u^{-1}L_{u\alpha\beta}, I, L_u^{-1}L_{u\alpha\beta}\rangle \tag{16}$$

is an autotopism of $(L,\cdot), \forall\, u \in L$. Therefore $\forall\, y \in L$ we have

$1L_u^{-1}L_{u\alpha\beta} \cdot y = (1 \cdot y)L_u^{-1}L_{u\alpha\beta}$

$(u\alpha\beta \cdot u^{-1}) \cdot y = yL_u^{-1}L_{u\alpha\beta}$

$yL_{u\alpha\beta\cdot u^{-1}} = yL_u^{-1}L_{u\alpha\beta}$

$$\Longrightarrow L_{u\alpha\beta\cdot u^{-1}} = L_{u^{-1}}L_{u\alpha\beta} \tag{17}$$

Thus, (17) into (16) gives

$$A(\alpha, u)A(\beta, u\alpha) = \langle L_{u\alpha\beta\cdot u^{-1}}, I, L_{u\alpha\beta\cdot u^{-1}}\rangle \tag{18}$$

From equation (18), $u\alpha\beta \cdot u^{-1} \in N_\lambda(L), \forall\, u \in L$, hence $u\alpha\beta \cdot u^{-1} \in N, u \in L$ so also $\alpha\beta \in S(L)$, since $(L,\cdot)$ is an inverse property loop.

If $\alpha \in S(L)$, then $A(\alpha, u)$ is an autotopism of $(L,\cdot)\ \forall\, u \in L$, and so is $A(\alpha, u\alpha^{-1})^{-1}\ \forall\, u \in L$, i.e.

$A(\alpha, u\alpha^{-1})^{-1} = \langle L_{u\alpha^{-1}}^{-1}L_{u\alpha^{-1}\cdot\alpha}, I, L_{u\alpha^{-1}}^{-1}L_{\alpha^{-1}\cdot\alpha}\rangle^{-1}$

$= \langle L_{u\alpha^{-1}}^{-1}L_u, I, L_{u\alpha^{-1}}^{-1}L_u\rangle^{-1}$

$= \langle L_u^{-1}L_{u\alpha^{-1}}, I, L_u^{-1}L_{u\alpha}^{-1}\rangle$

$= \langle L(u\alpha^{-1} \cdot u^{-1}), I, L(u\alpha^{-1} \cdot u^{-1})\rangle$

Hence it follows that $\alpha^{-1} \in S(L)$. Thus $S(L)$ is a subgroup of the automorphism group of $(L,\cdot)$.

Let $\alpha \in S(L)$, then $u\alpha \cdot \alpha^{-1} \in N_\lambda(L,\cdot), \forall\, u \in$ and

$(u\alpha \cdot u^{-1})xy = (u\alpha \cdot u^{-1})x \cdot y$

$\forall\, u, x, y \in L$, if γ is an automorphism of $(L,\cdot)$, then we have

$\{u\alpha\gamma \cdot (u\gamma)^{-1}\}(x\gamma \cdot y\gamma) = \{u\alpha\gamma \cdot (u\gamma)^{-1}\}x\gamma \cdot y\gamma$

$\forall\, u, x, y \in L$, and if we replace u by $u\gamma^{-1}$ in the last expression, we have

$(u\gamma^{-1}\alpha\gamma \cdot u^{-1})(x\gamma \cdot y\gamma) = (u\gamma^{-1}\alpha\gamma \cdot u^{-1})x\gamma \cdot y\gamma$

Thus, $u\gamma^{-1}\alpha\gamma \cdot u^{-1} \in N_\lambda(L,\cdot)$ and since L is an inverse property loop, the three nuclei coincide, then $u\gamma^{-1}\alpha\gamma \cdot u^{-1} \in N(L,\cdot)$ for all $u \in L$ and all automorphism γ of $(L,\cdot)$. Hence $\gamma^{-1}\alpha\gamma \in S(L)$ for all $\alpha \in S(L)$ and all automorphism γ of $(L,\cdot)$. So $S(L)$ is indeed normal in the automorphism group of $A(L)$ of $(L,\cdot)$.

2 Bryant-Schneider group

Theorem 4.1 Let $(L,\cdot)$ be a C-loop, an element θ of the Bryant-Schneider group of L is an automorphism of L provided

$\langle \theta R_{g^{-1}}, \theta L_{f^{-1}}, \theta\rangle$

is an autotopism of $(L,\cdot)$ if f and g are elements of the nucleus of $(L,\cdot)$.

