Journal of the Nigerian Association of Mathematical Physics
***Volume* 17 (November, 2010), pp 485 – 496**
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A Differential Equation for Downhill Compressible Flow in Pipes and Its Solution

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Abstract

A general differential equation for the downhill flow of any compressible fluid has been developed. The development of the equation was made by the combination of Euler equation for the steady flow of any fluid, the Darcy-Weisbach equation for the loss of pressure head during fluid flow in a pipe and the equation of continuity of a compressible fluid.

The differential equation was first numerically solved by use of the classical fourth order Runge-Kutta method in the form of a FORTRAN program. A test of the accuracy of the solution was made by changing the direction of flow to downhill, of a problem taken from the book ,"Natural Gas Production Engineering". The pressure drop when flow was uphill was given in the book. The pressure drop from the program was less than that in the book. This was to be expected because the pressure drop when gravity aided the flow should be less than when gravity opposed it. The program also showed that the length of a pipe can be up to 5700ft and the calculated pressure drop is still accurate. Next, the Runge-Kutta solution was translated a formula valid in any consistent set of units. The formula was tested with a problem from the book "Fluid Mechanics and Hydraulics". The test showed the formula to be accurate. The formula was also converted to oil field units and results from it were in close agreement wit the output of the computer program.

Keywords: Compressible flow, Pipes, down hill

Introduction

The problem of predicting downstream pressure of a compressible fluid in a pipe is encountered in several aspects of engineering. Some areas where this problem is encountered are transportation of gas to the market, simultaneous flow of gas and liquid petroleum from well head to separating plants, gas injection for storage and pressure maintenance and gas injection from the surface to gas lift valves in well tubing.

In the literature models for pressure prediction during downhill flow are rare and in many instances the same equations for uphill flow are used for downhill flow.

In this work a general differential equation suitable for pressure prediction during downhill flow of any compressible fluid is developed. The compressible fluid can be a liquid of constant compressibility, a gas or a combination of gas and liquid. The differential equation was formulated from the combination of Euler equation for the steady low of any fluid, the Darcy-Weisbach formula for the loss of pressure head during fluid flow in a pipe and the equation of continuity for a compressible fluid.

The new differential equation was tested with a problem of gas that flows in a vertical well. The problem came from the book of [3] called "Natural Gas Production Engineering". To adapt the problem to downhill flow, the flow was made to occur from the surface to the foot of the tubing. That is, the well was now considered as an injection well instead of a naturally flowing well. The method of solution was by programming the fourth order Runge-Kutta numerical solution to the differential equation of this work. The output of the program showed that pressure drop (difference between pressures at inlet end and exit end of the tubing pipe) reduced when the direction of flow was changed from uphill to downhill. This is to be expected because the pressure drop when gravity aids the flow is less than the drop when gravity opposes the flow.

The computer program was also used to study the maximum length of pipe for which the Runge Kutta solution to the downhill flow differential equation can be considered accurate. It was found that for the 1.9956 inch diameter pipe, the length can be as much as 5700 ft.

Next the Runge Kutta algorithm was used to generate a formula suitable for hand calculation of the pressure drop. The accuracy of the formula was tested by using it to solve the problem from "Natural Gas Production Engineering" which was initially programmed. The result from the formula was in close agreement with the computer solution. Another problem from the book of [2] called "Fluid Mechanics and Hydraulics" was also used to test the formula. The problem of Giles

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involved the flow of gas in a horizontal pipe. To make the problem suitable for downhill flow, the problem was modified to incline the pipe at 10^o downhill. The formula is such that when the angle of inclination is 0^o, it breaks down to horizontal flow. Giles obtained the pressure of the gas at 1800ft during horizontal flow to be 45.7 psia while formula obtained 45.6 psia. When the pipe was now inclined 10^o downhill, the formula gave the pressure at 1800ft of pipe came to be 46.2 psia. This confirmed a less pressure drop for downhill flow as compared with horizontal flow.

Development Of Equation For Compressible Flow In Pipes

The Euler equation which is generally accepted for the motion of any fluid in a pipe has been derived in several fluid mechanics books. The version according to [2] is:

$$\frac{dp}{\gamma} + \frac{vdv}{g} - d\ell \sin\theta + dh_L = 0 \tag{1}$$

The negative sign (-) preceding the incremental length of pipe shows that the difference in elevation of the pipe between two points in the direction of flow is negative.

