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On Some Generalization of the Eigenvectors Properties

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Abstract

In this paper the generalization of the classical mode orthogonality and normalization relationships known for undamped systems to non – classical and non – viscously damped systems were established and investigated. Classical mode orthogonality relationships known for undamped systems were generalized to non – viscously damped systems. It was shown that there exists unique relationship which relates the system matrices to the natural frequencies and modes of non – viscously damped systems. These relationships, in return, enable us to reconstruct the system matrices from full set of modal analysis.

The non – viscously damping model is such that the damping forces depends on the past history of motion via convolution integrals over some kernel functions. Classical modal analysis is extended to deal with general non – viscously damped of multiple degree- of- freedom (MDOF) linear dynamic systems. The concept (complex) of non – viscous mode was introduced and further shown that the system response can be obtained exactly in terms of these modes.

Key words: Eigenvectors, eigenvalues, viscously undamped, orthogonality, normalization.

1.0 INTRODUCTION

In [1], the eigenvalues, eigenvectors and transfer functions associated with multiple-degree-of-freedom non – viscously damped systems have been discussed. A method was outlined to obtain the eigenvectors and dynamic response of the system. Eventhough the method is analogous to classical modal analysis, unlike the classical mode, very little is known about qualitative properties of the modes of non – viscously damped systems. The objective of this paper is to develop some basic relationships which satisfied the eigensolutions and the system matrices of (2.3). Specifically, we have focused our attention to the normalization and orthogonality relationship of the eigenvectors.

2.0 BASIC CONCEPTS OF EIGENVALUES AND EIGENVECTORS

Modal analysis is the most popular and efficient method for solving engineering dynamic problems.The concept of modal analysis, as shown in [1] was originated from the linear dynamics of undamped systems. In [2], the undamped modes or classical normal modes satisfy an orthogonality relationship over the mass and stiffness matrices and uncouple the equation of motion, i.e., if $\boldsymbol{X} \in \mathbb{R}^{N\times N}$ is the modal matrix then $\boldsymbol{X^T M X}$ and $\boldsymbol{X^T K X}$ are both diagonal matrices. This significantly simplifies the dynamic analysis because complex multiple-degree-of-freedom system can be treated as a collection of SDOF oscillator.

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Real-life systems are however, not undamped, but possess some kind of energy dissipation mechanism or damping. In order to apply modal analysis of undamped systems to damped systems, it is common to assume the proportional damping, a special case (see [3] for examples) of viscous damping. The proportional damping model expresses the damping matrix as a linear combination of the mass and stiffness matrices, that is

$$\boldsymbol{C} = \alpha_1 \boldsymbol{M} + \alpha_2 \boldsymbol{K} \tag{2.1}$$

where α_1 , α_2 are real scalars.

The equations of motion of free vibration of a viscous damping system [1] can be expressed by

$$\boldsymbol{M}\ddot{\boldsymbol{q}}(t) + \boldsymbol{C}\dot{\boldsymbol{q}}(t) + \boldsymbol{K}\boldsymbol{q}(t) = 0 \tag{2.2}$$

In this case the equation of motion are characterized by three real symmetric matrices which brings additional complication compared to the undamped systems where the equations of motion are characterized by two matrices. We require a non – zero matrix $\boldsymbol{X} \in \mathbb{R}^{N\times N}$ such that it simultaneously diagonalizes $\boldsymbol{M}, \boldsymbol{C}\ and\ \boldsymbol{K}$ under a congruence transformation.
The concept of proportional damping was extended to non – viscously damped systems as in [1] and the conditions for existence of proportional damping in non – viscously damped systems were discussed. Thus, in general, non – viscously damped systems are non – proportionally damped. Adhikari [1] analyze non – viscously damped MDOF systems with non – proportional damping. We rewrite the equation of motion of force vibration of an N- degree-of-freedom linear system with non – viscous damping as

$$\boldsymbol{M}\ddot{\boldsymbol{q}}(t) + \int_0^t \boldsymbol{\mathcal{G}}(t-\tau)\, \dot{\boldsymbol{q}}(\tau) d\tau + \boldsymbol{K}\boldsymbol{q}(t) = \boldsymbol{f}(t) \tag{2.3}$$

The initial conditions associated with the above equation are

$$\boldsymbol{q}(0) = \boldsymbol{q_0} \in \mathbb{R}^N \text{ and } \dot{\boldsymbol{q}}(0) = \dot{\boldsymbol{q}}_{\boldsymbol{0}} \in \mathbb{R}^N \tag{2.4}$$

Considering the free vibration, that is $\boldsymbol{f}(t) = \boldsymbol{q_0} = \dot{\boldsymbol{q}}_{\boldsymbol{0}} = 0$, and taking the Laplace transformation of the equation of motion (2.3), one has

$$s^2 \boldsymbol{M}\overline{\boldsymbol{q}} + s\boldsymbol{G}(s)\overline{\boldsymbol{q}} + \boldsymbol{K}\dot{\boldsymbol{q}} = 0 \tag{2.5}$$