Proof: Let $(L,\cdot)$ be a C-loop then

$\langle R_{y^{-1}} R_{y^{-1}}, L_y L_y, I\rangle$

is an autotopism for all $x \in L. \theta \in BS(L,\cdot)$ imply that $\langle \theta R_{g^{-1}}, \theta L_{f^{-1}}, \theta\rangle$ is also an autotopism for some $g, f \in (L,\cdot)$. Hence $\langle \theta R_{g^{-1}}, \theta L_{f^{-1}}, \theta\rangle\langle R_{y^{-1}} R_{y^{-1}}, L_y L_y, I\rangle = \langle \theta R_{g^{-1}} R_{y^{-1}} R_{y^{-1}}, \theta L_{f^{-1}} L_y L_y, \theta\rangle$ is an autotopism of $(L,\cdot)$ for all $y \in L$ and some $g, f \in L$. Since $(L,\cdot)$ is an alternative property loop, then

$R_{y^{-1}} R_{y^{-1}} = R_{(y^{-1})^2} = R_{(y^2)^{-1}}$

and $L_y L_y = L_{y^2}$ therefore $\langle \theta R_{g^{-1}} R_{y^{-1}} R_{y^{-1}}, \theta L_{f^{-1}} L_y L_y, \theta\rangle = \langle \theta R_{g^{-1}} R_{(y^2)^{-1}}, \theta L_{f^{-1}} L_{y^2}, \theta\rangle$. If $g = (y^2)^{-1}$ and $f = y^2$ implies that$f = g^{-1} = y^2$. Then it follows that f and g are elements of $N(L,\cdot)$ the nucleus of $(L,\cdot)$ since the square of every element $y \in L$ belongs to $N(L,\cdot)$.

Theorem 4.2 Let $(L,\cdot)$ be a C-loop and let $\theta \in S(L,\cdot)$ (the symmetric group of L). Then $\theta \in BS(L,\cdot)$ if there is a unique $\alpha \in P(L,\cdot)$ (the set pseudo-automorphisms of $(L,\cdot)$) and a unique $f \in N(L,\cdot)$ such that $\theta = \alpha R_f (\alpha = \theta R_f^{-1})$.

Proof:

Let $(L,\cdot)$ be a C-loop then

$A = \langle R_{x^{-1}} R_{x^{-1}}, L_x L_x, I\rangle$

An autotopism of $(L,\cdot)$ for all $x \in L$.

$B = \langle I, R_{x^2}, R_{x^2}\rangle = \langle R_{x^2}, \rho R_{x^2 \rho}, I\rangle$ is also an autotopism for all $x \in L$. Therefore by Bruck[4]

$BA = \langle R_{x^2}, \rho R_{x^2 \rho}, I\rangle\langle R_{x^{-1}} R_{x^{-1}}, L_x L_x, I\rangle = \langle I, \rho R_{x^2} \rho L_x L_x, I\rangle$

is an autotopism for all $x \in L. \theta \in BS\,(L,\cdot)$ implies that $C = \langle \theta R_{f^{-1}}, \theta L_{g^{-1}}, \theta\rangle$ is an autotopism for some $f, g \in L$

$CBA = \langle \theta R_{f^{-1}}, \theta L_{g^{-1}}, \theta\rangle\langle I, \rho R_{x^2} \rho L_x L_x, I\rangle = \langle \theta R_{f^{-1}}, \theta L_{g^{-1}} \rho R_{x^2} \rho L_x L_x, \theta\rangle$

which implies that $\langle \alpha, \theta L_{g^{-1}} \rho R_{x^2} \rho L_x L_x, \alpha R_f\rangle$ is autotopism of $(L,\cdot)$ for some $f, g \in Q$ and all $x \in L$. Now if

$\langle \alpha, \theta L_{g^{-1}} \rho R_{x^2} \rho L_x L_x, \alpha R_f\rangle$

is an autotopism we have $s\alpha \cdot t\beta = (s \cdot t)\alpha R_f$ for all $s, t \in L$ where $\beta = \theta L_{g^{-1}} \rho R_{x^2} \rho L_x L_x$.