The generally accepted equation for the loss of head in a pipe transporting a fluid is that of Darcy-Weisbach. The equation is:

$$H_L = \frac{f\,Lv^2}{2gd} \tag{2}$$

The equation of continuity for compressible fluid flow is

$$W = A\ \gamma\ \nu \tag{3}$$

Differentiation of equation (3) and simultaneous solution with equation (2) and (3) lead to:

$$\frac{dp}{d\ell} = -\frac{\left[\dfrac{f\,W^2}{2\gamma A^2 dg} - \gamma\sin\theta\right].}{\left[1 - \dfrac{W^2}{\gamma^2 A^2 g}\dfrac{d\gamma}{dp}\right]} \tag{4}$$

The minus sign that precedes the numerator of equation (4) shows that pressure decreases with increasing length of pipe

The numerator of differential equation (4) accounts for kinetic effect contribution to the pressure gradient.

Equation (4) is a general differential equation that describes downhill fluid flow in a pipe.

The compressibility of a fluid (C_f) is:

$$C_f = \frac{1}{\gamma}\frac{d\gamma}{dp} \tag{5}$$

Equation (4) can be written as:

$$\frac{dp}{d\ell} = -\frac{\left[\dfrac{f\,W^2}{2\gamma A^2 dg} - \gamma\sin\theta\right]}{\left[1 - \dfrac{W^2 C_f}{\gamma A^2 g}\right]} \tag{6}$$

Differential equation (4) or (6) is valid for any compressible fluid, including,

compressible liquid, gas and multiphase (combination of liquid and gas).

When applied to a gas equation (6) can be simplified further. The equation of state for a non-ideal gas can be written as:

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$$\gamma = \frac{p\ \mathrm{M}}{zR\mathrm{T}} \tag{7}$$

Multiply differential equation (6) by γ, substitute for γ as in equation (7) and use the fact that;

$$\frac{\mathrm{pdp}}{\mathrm{d}\ell} = \frac{1}{2}\frac{\mathrm{dp}^2}{\mathrm{d}\ell}, \text{ gives}$$

$$\frac{dp^2}{d\ell} = -\frac{\left[\frac{fW^2\ zR\mathrm{T}}{\mathrm{A}^2 d\mathrm{M}g} - \frac{2p^2\mathrm{M}\sin\theta}{zR\mathrm{T}}\right]}{\left[1 - \frac{W^2\ zR\mathrm{T}C_f}{\mathrm{MA}^2 g\ p}\right]} \tag{8}$$

The denominator of equation (8) is close to 1 and can be neglected as pointed out by previous researchers such as [3] and [9]. If the expression is to be evaluated, the compressibility of the gas (C_f) can be calculated as follows:

The cross-sectional area (A) of a pipe is

$$\mathrm{A}^2 = \left(\frac{\pi d^2}{4}\right)^2 = \frac{\pi^2\ d^4}{16} \tag{9}$$

For an ideal gas such as air,

$$\mathrm{C_f} = \frac{1}{\mathrm{p}}. \text{ For a non ideal gas, } C_f = \frac{1}{p} - \frac{1}{z}\frac{\partial z}{\partial p}$$

Matter [4] have proposed an equation for the calculation of the compressibility of hydrocarbon gases. [7] has expressed the compressibility of a real gas (C_f) as:

$$C_f = \frac{\mathrm{K}}{p} \tag{10}$$

For a sweet natural gas (natural gas that contains CO_2 as major contaminant), K = 1.0328 when p is in psia. The denominator of equation (8) can then be written as

$$1 - \frac{\mathrm{KW^2zRT}}{\mathrm{M\,g\,d^4P^2}} \text{ Where K = constant.}$$

Then equation (8) can be written as

$$\frac{dy}{d\ell} = \frac{(A - By)}{(1 - \frac{G}{y})} \tag{11}$$

Where $\mathrm{y} = \mathrm{p}^2$

$$\mathrm{A} = \frac{1.621139\ \mathrm{fW^2zRT}}{\mathrm{gd^5M}}, \mathrm{B} = \frac{2\mathrm{M}\sin\theta}{\mathrm{zRT}}, \mathrm{G} = \frac{\mathrm{KW^2zRT}}{\mathrm{gMd^4}}$$

Also, the molecular weight (M) can be expressed as M = 28.97Gg.