Here $\overline{\boldsymbol{q}}(s) = \boldsymbol{\mathcal{L}}[\boldsymbol{q}(t)] \in \mathbb{C}^N$, $\boldsymbol{G}(s) = \boldsymbol{\mathcal{L}}[\boldsymbol{\mathcal{G}}(t)] \in \mathbb{C}^{\boldsymbol{N\times N}}$ and $\boldsymbol{\mathcal{L}}[\bullet]$ denotes the Laplace transform. In the context of structural dynamics, $s = iw$, where $i = \sqrt{-1}$ and $w \in \mathbb{R}^+$ denotes frequency. It is assumed that: 1) $\boldsymbol{M^{-1}} exists$ and 2) all the eigenvalues of $\boldsymbol{M^{-1}K}$ are distinct and positive. Because $\boldsymbol{\mathcal{G}}(t)$ is a real function, $\boldsymbol{G}(s)$ is also a real function of the parameters. For the linear viscoelastic case it can be shown that (see [4]), in general, the elements of $\boldsymbol{G}(s)$ can be represented as

$$G_{jk}(s) = \frac{P_{jk}(s)}{q_{jk}(s)} \tag{2.6}$$

where $P_{jk}(s)$ and $q_{jk}(s)$ are finite – order polynomials in s. Here, we do not assume any specific functional form of $G_{jk}(s)$ but assume that $|G_{jk}(s)| < \infty$ when $s \to \infty$. Thus in turn implies that the elements of $\boldsymbol{G}(s)$ are at the most of order $\frac{1}{s}$ in s or constant, as in the case of viscous damping. The eigenvalues, s_j; associated with equation (2.5) are roots of the characteristic equation

$$det[s^2 \boldsymbol{M} + s\boldsymbol{G}(s) + \boldsymbol{K}] = 0 \tag{2.7}$$

If the element of $G(s)$ have simple forms, for example – as in equation (2.6) then the characteristic equation becomes a polynomial equation of finite order. However, for practical purposes as in [8] the Taylor expansion of $G(s)$ can be truncated to a finite series to make the characteristic equation a polynomial equation of finite order. Suppose the order of the characteristic polynomial is m. In general $m > 2N$, that is $m = 2N + p; p \geq 0$. Thus, although the system has N degree-of-freedom, the number of eigenvalues is more than $2N$. This is a major difference between the non-viscously damped systems where the number of eigenvalues is exactly $2N$, including any multiplicities. It is assumed all m eigenvalues are distinct.

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For convenience the eigenvalues are as

$$s_1, s_2, \dots, s^*{}_N, s_{2N+1}, \dots, s_m \tag{2.8}$$

where $(\bullet)^*$ denotes complex conjugation. The eigenvalue problem associated with equation (2.3) can be defined from (2.5) as

$$\boldsymbol{D}(s_j)\boldsymbol{z}_j = 0, \quad for\ j = 1, \dots, m. \tag{2.9}$$

where

$$\boldsymbol{D}(s_j) = s_j{}^2\boldsymbol{M} + s_j\boldsymbol{G}(s_j) + \boldsymbol{K} \tag{2.10}$$

is the dynamic stiffness matrix corresponding to the $j-th$ eigenvalue and z_j is the $j-th$ eigenvectors. Here $(\bullet)^T$ denotes the matrix transpose.

3.0 NATURE OF THE EIGENSOLUTIONS

The eigenvalue problem associated with equation (2.3) can be defined as

$$[s_j{}^2\boldsymbol{M} + s_j\boldsymbol{G}(s_j) + \boldsymbol{K}]\boldsymbol{z}_j = 0 \quad \text{or} \quad \boldsymbol{D}(s_j)\boldsymbol{z}_j = 0,\ for\ all\ k = 1, \dots, m \tag{3.1}$$

where the dynamic stiffness matrix

$$\boldsymbol{D}(s) = s^2\boldsymbol{M} + s\boldsymbol{G}(s) + \boldsymbol{K} \quad \epsilon\ \mathbb{C}^{N\times N} \tag{3.2}$$

Here $\boldsymbol{z}_j$ is the $j-th$ eigenvector and s_j is the $j-th$ eigenvalue. In general the number of eigenvalues, $m = 2N + p$; $p \geq 0$. It is assumed that all m eigenvalues are distinct. We consider the damping to be 'non-proportional', that is, the mass and stiffness matrices as well as the matrix of the kernel function cannot be simultaneously diagonalized by any linear transformation (see numerical examples in [1]). It is assumed that $|G_{jk}(s) < \infty|$ when $s \to \infty$. This in turn implies that the elements of $\boldsymbol{G}(s)$ are at the most of order $\frac{1}{s}$ in s or constant, as in the case of viscous damping. Now, we construct the diagonal matrix containing the eigenvalues as

$$S = diag[s_1, s_2, \dots, s_m] \ \in\ \mathbb{C}^{m\times m} \tag{3.3}$$

and the matrix containing the eigenvectors (the modal matrix) as

$$\boldsymbol{Z} = [\boldsymbol{z}_1, \boldsymbol{z}_2, \dots, \boldsymbol{z}_m] \ \in\ \mathbb{C}^{N\times m} \tag{3.4}$$