If s is set to be e in the last autotopism and noting that $e\alpha = e\theta R_{e\theta} = e$ we get $\beta = \alpha R_f$ therefore $\langle \alpha, \alpha R_f, \alpha R_f\rangle$ is an autotopism of $(L,\cdot)$ for some $f \in L$ hence α is a pseudo-automorphism with companion f. $\theta = \alpha R_f$ implies that the elements of the Bryant-Schneider group of a C-loop $(L,\cdot)$ can be expressed in terms of pseudo-automorphisms $P(L,\cdot)$ and right translations of elements of the nucleus of $(L,\cdot)$. To show uniqueness, let $\alpha_1 R_{x_1} = \alpha_2 R_{x_2}$ where $\alpha_1, \alpha_2 \in P(L,\cdot)$ and $x_1, x_2 \in N(L,\cdot)$. Then $\alpha_2^{-1}\alpha_1 = R_{x_2} R_{x_1}^{-1}$ which implies that $e\alpha_2^{-1}\alpha_1 = eR_{x_2} R_{x_1}^{-1}$. Then we observe that $e = x_2 x_1^{-1}$ and therefore $x_1 = x_2$. It follows that $\alpha_1 = \alpha_2$.

Remark 4.1 Robinson[12] considered the Bryant-Schneider group of a Bol loop and found out that they can be expressed as a product of pseudo-automorphisms and right translations. Theorem 2.2 above shows that the Bryant-Schneider group of a C-loop can also be expressed as in the same way. This further emphasis the fact that C-loops are analogous to Moufang loops since Moufang loops satisfies the Bol identities (right and left).

Theorem 4.3 Let $(L,\cdot)$ be a C-loop . If $x, y \in Q$, let $\odot$ be a binary operation defined on the pseudo-automorphism$PS(L,\cdot)$ by

$\alpha \odot \beta = \alpha R_x \beta R_y R_{(x\beta \cdot y)^{-1}}$ *for all* $\alpha\beta \in PS(L,\cdot)$. Let $H = PS(L,\cdot) \times Q$ and for

$(\alpha, x) \circ (\beta, y) = (\alpha \odot \beta, x\beta \cdot y)$

Then $(H,\circ)$ a group which is isomorphic to $BS(L,\cdot)$.

Proof:

Let $\alpha, \beta \in PS(L,\cdot)$ and let $x, y \in N(L,\cdot)$ the nucleus of $(L,\cdot)$. Then we know from the immediate preceding theorem that there exist unique $\delta \in PS(L,\cdot)$ and unique $z \in N(L,\cdot)$ such that $\alpha R_x \beta R_y = \delta R_z$. Thus we observe that

$(u\alpha \cdot x)\beta y = u\delta \cdot z$

For all $u \in L$. If we set $u = e$ we obtain $x\beta \cdot y = z$. Therefore $\alpha R_x \beta R_y = \delta R_{(x\beta \cdot y)^{-1}}$ and so

$\delta = \alpha R_x \beta R_y R_{(x\beta \cdot y)^{-1}} = \alpha \odot \beta$

Hence $\odot$ is a closed binary operation of $PS(L,\cdot)$. It is also obvious now that $(\alpha, x) \to \alpha R_x$ provided $x \in N(L,\cdot)$ gives an isomorphism of $(H,\circ)$ onto the $BS(L,\cdot)$ of a C-loop. Hence the Bryant-Schneider group of a C-loop is a form generalized holomorph of the loop.

Theorem 4.4 A finite C-loop is isomorphic to all its loop isotopes if

$[(L,\cdot): N(L,\cdot)]^2 = [PS(L,\cdot): A(L)]$

where $A(L)$ is the automorphism group of $(L,\cdot)$

Proof:

By Theorem 4.2 it is clear that $|BS(L,\cdot)| = |L \| PS(L,\cdot)|$. By Bryant & Schneider[2] $(L,\cdot)$ is isomorphic to all its loop isotopes if

$|L|^2|A(L,\cdot)| = |BS(L,\cdot)||N_\mu(L,\cdot)|$

But in a C-loop the nuclei coincide hence $|N_\mu(L,\cdot)| = |N(L,\cdot)|$. Now by Theorem 4.2 $|BS(L,\cdot)| = |PS(L,\cdot)||N(L,\cdot)|$ and therefore we have

$|L|^2|A(L,\cdot)| = |PS(L,\cdot)||N(L,\cdot)|^2$

which implies that

$$\left[\frac{|L|}{|N(L,\cdot)|}\right]^2 = \frac{|PS(L,\cdot)|}{|A(L,\cdot)|}$$

which is the same as

$[L: N(L,\cdot)]^2 = [PS(L,\cdot): A(L,\cdot)]$

As required.

Corollary 4.1 let $(L,\cdot)$ be a C-loop then

$[PS(L,\cdot) : A(L,\cdot)] \neq 4$

Proof:

The proof follows directly from lemma 2.9 of [20] and Theorem 4.4

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