The differential equation (11) can be written as

$$\frac{d\mathrm{P}^2}{d\ell} = \frac{\left[\frac{0.0559592\quad f\,zR\mathrm{T}W^2}{gd^5G_g} - \frac{59.940\ G_g\ \sin\theta\,\mathrm{P}^2}{zR\mathrm{T}}\right]}{\left[1 - \frac{0.0559592\quad zR\mathrm{T}W^2\mathrm{K}}{g\,d^5 G_g\,\mathrm{P}^2}\right]} \tag{12}$$

The differential equation (12) is valid in any consistent set of units. The relationship between weight flow rate (W) and the volumetric flow rate measured at a base condition of pressure and temperature (Q_b) is;

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$$W = \gamma_b Q_b \quad (13)$$

Since the specific weight at base condition is:

$$\gamma_b = \frac{p_b M}{z_b T_b R} = \frac{28.97 G_g p_b}{z_b T_b R} \tag{14}$$

Substitution of equation (13) and (14) into differential equation (12) gives;

$$\frac{dp^2}{dl} = \frac{\left[\frac{46.9583259 \quad f z T G_g P_b^2 Q_b^2}{g d^5 z_b^2 T_b^2 R} - \frac{59.940 G_g \sin\theta P^2}{z R T}\right]}{\left[1 - \frac{46.95832593 \quad G_g Q_b^2 K}{g R d^4}\left(\frac{P_b}{T_b}\right)^2\left(\frac{T}{P^2}\right)\right]} \tag{15}$$

The differential equation (15) is also valid in any consistent set of units.

Solution To The Differential Equation For Downhill Flow

In order to find a solution to the differential equation for downhill flow (as presented in Equation (6), (8), (12) and (15) we need equations or charts that can provide values of the variables z and f. The widely accepted chart for the values of the gas deviation factor (z) is that of [11]. The chart has been curve fitted by some researchers. The version used in this work is that of [5]. The Ohirhian set of equations are able to read the chart within $\pm 0.777\%$ error. The Standing and Katz charts require reduced pressure (Pr) and reduced temperature (T r). The Pr is defined as Pr = P/Pc and the T r is defined as T r = T / T c; where Pc and T c are pseudo critical pressure and pseudo critical temperature, respectively.

Standing [12] has presented equations for Pc and T c as functions of gas gravity (G g). The equations are;

$$Pc = 677 + 15.0 G_g - 37.5 G_g^2 \quad (16)$$

$$Tc = 168 + 325 G_g - 12.5 G_g^2 \quad (17)$$

The generally accepted equation for the calculation of the dimensionless friction factor (f) is that of Colebrook (1938). The equation is;

$$\frac{1}{\sqrt{f}} = -2\log\left(\frac{\in}{3.7d} + \frac{2.51}{R_N\sqrt{f}}\right) \tag{18}$$

The Colebrook equation is non-linear and requires an iterative solution. There are several attempts in the literature to bring the Colebrook equation to an explicit form. Ohirhian [6] has presented some explicit forms of the Colebrook. The equations have absolute error that ranges from 1.303 to 0.026%. The Ohirhian equation used in this work is

$$f = \left[-2\log\left(a - 2b\ \log\ (a + bh)\right)\right]^{-2} \tag{19}$$

Where:

$$a = \frac{\in}{3.7d}\ ,\ b = \frac{2.51}{R_N}\ ,\ c = a + bh$$

$$h = -1.14\ \log\left(\frac{\in}{d} + 0.30558\right) + 0.57\ \log\ R_N\left(0.1772\ \log\ R_N + 1.0693\right)$$

Equation (19) can calculate f within an absolute error of 1.303 %. The highest error occurring in the lowest Reynolds number of 4000 and relative roughness of zero (smooth pipe).

The Reynolds number is defined as:

$$R_N = \frac{\rho v d}{\mu} = \frac{W\ d}{A g\ \mu} \tag{20}$$

Substitution of equation (13) for the weight flow rate (W) and equation (14) for the density of a gas at standard condition (r_b) into equation (20) gives

The Reynolds number can be written as:

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$$R_N = \frac{36.88575 G_g P_b Q_b}{Rgd\,\mu_g\, z_b T_b} \tag{21}$$

By use of a base pressure (p_b) = 14.7psia, base temperature (T_b) = 520^{o}R and R = 1545

$$R_N = \frac{20071 Q_b G_g}{\mu_g\, d} \tag{22}$$

Where d is expressed in inches, Q_b = MMSCf / Day and μ_g is in centipoise

Ohirhian and Abu [7] have presented a formula for the calculation of the viscosity of natural gas. The gas may be sweet or sour. The formula is

$$\mu_g = \frac{0.0109388 - 0.0088234xx - 0.00757210xx^2}{1.0 - 1.3633077xx - 0.0461989xx^2} \tag{23}$$