4.0 NORMALIZATION OF THE EIGENVECTORS

We now considered the normalization relationship of eigenvectors as in [4]. Pre-multiplying equation (3.1) by $\boldsymbol{z}_k^T$, applying equation (3.1) for $k-th$ the set and post-multiplying by $\boldsymbol{z}_j$ and subtracting one from the other we obtain

$$\boldsymbol{z}_k^T[(s_J^2 - s_k^2)\boldsymbol{M} + s_j\boldsymbol{G}(s_j) - s_k\boldsymbol{G}(s_k)]\boldsymbol{z}_j = 0 \tag{4.1}$$

Since s_j and s_k are distinct for different j and k are, the above equation can be divided by $(s_j - s_k)$ to obtain

$$\boldsymbol{z}_k^T\left[(s_j + s_k)\boldsymbol{M} + \frac{s_J\boldsymbol{G}(s_j) - s_k\boldsymbol{G}(s_k)}{s_j - s_k}\right]\boldsymbol{z}_j = 0,\ for\ all\ j,k;\ j \neq k \tag{4.2}$$

This equation may be regarded as the orthogonality relationship of the eigenvectors. It is easy to verify that, in the undamped limit equation (4.2) degenerates to the familiar mass orthogonality relationship of the undamped eigenvectors. However, this orthogonality relationship is not very useful because it is expressed

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in terms of natural frequencies. A frequency-independent orthogonality relationship of the eigenvector is derived later in this paper. Assuming $\delta_s = s_j - s_k$, we rewrite (4.2) as

$$\boldsymbol{z}_k^T \left[(\delta_s + 2s_k)\boldsymbol{M} + \frac{(s_k+\delta_s)\boldsymbol{G}(s_k+\delta_s) - s_k\boldsymbol{G}(s_k)}{\delta_s} \right] \boldsymbol{z}_J = 0 \tag{4.3}$$

Consider the case when $s_j \to s_k$, that is, $\delta_s \to 0$. For this limiting case, equation (4.3) reads

$$\boldsymbol{z}_k^T \left[2s_k\boldsymbol{M} + \frac{\partial[s\boldsymbol{G}(s)]}{\partial s}\bigg|_{s_k} \right] \boldsymbol{z}_k = \theta_k \tag{4.4}$$

$$\text{or } \boldsymbol{z}_k^T [2s_k\boldsymbol{M} + \boldsymbol{G}(s_k) + s_k\boldsymbol{G}'(S_K)]\boldsymbol{z}_k = \theta_k, \;\; for\ all\ k = 1, \ldots, m \tag{4.5}$$

for some non-zero $\theta_k \epsilon\, \mathbb{C}$. Equation (4.5) is the normalization relationship for the eigenvectors of the non-viscously damped system (2.3). From the expression of the dynamic stiffness matrix in (3.2), the normalization condition in equation (4.5) can be expressed as

$$\boldsymbol{z}_z^T \boldsymbol{D}'(s_k)\boldsymbol{z}_k = \theta_k, \;\; for\ all\ \ k = 1, \ldots, m \tag{4.6}$$

Equation (4.5) and (4.6), can be regarded as the generalization of the mass normalization relationship used in structural dynamics (see [6], [7] *and* [8]). In the undamped limit when $\boldsymbol{G}(s)$ is a null matrix, equation (4.5) reduces to the familiar mass normalization relationship for the undamped eigenvectors. For viscously damped system (see section 7.0 for details), relationship analogous to (4.5) was obtained using state-space approach by [5] and using second order equation of motion by (7) and (8). We define the normalization matrix, $\boldsymbol{\Theta}$, as

$$\boldsymbol{\Theta} = diag[\theta_1, \theta_2, \ldots, \theta_m] \in \mathbb{C}^{m\times m} \tag{4.7}$$

Numerical value of θ_k can be selected in various ways:

Choose $\theta_k - 2s_k,\ for\ all\ k$ that is $\boldsymbol{\Theta} - 2\boldsymbol{S}$. This reduces to $\boldsymbol{z}_k^T\boldsymbol{M}\boldsymbol{z}_k = 1,\ for\ all\ k$ when the damping is zero. This is consistent with the unity modal mass convention, often used in experimental modal analysis and FEM.

Choose $\theta_k = 1 + 0i,\ for\ all\ k$ that is, $\boldsymbol{\Theta} = \boldsymbol{I}_m$. Theoretical analysis becomes easiest with this normalization. However, as pointed out by [1], [2] and [3] in the context of viscously damped systems, this normalization is inconsistent with undamped or classically damped modal theories.

5.0 ORTHOGONALITY OF THE EIGENVECTORS

Equation (4.2) is the orthogonality relationship of the eigenvector and is not very useful because it is expressed in terms of the eigenvalues system. In this section, we developed an orthogonality relationship that is independent of the eigenvalues. Expressions equivalent to the orthogonality relationship of undamped eigenvectors with respect to mass and stiffness matrices are also established. We derived these results by first recalling the expression of the transfer function matrix.

The transfer function (matrix) of a system completely defines its input-output relationship in steady-state. For any linear system, if the force function is harmonic, that is $\boldsymbol{f}(t) = \bar{\boldsymbol{f}}exp[st]$ with $s = iw$ and amplitude vector $\bar{\boldsymbol{f}} \in \mathbb{R}^N$, the steady-state response will also be harmonic at frequency $w \in \mathbb{R}^+$. Now, we find a solution of the form $\boldsymbol{q}(t) = \bar{\boldsymbol{q}}exp[st]$, where $\bar{\boldsymbol{q}} \in \mathbb{C}^N$ is the response vector in the frequency domain. Substituting $\boldsymbol{q}(t)$ and $\boldsymbol{f}(t)$ in equation (2.3) yields

$$s^2\boldsymbol{M}\bar{\boldsymbol{q}} + s\boldsymbol{G}(s)\bar{\boldsymbol{q}} + \boldsymbol{K}\bar{\boldsymbol{q}} = \bar{\boldsymbol{f}} \quad \text{or} \quad \boldsymbol{D}(s)\bar{\boldsymbol{q}} = \bar{\boldsymbol{f}} \tag{5.1}$$

Here the dynamic stiffness matrix

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$$D(s) = s^2M + sG(s) + K \in \mathbb{C}^{N\times N} \qquad (5.2)$$

From equation (5.1) the response vector can be obtained as

$$\bar{q} = D^{-1}(s)\bar{f} = H(s)\bar{f} \qquad (5.3)$$

where

$$H(s) = D^{-1}(s) \in \mathbb{C}^{N\times N} \qquad (5.4)$$

is the transfer function matrix. From this equation, it is implies that

$$H(s) = \frac{adj[D(s)]}{det[D(s)]} \qquad (5.5)$$

The poles of $H(s)$, denoted by s_j are the eigenvalues system. Each pole is a simple pole because it is assumed that all the m eigenvalues are distinct. From the residue theorem (see [1]), it is known that any complex function can be expressed in terms of the poles and residues, that is the transfer function now has the form

$$H(s) = \sum_{j=1}^{m} \frac{R_j}{s-s_j} \qquad (5.6)$$

Here, we have

$$R_j = \operatorname*{res}_{s=s_j}[H(s)] \stackrel{\text{def}}{=} \lim_{s\to s_j}(s-s_j)[H(s)] \qquad (5.7)$$

is the residue of the transfer function matrix at the pole s_j. It may be noted that equation (5.6) is equivalent to expressing the right - hand side of equation (5.5) in the partial-fraction form. Here we try to obtain the residues, that is the coefficients in the partial-fraction form, in terms of the systems eigenvectors.

In the pole-residue form the inverse of the dynamic stiffness matrix (transfer function matrix) can be expressed as

$$D^{-1}(s) = \frac{adj[D(s)]}{det[D(s)]} = \sum_{j=1}^{m} \frac{R_j}{s-s_j} \qquad (5.8)$$

Here R_j, the residue of $D^{-1}(s)$ at the pole s_j is obtained by using the normalizing conditions and l'Hospital rule (see [1]) as

$$R_j = \frac{z_j z_j^T}{z_j^T \frac{\partial D(s_j)}{\partial s_j} z_j} \in \mathbb{C}^{N\times N} \qquad (5.9)$$

Now using equations (4.6) and (5.9) one finally obtained the residue as

$$R_j = \frac{z_j z_j^T}{\theta_j} \qquad (5.10)$$

THEOREM 5.1 The modal matrix of a non-viscously damped system, $Z \in \mathbb{C}^{N\times m}$, satisfy the orthogonality relationship $Z\Theta^{-1}Z^T = 0_N$.

Proof. From equation (5.8) and (5.10) one obtains

$$\frac{adj[D(s)]}{det[D(s)]} = \sum_{j=1}^{m} \frac{1}{s-s_j} \frac{z_j z_j^T}{\theta_j} \qquad (5.11)$$

Multiplying both sides of the above equation by s and taking limit as $s \to \infty$ we obtain

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$$\lim_{s\to\infty} s\frac{adj[\boldsymbol{D}(s)]}{det[\boldsymbol{D}(s)]} = \lim_{s\to\infty}\sum_{j=1}^{m}\frac{s}{s-s_j}\ \frac{\boldsymbol{z}_j\boldsymbol{z}_j^T}{\theta_j} = \sum_{j=1}^{m}\frac{\boldsymbol{z}_j\boldsymbol{z}_j^T}{\theta_j} \tag{5.12}$$

It is easier to observe that the order of the element of $adj[\boldsymbol{D}(s)]$ is at most $(m-2)$ in s. Since the order of the determinant, $det[\boldsymbol{D}(s)]$ is m, after taking the limit every element of the left-hand side of equation (5.12) reduces to zero. Thus, in the limit, the left-hand side of equation (5.12) approaches to an $N\times N$ null matrix. Finally, writing equation (5.12) in the matrix form we obtained

$$\boldsymbol{Z}\boldsymbol{\Theta}^{-1}\boldsymbol{Z}^T = \boldsymbol{0}_N \tag{5.13}$$

and the theorem is proved.