Where

$$xx = \frac{0.0059723\ p}{z\left(16.393443 - 1/p\right)}$$

In equation (23) μ_g is expressed in centipoise(c_p) , p in (psia) and Tin (^{o}R)

The differential equation for the downhill compressible flow can be solved by the classical fourth order Runga Kutta method. The Runga Kutta method allows large increment in the independent variable when used to solve a differential equation. The Runge Kutta solution to the differential equation

$$\frac{dy}{dx} = f(x, y) \qquad at \;\; x = x_n \tag{24}$$

$$given\;\; that\;\; y = y_o \;\; when\;\; x = x_o$$

$$is$$

$$y = y_o + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4) \tag{25}$$

$$where$$

$$k_1 = Hf(x_o, y_o)$$

$$k_2 = Hf(x + \frac{1}{2}H, y_o + \frac{1}{2}k_1)$$

$$k_3 = Hf(x_o + \frac{1}{2}H,\ y_o + \frac{1}{2}k_1)$$

$$k_4 = Hf(x_o + H,\ y + k_3)$$

$$H = \frac{x_n - x_o}{n}, \mathrm{n = number\ of\ \ applications}$$

A large value of H can be used in the Runge Kutta algorithm and the calculated y can still be accurate. The downhill flow differential equation was tested by reversing the direction of flow in a problem taken from the book of Ikoku.

Example 1

Calculate the sand face pressure (P wf) of an injection gas well from the following surface measurements.

Flow rate (Q) = 5.153 MMscfd
Tubing internal diameter (d) = 1.9956in
Gas gravity (G g) = 0.6
Depth (L) = 5790ft (bottom of casing)
Temperature at foot of tubing (T_{wf}) = 160^{o}F

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Surface temperature (T_{sf}) = 83^oF

Tubing head pressure ($P_{sf'}$) = 2545 psia

Absolute roughness of tubing ($\in$) = 0.0006in

Length of tubing (L) = 5700ft (well is vertical)

Answer

Here, $(x_o, y_o) = (0, 2545)$

By use of I step Runge Kutta.

$$H = \frac{(5700-0)}{1} = 5700$$

The Runge Kutta algorithm was programmed in Fortran 7 and used to solve this problem. The output from the program by use one step Runge Kutta (Length = 5700 ft) is as follows

```
TUBING HEAD PRESSURE =      2545.0000000 PSIA
SURFACE TEMPERATURE =       543.0000000 DEGREE RANKINE
TEMPERATURE AT TOTAL DEPTH =       620.0000000 DEGREE RANKINE
GAS GRAVITY =   6.000000E-001
GAS FLOW RATE =          5.1530000 MMSCFD
DEPTH AT SURFACE =          .0000000 FT
TOTAL DEPTH =      5700.0000000 FT
INTERNAL TUBING DIAMETER =        1.9956000 INCHES
ROUGHNESS OF TUBING =  6.000000E-004 INCHES
INCREMENTAL DEPTH =      5700.0000000 FT


    PRESSURE PSIA       DEPTH FT

     2545.000              .000
     2327.930          5700.000
```

To check that the length increment of 5700 ft gave an accurate result, another run in which the length increment was 1000 ft was made. The result is as follow;

```
TUBING HEAD PRESSURE =      2545.0000000 PSIA
SURFACE TEMPERATURE =       543.0000000 DEGREE RANKINE
TEMPERATURE AT TOTAL DEPTH =       620.0000000 DEGREE RANKINE
GAS GRAVITY =   6.000000E-001
GAS FLOW RATE =          5.1530000 MMSCFD
DEPTH AT SURFACE =          .0000000 FT
TOTAL DEPTH =      5700.0000000 FT
INTERNAL TUBING DIAMETER =        1.9956000 INCHES
ROUGHNESS OF TUBING =  6.000000E-004 INCHES
INCREMENTAL DEPTH =      1000.0000000 FT

    PRESSURE PSIA       DEPTH FT

     2545.000              .000
     2489.816          1140.000
     2440.945          2280.000
     2397.941          3420.000
     2360.347          4560.000
     2327.913          5700.000
```

Comparing the P_{wf} = 2327.930 psia by use of L = 5700 ft and P_{wf} = 2327.913 psia by use of L = 1000 ft, it is seen that the P_{wf} by use of L = 5700 ft in accurate enough.

Next a run was made to test the contribution of kinetic effects to the pressure drop. In this run, the denominator of the differential equation (that accounts for kinetic effects) was ignored. The output is as follows

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TUBING HEAD PRESSURE = 2545.0000000 PSIA
SURFACE TEMPERATURE = 543.0000000 DEGREE RANKINE
TEMPERATURE AT TOTAL DEPTH = 620.0000000 DEGREE RANKINE
GAS GRAVITY = 6.000000E-001
GAS FLOW RATE = 5.1530000 MMSCFD
DEPTH AT SURFACE = .0000000 FT
TOTAL DEPTH = 5700.0000000 FT
INTERNAL TUBING DIAMETER = 1.9956000 INCHES
ROUGHNESS OF TUBING = 6.000000E-004 INCHES
INCREMENTAL DEPTH = 5700.0000000 FT

PRESSURE PSIA	DEPTH FT
2545.000	.000
2327.965	5700.000

Pressure drop with inclusion of kinetic effects is 2545 psia – 2327.930 psia = 21707 psia. Pressure drop without kinetic effects is 2545psia - 2327.965 psia = 217.035 psia.
Indeed, the kinetic effects are small and can be neglected.
Neglecting the kinetic effect, the Runge Kutta algorithm can be used to provide a solution to the differential equation (15) as follows

$$p_1 = \sqrt{p_2^2 - \left|\bar{y}\right|} \tag{25}$$

Here

$$\bar{y} = \frac{aa}{6}\left(1 - x + 0.5x^2 + 0.3x^3\right) + \frac{p_1^2}{6}\left(-5.2x + 2.2x^2 - 0.6x^3\right) + \frac{u}{6}\left(5.2 - 2.2x + 0.6x^2\right)$$

$$aa = \left(\frac{46.958326\ f_1 z_1 T_1 G_g P_b^2 Q_b^2}{gd^5 z_b^2 T_b^2 R} - \frac{57.940\ G_g \sin\theta p_1^2}{z_1 T_1 R}\right) L$$

$$u = \frac{46.958326\ f_1 z_{av} T_{av} G_g P_b^2 Q_b^2 L}{gd^5 z_b^2 T_b^2 R}$$

$$x = \frac{57.940\ G_g \sin\theta L}{z_{av} T_{av} R}$$

L = length of pipe
G_g = gas gravity (air =1)
f_1 = moody friction factor evaluated at inlet end pipe
R = universal gas constant in any consistent set of units
g = acceleration due to gravity
d = internal diameter of pipe
T_1 = temperature at inlet end of pipe
T_{av} = temperature at mid section of pipe
p_1 = pressure at inlet end of pipe
z_1 = gas deviation factor evaluated with p_1 and T_1
z_{av} = gas deviation factor calculated with temperature at mid section (T_{av}) and pressure at the mid section of pipe (p_{av}) given by $p_{av} = \sqrt{p_1^2 - 0.5\left|aa\right|}$
θ = angle of pipe inclination with the horizontal.
p_2 = pressure at exit end of pipe, psia
p_1 = pressure at inlet end of pipe
Q_b = volumetric rate measure at P_b and T_b
Note that $p_1 > p_2$ and flows occurs from point (1) to point (2)

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In equation (25), the component k_4 in the Runge Kutta method given by $k_4 = H f(x_o + H, y + k_3)$ was given some weighting to compensate for the fact that the temperature and gas deviation factor vary between the mid section and the exit end of the pipe.

Equation (25) can be converted to oil field units. In oil field units in which L is in feet, R = 1545, temperature(T) is in ^oR, g = 32.2 ft/sec^2, diameter (d) is in inches, pressure (p) is in pounds per square inches (psia), flow rate (Q_b) is in MMscfd and P_b = 14.7 psia, $T_b = 520^o$ R. The variables aa, u and x that occur in equation (25) can be written as:

$$aa = \left(\frac{25.1472069 \; G_g f_1 z_1 T_1 Q_b^2}{d^5} - \frac{0.0375016 \; G_g \sin\theta \; p_1^2}{z_1 T_1} \right) L \qquad (26)$$

$$u = \frac{25.1472069 \; G_g f_1 z_{av} T_{av} Q_b^2 L}{d^5}$$

$$x = \frac{0.0375016 \; G_g \mathrm{Sin}\theta \; L}{z_{av} T_{av}}$$

Example 2

Use equation (25) to solve the problem of example 1

Answer

Step 1: obtain the gas deviation factor at the inlet end

$T_1 = 83^oF = 543^o$ R

P_1 = 2545 psia

Gg = 0.6

By use of equation (16) and (17)