THEOREM 5.2 The modal matrix of a non-viscously damped system, $\boldsymbol{Z}\in\mathbb{C}^{N\times m}$, satisfy the relationship $\boldsymbol{Z}\boldsymbol{\Theta}^{-1}\boldsymbol{S}\boldsymbol{Z}^T = \boldsymbol{M}^{-1}$.

Proof. First we considered the function $s\boldsymbol{D}^{-1}(s)$ and using the residue theorem one obtains

$$s\boldsymbol{D}^{-1}(s) = \sum_{j=1}^{m}\frac{\boldsymbol{Q}_j}{s-s_j} \tag{5.14}$$

Here the residues $\boldsymbol{Q}_j$ can be obtained as

$$\boldsymbol{Q}_j \stackrel{\text{def}}{=} \lim_{s\to s_j}(s-s_j)\,[s\boldsymbol{D}^{-1}(s)] = s_j\frac{\boldsymbol{z}_j\boldsymbol{z}_j^T}{\theta_j} \tag{5.15}$$

Using the expression of the dynamic stiffness matrix in equation (3.2) we can deduce

$$\lim_{s\to\infty}\frac{\boldsymbol{D}(s)}{s^2} = \lim_{s\to\infty}\left[\boldsymbol{M}+\frac{\boldsymbol{G}(s)}{s}+\frac{\boldsymbol{K}}{s^2}\right] = \boldsymbol{M} \tag{5.16}$$

Taking the inverse of the above equation results

$$\lim_{s\to\infty}[s^2\boldsymbol{D}^{-1}(s)] = \boldsymbol{M}^{-1} \tag{5.17}$$

Now multiplying equation (5.14) by s and taking the limit as $s\to\infty$ we obtain

$$\lim_{s\to\infty}[s^2\boldsymbol{D}^{-1}(s)] = \lim_{s\to\infty}\sum_{j=1}^{m}\frac{s}{s-s_j}\ s_j\frac{\boldsymbol{z}_j\boldsymbol{z}_j^T}{\theta_j} = \sum_{j=1}^{m}s_j\frac{\boldsymbol{z}_j\boldsymbol{z}_j^T}{\theta_j} \tag{5.18}$$

Putting the right-hand side of the above equation in matrix form and equating it with (5.17) results

$$\boldsymbol{Z}\boldsymbol{\Theta}^{-1}\boldsymbol{S}\boldsymbol{Z}^T = \boldsymbol{M}^{-1} \tag{5.19}$$

and the theorem is proved.

REMARK: Since $\boldsymbol{\Theta}$ and $\mathbf{S}$ are diagonal matrices, they commute in product. For this reason the above result can also be expressed as $\boldsymbol{Z}\boldsymbol{S}\boldsymbol{\Theta}^{-1}\boldsymbol{Z}^T = \boldsymbol{M}^{-1}$.

By considering the normalization matrix $\boldsymbol{\Theta}$ as the identity matrix. It might be thought that by taking the inverse of equation (5.19) and rearranging the conventional mass-orthogonality relationship

$$\boldsymbol{Z}^T\boldsymbol{M}\boldsymbol{Z} = \boldsymbol{S}^{-1}\boldsymbol{\Theta} \tag{5.20}$$

could be obtained. We now emphasized that the representation of equation (5.19) in the form of equation (5.20) is not always possible. To show this, we premultiply equation (5.20) by $\boldsymbol{Z}\boldsymbol{\Theta}^{-1}$ to obtain

$$\boldsymbol{Z}\boldsymbol{\Theta}^{-1}\boldsymbol{Z}^T\boldsymbol{M}\boldsymbol{Z} = \boldsymbol{Z}\boldsymbol{\Theta}^{-1}\boldsymbol{S}^{-1}\boldsymbol{\Theta} = \boldsymbol{Z}\boldsymbol{S}^{-1} \tag{5.21}$$

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Due to theorem 5.1, the left-hand side of equation (5.21) is a null matrix, while its right-hand side is not. Thus, (5.20) cannot be a valid equation. However, for a special case, when the system is undamped, the modal matrix $\boldsymbol{Z}$ can be expressed by a square matrix and equation (5.19) can be represented by the classical mass-orthogonality relationship in (5.20). Thus, theorem 5.2 provide the results equivalent to the classical mass-orthogonality relationship for general cases.

Like the mass-orthogonality relationship of the eigenvectors, the orthogonality relationship with respect to the stiffness mass can also be obtained. Assuming that $\boldsymbol{K}^{-1}$ exists we have the following results:

THEOREM 5.3 The modal matrix of a non-viscously damped system, $\boldsymbol{Z} \in \mathbb{C}^{N\times m}$, satisfy the relationship $\boldsymbol{Z\theta}^{-1}\boldsymbol{S}^{-1}\boldsymbol{Z}^T = -\boldsymbol{K}^{-1}$.

Proof. Using the expression of the dynamic stiffness matrix in equation (3.2) we can easily deduce

$$\lim_{s\to 0} \boldsymbol{D}(s) - \boldsymbol{K} \qquad (5.22)$$

Taking the inverse of the above equation results

$$\lim_{s\to 0} \boldsymbol{D}^{-1}(s) = \boldsymbol{K}^{-1} \qquad (5.23)$$

From the equations (5.8) and (5.10) one obtains

$$\boldsymbol{D}^{-1}(s) = \sum_{j=1}^{m} \frac{1}{s-s_j}\frac{\boldsymbol{z}_j\boldsymbol{z}_j^T}{\theta_j} \qquad (5.24)$$

Taking the limit as $s \to 0$ in equation (5.24) we obtain

$$\lim_{s\to 0} \boldsymbol{D}^{-1}(s) = \sum_{j=1}^{m} \frac{1}{-s_j}\frac{\boldsymbol{z}_j\boldsymbol{z}_j^T}{\theta_j} \qquad (5.25)$$

Now, putting the right-hand side of the preceding equation in the matrix form and equating it with (5.23) results

$$Z\theta^{-1}S^{-1}S^{T} = -K^{-1} \qquad (5.26)$$

and the theorem is proved.

6.0 EIGENSOLUTIONS AND DAMPING RELATIONSHIP

Some direct relationships have been established between the mass and stiffness matrices and eigensolutions in the last section. In this section, the relationships between the damping matrix and eigensolutions are established. However, a major difficulty in this regards is that, unlike the mass and stiffness matrices, the damping matrix, $G(s)$, is a function of s. To simplify the problem we have considered two limiting cases, (a) when $s \longrightarrow \infty$ (b) when $s \longrightarrow 0$. Suppose

$$\lim_{s\to\infty} G(s) = G_{\infty} \in R^{N\times N} \qquad (6.1)$$

and

$$\lim_{s\to 0} G(s) = G_0 \in R^{N\times N} \qquad (6.2)$$

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where $\|G_\infty\|, \|G_0\| < \infty$.

6.1 **RELATIONSHIP IN TERMS OF** M^{-1}

Putting equation (5.24) into matrix form one obtains

$$D^{-1}(s) = Z\theta^{-1}(sI_m - S)^{-1}Z^T \qquad (6.3)$$

The proceding equation can be expressed as

$$D^{-1}(s) = \frac{1}{s}Z\theta^{-1}\left(I_m - \frac{S}{s}\right)^{-1} z^T = \frac{1}{s}\left(Z\theta^{-1}SZ^T\right) + \frac{1}{s^3}\left(Z\theta^{-1}SZ^T\right) + \frac{1}{s^3}\left(Z\theta^{-1}S^2Z^T\right) + \frac{1}{s^4}\left(Z\theta^{-1}S^{-3}Z^T\right) + \ldots \qquad (6.4)$$

Now, we rewrite the expression of the dynamic stiffness matrix in equation (3.2) as

$$D(s) = s^2M\left[I_N + \frac{M^{-1}}{s}\left(G(s) + \frac{K}{s}\right)\right] \qquad (6.5)$$

Taking the inverse of the preceding equation and expanding the right-hand side one obtains

$$D^{-1}(s) = \left[I_N - \frac{M^{-1}}{s}\left(G(s) + \frac{K}{s}\right) + \left\{\frac{M^{-1}}{s}\left(G(s) + \frac{K}{s}\right)\right\}^2 - \ldots\right]\frac{M^{-1}}{s^2} \qquad (6.6)$$

Equation (6.6) can be further simplified to obtain

$$D^{-1}(s) = \frac{M^{-1}}{s^2} + \frac{1}{s^3}\left(-M^{-1}G(s)M^{-1}\right) + \frac{1}{s^4}\left(M^{-1}\left[G(s)M^{-1}G(s) - K\right]M^{-1}\right) + \ldots \qquad (6.7)$$

Comparing equations (6.4) and (6.7) it is clear that their right-hand sides are equal. Theorems 5.1 and 5.2 can be alternatively proved by multiplying these equations by s and s^2 respectively and taking the limit as $s \to \infty$. Observe that, the coefficients associated with the corresponding (negative) powers of s in the series expansions (6.4) and (6.7) cannot be equated because $G(s)$ is also a function of s. However, in the limit when $s \to \infty$ the variation of $G(s)$ becomes negligible as by equation (6.1) it approaches to G_∞ Considering the second term of the right-hand side of equation (6.7), equating it with the corresponding terms of equation (6.4) and taking the limit as $s \to \infty$ one obtains

$$Z\theta^{-1}S^2Z^T = -M^{-1}G_\infty M^{-1} \qquad (6.8)$$

Were the system viscously damped $G(s)$ would be a constant matrix and equating the coefficients associated with different powers of s one could obtain several relationships between the eigensolutions and the system matrices.