Pc (psia) = 677 + 15 x 0.6 – 37.5 x 0.6^2 = 672.5 psia

Tc (^{o}R) = 168 + 325x0.6 – 12.5 x 0.6^2 = 358.5 ^{o}R

Then, P_{1r} = 2545/672.5 = 3.784, T_{1r} = 543/358.5 = 1.515

The required Ohirhian equation is

$$z = \left(z_1 + \left(1.39022 + \Pr\left(0.06202 - 0.02113 \times \Pr\right)\right) \times \log \mathrm{Tr}\right) \mathrm{Fc}$$

Where

$$z_1 = 0.60163 + \Pr(-0.06533 + 0.0133\Pr)$$

$$Fc = 20.208372 + Tr(-44.0548 + Tr(37.55915 + Tr(-14.105177 + 1.9688Tr)))$$

Substitution of values of Pr = 3.784 and Tr = 1.515 gives z = 0.780588

Step 2

Evaluate the viscosity of the gas at inlet condition. By use of Ohirhian and Abu formula (equation 23)

$$xx = \frac{0.0059723 \times 2545}{0.780588 \; (16.393443 - 543/2545)} = 1.203446$$

$$\mu_g = \frac{0.0109388 - 0.008823 \; (1.203446) - 0.0075720 \; (1.203446)^2}{1.0 - 1.3633077 \; (1.203446) - 0.0461989 \; (1.203446)^2} - 0.015045 \text{ Cp}$$

Step 3

Evaluation of Reynolds number (R_N) and dimensionless friction factor (f). From eqn. (22)

$$R_N = \frac{20071 \times 5.153 \times 0.6}{0.015045 \times 1.9956} = 2066877$$

The dimensionless friction factor can be explicitly evaluated by use of Ohirhian formula (equation 19)

$$\in/d = 0.0006/1.9956 = 3.066146 \text{ E} - 4$$

$$a = 3.066146 \text{ E} - 4/3.7 = 8.125985 \text{ E} - 5$$

$$b = 2.51/2066877 = 1.21393 \text{ E} - 6$$

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$$h = -1.14\log(3.066146E-4+0.30558)+0.57\times\log 2066877(0.01772\log 2066877+1.0693)$$
$$= 4.838498$$

Substitute of values of a, b and h into $f = [-2\log(a-2b\log(a+bh))]^{-2}$ gives
f = 0.01765

Step 4

Evaluate the coefficient aa in the formula. This coefficient depends only on surface (inlet) properties. Note that the pipe is vertical $\theta = 90^o$ and sin 90^o = 1

$$aa = (25.147207 \times 0.6\times 0.017650 \times 0.780588 \times 543\times 5.153^2\times 5700)/1.9956^5$$
$$-(0.037502 \times 0.6\times 1\times 2545 \times 5700)/(0.780588 \times 543)$$
$$= 539803 - 1959902 = -1420099$$

Step 5

Evaluate the average pressure (p_{av}) at the mid section of the pipe given by

$$p_{av} = \sqrt{p_1^2 - 0.5|aa|} = \sqrt{2545^2 - 0.5\times 1420099} = 2401.5$$

Step 6:
Evaluate the average gas deviation factor (z_{av}). Reduced average pressure(p_{avr}) = 2401.5/672.5 = 3.571. $T_{av} = T_1 + \alpha L/2$ where α = geothermal gradient given by :
$\alpha = (T_2 - T_1)/L = (620-543)/5700 = 0.013509$
T_{av} at mid section of pipe (2850 ft) then, is: $T_{av} = 543 + 0.013509\times 2850 = 581.5^o R$
Reduced T_{av} = 581.5/358.5 = 1.622
Substitution into the Ohirhian equation used in step 1, gives z = 0.821102
Step 7:

Evaluate the coefficients x and u

$$x = \frac{0.0375016 \times 0.6\times 1\times 5700}{0.821102 \times 581.5} = 0.268614$$
$$u = (25.147207 \times 0.6\times 0.017650 \times 0.821102 \times 581.5\times 5.153^2\times 5700)/1.9956^5 = 608079$$

Step 8: Evaluate $\bar{y}$

$$\bar{y} = \frac{u}{6}(5.2 - 2.2x + 0.6x^2) + \frac{p_1^2}{6}(-5.2x + 2.2x^2 - 0.6x^3) + \frac{aa}{6}(1 - x + 0.5x^2 - 0.3x^3)$$

Substitution of u = 608079, p_1 = 2545 psia, aa = -1420099 and x = 0.268614 gives
$\bar{y} = 471499 - 1349039 - 180269 = -1057809$
Step 9: Evaluate p_2, the pressure at the exit end of the pipe
$P_2 = \sqrt{(2545^2 - 1057809)} = 2327.92\ psia \approx 2328\ psia$
Pressure drop across 5700 ft of tubing is 2545 psia – 2328 psia = 217 psia

This pressure drop may be compared with the pressure drop across the 5700 ft of tubing when gas flows uphill against the force of gravity. From the book of Ikoku and computer program for uphill flow (not shown here), tubing pressure at the surface = 2122 psia when the bottom hole pressure (inlet pressure) = 2545 psia. Then pressure drop = 2545 psia – 2122 psia = 423 psia. Models that predict pressure transverse during compressible flow in inclined pipes that use the same equation for uphill flow and downhill flow are bound to overestimate the drop occurring in downhill flow.

The general solution (valid in any system of units) to the differential equation for downhill flow was tested with slight modification of a problem from the book of Giles. In the original problem the pipe was horizontal. In the modification used in this work, the pipe was made to incline at 10 degrees from the horizontal in the downhill direction. Other data remained as they were in the book of Giles. The data are as follows:

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Example 3

Length of pipe (L) = 1800ft
$z_2 = z_1 = z_{av} = 1$ (air is flowing fluid)
$p_1 = 49.5$ psia = 49.5×144 psf = 7128 psf
W = 0.75 lb/sec
$Q_b = 9.81937$ ft^3/sec
$P_b = 14.7$ psia = 2116.8 psf
$T_b = 60\,^oF = 520\,^oR$
$T_1 = T_{av} = 90\,^oF = 550\,^oR$
G g = 1.0 (air)
R = 1544
$\mu = 390 \times 10^{-9}$ lb sec/ft^2
d = 4 inch = 0.333333 ft
Absolute Roughness ($\in$) = 0.0003 ft
The question is to find the exit pressure (p_2) at 1800 ft of pipe.

Answer

Step 1: Obtain the gas elevation factor at inlet end, $z_1 = 1.0$, air in flowing fluid
Step 2: Obtain the viscosity of the gas at inlet condition. Viscosity of gas is 390×10^{-9} lb sec / ft ^{2} (given)
Step 3: Evaluate the Reynolds number and friction factor.

$$R_N = \frac{36.88575\ G_g P_b Q_b}{g\,R\,d\mu\ z_b T_b}$$

Here, $G_g = 1.0\,(\text{air})$, $P_b = 2116.8$ psf, $Q_b = 9.81937$ ft^3/sec, $g = 32.2$ ft/sec^2, $d = 0.3333$ ft, Then $R = 1544$, $\mu = 390 \times 10^{-9}$, $z_b = 1.0$ (air), $T_b = 520\,^oR$

$$R_N = \frac{36.88575 \times 1 \times 2116.8 \times 9.81937 \times 10^9}{32.2 \times 1544 \times 0.3333 \times 390 \times 1 \times 520} = 2281249$$

$$\frac{\in}{d} = \frac{0.0003}{0.33333} = 0.0009$$

From Moody chart, $f_1 = 0.0205$

Step 4: Evaluate the coefficient aa in the formula

$$aa = \left(\frac{46.958326\ f_1 z_1 T_1 Gg P_b^2 Q_b^2}{gd^5 z_b^2 T_b^2 R} - \frac{57.940\ GgSin\ \theta P_1^2}{z_1 T_1 R} \right) L$$

$$\frac{46.958326 \times 0.0205 \times 1 \times 550 \times 1 \times 2116.8^2 \times 9.81937^2}{32.2 \times 0.33333^5 \times 520^2 \times 1544} = 4134.70$$

$$\frac{57.94 \times 1 \times 0.173648 \times 7128^2}{1 \times 550 \times 1544} = \frac{511191540.9}{849200} = 601.97$$

Then aa = (4134.70 – 601.97) $\times$ 1800 = 6358914

Step 5

Evaluate the average pressure (p_{av}) at the mid section of the pipe

$$p_{av} = \sqrt{p_1^2 - 0.5 \times 6358914} = 6901.4\ \text{psf}$$

Step 6: Evaluate average gas derivation factor (z_{av}) for air, $z_{av} = 1.0$

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Step 7: Evaluate the coefficients x and u

$$x = \frac{57.94\, G_g \operatorname{Sin}\theta L}{z_{av} T_{av} R} = \frac{57.94 \times 1 \times 0.173648 \times 1800}{1 \times 550 \times 1544} = 0.021326$$

$$u = \frac{46.958326\, f_1 z_{av} T_{av} Gg P_b^2 Q_b^2 L}{gd^5 z_b^2 T_b^2 R}$$

$$= \frac{46.958326 \times 0.0205 \times 1 \times 550 \times 1 \times 2116.8^2 \times 9.81937^2 \times 1800}{32.2 \times 0.33333^5 \times 1 \times 520^2 \times 1544} = 7442642.4$$