6.2 RELATIONSHIP IN TERMS OF K^{-1}

Now, rewriting equation (6.3) as

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$$D^{-1}(s) = -Z\theta^{-1}S^{-1}\left(I_m - sS^{-1}\right)^{-1} z^T \tag{6.9}$$

Expanding equation (6.9) one obtains

$$D^{-1}(s) = -Z\theta^{-1}S^{-1}Z^T - s\left(Z\theta^{-1}S^{-2}Z^T\right) - s^2\left(Z\theta^{-1}S^{-3}Z^T\right) - s^3\left(Z\theta^{-1}S^{-4}Z^T\right) - \ldots \tag{6.10}$$

The expression of the dynamic stiffness matrix equation (3.2) can be rearranged as

$$D(s) = K\left[I_N + s\left(sK^{-1}M + K^{-1}G(s)\right)\right] \tag{6.11}$$

Taking the inverse of equation (6.11) and expanding the right-hand side one obtains

$$D^{-1}(s) = \left[I_N + s\left(sK^{-1}M + K^{-1}G(s)\right) + \left\{s\left(sK^{-1}M + K^{-1}G(s)\right)\right\}^2 - \ldots\right]K^{-1} \tag{6.12}$$

The preceding equation can further be simplified to obtain

$$D^{-1}(s) = K^{-1} + s\left(-K^{-1}G(s)K^{-1}\right) + s^2\left(K^{-1}\left[G(s)K^{-1}G(s) - M\right]K^{-1}\right) + \ldots \tag{6.13}$$

Comparing the right-hand side of equation (6.10) and (6.13), theorem 5.3 can be proved alternatively by taking the limit as $s \rightarrow 0$. Considering the second term of the right-side of equation (6.13), equating it with the second term of equation (6.10) and taking the limit as $s \rightarrow 0$ one obtains

$$Z\theta^{-1}S^{-2}Z^T = K^{-1}G_0K^{-1} \tag{6.14}$$

Theorem 5.2, 5.3 and equations (6.8), (6.14) allowed us to represent the system property matrices explicitly in terms of eigensolutions. This might be useful in system identification problems where the eigensolutions of a structure can be measured from experiments. Using the eigensolutions we defined two matrices as follows:

$$P_1 = Z\theta^{-1}SZ^T \tag{6.15}$$

and

$$P_2 = Z\theta^{-1}SZ^T \tag{6.16}$$

Using the equations, from equation (5.19) one obtains the mass matrix as

$$M = P_1^{-1} \tag{6.17}$$

Similarly from equation (5.26) the stiffness matrix can be obtained as

$$K = -P_2^{-1} \tag{6.18}$$

The damping matrix in the Laplace domain, $G(s)$ can be obtained only at the two limiting values when $s \rightarrow 0$ and $s \rightarrow 0$. From equations (6.8) and (6.14) one obtains

$$G_\infty = -P_1^{-1}\left[Z\theta^{-1}S^2Z^T\right]P_1^{-1} \tag{6.19}$$

and

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$$G_0 = -P_2^{-1}\left[Z\theta^{-1}S^{-2}Z^T\right]P_2^{-1} \quad (6.20)$$

7.0 EIGENRELATIONSHIPS FOR VISCOUSLY DAMPED SYSTEMS

Viscously damped systems arise as a special case of the more general non-viscously damped systems when the damping matrix become a constant matrix, that is

$G(s) = C \in R^{N\times N}$, *for all* s. For viscously damped systems the order of the characteristic polynomial $m = 2N$, and consequently the modal matrix $Z \in C^{N\times 2N}$ and the diagonal matrices $S, \theta \in C^{2N\times 2N}$. From equation (4.5), the normalization relationships reads

$$z_k^T\left[2s_k M + C\right]z_k = \theta_k, \quad \textit{for all } k = 1,\dots,2N \quad (7.1)$$

Now, considering the series expansion of $D^{-1}(s)$ given by (6.4) and (6.7). Equating the coefficients of $\frac{1}{s}$ we obtain the mode orthogonality relationship

$$Z\theta^{-1}Z^T = O_N \quad (7.2)$$

Also, equating the coefficients of $\frac{1}{s^2},\dots,\frac{1}{s^5}$ in the right-hand sides of equating (6.4) and (6.7), several relationships involving the eigensolutions and M^{-1}, C *and* K may be obtained:

$$Z\theta^{-1}SZ^T = M^{-1} \quad (7.3)$$

$$Z\theta^{-1}S^2Z^T = -M^{-1}CM^{-1} \quad (7.4)$$

$$Z\theta^{-1}S^3Z^T = M^{-1}\left[CM^{-1}C - K\right]M^{-1} \quad (7.5)$$

and

$$Z\theta^{-1}S^4Z^T = M^{-1}\left[KM^{-1}C + K - CM^{-1}CM^{-1}C\right]M^{-1} \quad (7.6)$$