Step 8: Evaluate $\bar{y}$

$$\bar{y} = \frac{u}{6}\left(5.2 - 2.2x + 0.6x^2\right) + \frac{p_1^2}{6}\left(-5.2x + 2.2x^2 - 0.6x^3\right) + \frac{aa}{6}\left(1 - x + 0.5x^2 - 0.3x^3\right)$$

Where x = 0.021326, $(5.2 - 2.2x + 0.6x^2)$ = 5.153356, $(-5.2x + 2.2x^2 - 0.6x^3)$ = - 0.10990

$(1 - x + 0.5x^2 - 0.3x^3)$ = 0.9789. Then,

$$\bar{y} = \frac{7442642.4}{6} \times 5.153356 + \frac{7128^2}{8}(-0.1099) + \frac{6358914}{6}(0.9789)$$

= 6392431 – 930644 + 1037457 = 6499244

Step 9: Evaluate p_2, the pressure at the exit end of the pipe

$$p_2 = \sqrt{7128^2 - 6499244} = 6656.5\,\text{psf} = \frac{6656.5\ \text{psia}}{144} = 46.2\ \text{psia}$$

Pressure drop = 49.5 psia – 46.2 psia = 3.3 psia

When the pipe is horizontal, p_2 (from Fluid Mechanics and Hydraulics is 45.7 psia. The pressure drop = 49.5 psia – 45.7 psia = 3.8 psia

CONCLUSIONS

1. A differential equation for the downhill flow of any compressible fluid in a pipe has been developed.
2. The classical fourth order Runge-Kutta algorithm was used to provide a solution to the new differential equation.
3. The formula(solution to the new differential equation) is suitable for wells and pipelines with large temperature gradients
4. The formula is suitable for hand calculation and is accurate for a pipe length as large as 5700ft.

SI Metric Conversion Factors

$\left({}^0F - 32\right)/1.8 \quad = {}^0C$

$ft \times 3.048000 \ * E - 01 \quad = m$

$in \times 2.540 \ * E + 00 \quad = cm$

$lbf \times 4.448222 \ E + 00 \quad = N$

$lbm \times 4.535924 \ E - 01 \quad = kg$

$psi \times 6.894757 \ E + 03 \quad = Pa$

$cp \times 1.0 \ E - 03 \quad = Pa.s$

- Conversion factor is exact.

Numenclature

p = Pressure

γ = Specific weight of flowing fluid

v = Average fluid velocity

g = Acceleration due to gravity in a consistent set of umits.

$d\ell$ = Incremental length of pipe

θ = Angel of pipe inclination with the horizontal, degrees

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dh_L = Incremental pressure head loss
f = Dimensionless friction factor
L = Length of pipe
d = Internal diameter of pipe
W = Weight flow rate of fluid

C_f = Compressibility of a fluid
C_g = Compressibility of a gas
K = Constant for expressing the compressibility of a gas
M = Molecular weight of gas
T= Temperature
z = Gas deviation factor
R = Universal gas constant in a consistent set of units .
Q_b = Gas volumetric flow rate referred to P_b and T_b,
γ_b = specific weight of the gas at p_b and T_b
p_b = Base pressure, absolute unit
T_b = Base temperature, absolute unit
z_b = Gas deviation at p_b and T_b usually taken as 1
G_g = Specific gravity of gas (air = 1) at standard condition
R_N = Reynolds number
$\in$ = Absolute roughness of tubing
ρ = Mass density of a fluid
μ =Absolute viscosity of a fluid
μ_g = Absolute viscosity of a gas
GTG = Geothermal gradient
f_1 = Moody friction factor evaluated at inlet end pipe
T_1 = Temperature at inlet end of pipe
T_{av} = Temperature at mid section of pipe
p_1 = Pressure at inlet end of pipe
z_1 = Gas deviation factor evaluated with p_1 and T_1
z_{av} = gas deviation factor calculated with temperature at mid section (T_{av}) and pressure at the mid section of pipe (p_{av}).

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