This procedure can be extended to obtain further higher order terms involving S. Similarly, equating the coefficients of $s^0,\dots,s^3$ in the right-hand sides of equations (6.10) and (6.13), several relationships involving the eigensolutions and K^{-1}, C *and* M may be obtained:

$$Z\theta^{-1}S^{-1}Z^T = -K^{-1} \quad (7.7)$$

$$Z\theta^{-1}S^{-2}Z^T = K^{-1}CK^{-1} \quad (7.8)$$

$$Z\theta^{-1}S^{-3}Z^T = K^{-1}\left[M - CK^{-1}C\right]K^{-1} \quad (7.9)$$

and

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$$Z\theta^{-1}S^{-4}Z^{T} = K^{-1}\left[CM^{-1}CK^{-1}C - MK^{-1}C + CK^{-1}M\right]K^{-1} \qquad (7.10)$$

This procedure can be extended to obtain further lower order terms involving s (see $[1], [2]$ *and* $[3]$ *for numerical examples*.

8.0 CONCLUSION

In this paper we have developed several eigenrelations for non-viscously damped MDOF linear dynamic systems. It has been assumed that, in general, the mass and stiffness matrices as well as the matrix of the kernel function cannot be simultaneously diagonalized by any linear transformation. The analysis is, however, restricted to systems with non-repetive eigenvalues and non-singular mass matrices.

Relationship regarding the normalization and the orthogonality of the (complex) eigenvectors have been established (theorem 5.1). Expressions equivalent to the orthogonality of the undamped modes over the mass and stiffness matrices have been proposed (theorems 5.2 and 5.3). It was shown that the classical relationships can be obtained as a special case of these general results. Based on these results, we have shown that the mass and stiffness matrices can be uniquely expressed in terms of the eigensolutions. The damping matrix, $G(s)$, cannot be reconstructed using this approach because it is not a constant matrix. However, we have provided expressions which relate the damping matrix to the eigensolutions for the case when $s \to \infty$ *and* $s \to 0$. Whenever applicable, viscously damped counterparts of the newly developed results were also provided.

NOMENCLATURE

C Viscously damping matrix

M Mass matrix

K Stiffness matrix

I_N Identity matrix of size N

N Degree -of- freedom of the system

S Diagonal matrix containing s_k

O_N Null matrix of size N

$D(s)$ Dynamic stiffness matrix

$G(s)$ Damping function in the Laplace domain

$G'(s)$ Damping function in the Laplace domain in the modal coordinates

$H(s)$ Transfer function matrix in the Laplace domain

Z Matrix of the eigenvectors

p Number of non-viscous modes, $p = m - 2N$

Laplace transformation of the system

R_j Residue matrix corresponding to the pole s_j

Q_j Residue matrix, Q-FACTOR

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$f(t)$ Force vector

i Unit imaginary number, $i=\sqrt{-1}$

$q(t)$ Vector of the generalized coordinates

X Matrices of the eigenvectors

$y(t)$ Modal coordinates

$G(t)$ Damping function in the time domain

w_j, w_k j-th and k-th undamped natural frequencies

t Time

q_0 Vector initial displacements

$\dot{q}_0$ Vector initial velocities

m Order of the characteristic polynomial

s Laplace domain parameter

z_j j-th eigenvector of the system

s_j j-th eigenvalue of the system

x_j j-th undamped vector

$\Phi(s)$ Matrix of the eigenvectors of $D(s)$

Θ Normalization matrix

$(\bullet)_e$ Elastic modes

$(\bullet)_n$ Non-viscous modes

$(\bullet)'$ Derivative of $(\bullet)$ with respect to s

$(\bullet)^T$ Matrix transpose of $(\bullet)$

$(\bullet)^{-1}$ Matrix inverse of $(\bullet)$

$(\bullet)^{-T}$ Inverse transpose of $(\bullet)$

$(\bullet)^*$ Complex conjugate of $(\bullet)$

$|\bullet|$ Absolute value of $(\bullet)$

$\overline{C}$ Coefficient matrix associated with the non-viscous damping function

θ Characteristic time constant

$\det(\bullet)$ Determinant of $(\bullet)$

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$adj(\bullet)$ Adjoint of $(\bullet)$

$diag(\bullet)$ A diagonal matrix

DOF Degree of freedom

FEM Finite Element Method

$MDOF$ Multiple-Degree-of-freedom System

$SDOF$ Single-Degree-of-freedom System

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