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An Optimal Stochastic Investment and Consumption Strategy with Log Utility

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Abstract

This paper considers a single investor who owns a production plant that generates units of consumption goods in a capitalist economy. The goal is to choose optimal investment and consumption policies that maximize the finite horizon expected discounted logarithmic utility of consumption and terminal wealth. A dynamical programming principle is used to derive the optimal investment and consumption strategy when the state variable follows a mean-reverting square root process.

Keywords: Mean-reverting square root process, Control process, Optimal Strategies, Dynamic Programming Principle

1.0 Introduction

This paper treats a consumption and investment decision problem for a single economic investor who owns a production plant that generates units of consumption goods for the economy.

The economic situation and the investment therein is characterized by a lot of uncertainties as the case of the productivity of the investor's plant that fluctuates randomly and is represented by a filtered probability space (Ω, f, F, P), where $F=\{f_t\}t \geq 0$ is the natural filtration generated by a one-dimensional Brownian motion z, see [9].

In each period, the plant works with the capital stocks to produce the output. This output is then used either to satisfy the needs and wants of the first (consumption) or to replace and augment the second (investment) see [1].

Both production and the allocation of output between consumption and investment are carried out efficiently; that is, in any given period, no output could be increased without another being decreased, and at the end of any span of periods, no capital stock could be increased without another or some intermediate consumption being decreased. Such an evolution starts from a given profile of capital stocks and continues within the planning horizon.

We assume that the value of the production plant follows the dynamics defined in equation (2.2), see [5]. The objective is to find optimal controls that maximize the total expected discounted logarithmic utility of consumption and investment; thus, identify the path of capital accumulation that maximizes discounted per capita consumption over a finite planning horizon.

We apply the dynamical programming principle and derive the Hamilton-Jacobi-Bellman (HJB) equation associated with the utility maximization problem. Since the HJB-equation turns out to be a highly non-linear second order equation, we therefore conjecture a solution that solves the HJB-equation and obtain ordinary differential equation with deterministic time dependent real valued functions, which satisfies the terminal wealth condition see for example [7].

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2.0 The Problem And Its Formulation

In this section, we describe the general economic model and state the utility maximization problem of an individual investor.

The general form of the productivity of her plant is formulated as a stochastic control problem, which depends on a state variable Y_t that follows a mean-reverting process; see [8] as:

$$dY_t = (b - kY_t)dt + q\sqrt{Y_t} \quad 2.1$$
$$Y_0 = y.$$

The state variable follows a mean-reverting square root process with reversion parameter $k > 0$, with $\mathrm{E}[y_t] = b$, and $Var(y_t) = q^2 b/2$ see [3]. In order to satisfy standard integrability conditions, we assume that $2b > q^2$.

Let $c_t \geq 0$ denote the rate by which the investor withdraws consumption goods from the production plant and $\pi_t \geq 0$ denote the rate the investor re-invest in the production process or the path of capital accumulation see for example [6]. Also let X_t^A be the value of the production plant at time t, given the admissible control process A. . We assume that the stochastic differential equation of X_t^A are:

$$dX_t^A = (X_t^A \pi h Y_t - c_t)dt + X_t^A \pi \varepsilon \sqrt{Y_t} \quad 2.2$$
$$X_0^A = x.$$

where h and ε are positive constants with h $> \varepsilon^2$ see [4]. The pair $(c_t, \pi_t) \in A_t, t \geq 0$ represents our control variables. For initial endowment $x \geq 0$ and admissible control process $(A_t) t \geq 0$, we introduce the indirect utility function as:

$$V(x,y,t) = \sup_{(c,x)\in A} E_{x,y,t}\left[\int_t^T e^{-\delta(s-t)} Inc_s ds + e^{-\delta(T-t)} InX_T\right] \quad 2.3$$

where X_T is the terminal value of the production plant. $E_{x,y,t}$ denote the expectations operator given $X_t^A = X$. Our goal is to choose optimal strategies to maximize the total expected discounted log utility of consumption and wealth at the end of the planning horizon $t \leq T$.

2.1 Method Of Solution By Dynamic Programming Principle (DPP)

By the dynamical programming method, the optimal solution to the control problem will satisfy the HJB-equation:

$$\delta V(x,y,t) = \sup_{A\geq 0}\left\{Inc + \frac{\partial V}{\partial t} + V_x(X^A \pi h Y - c) + \frac{1}{2}V_{xx}X^{2A}\pi^2\varepsilon^2 Y + V_y(b-kY) + \frac{1}{2}V_{yy}q^2 Y + V_{xy}X^A \pi\varepsilon q Y\right\}$$

2.4

For simplicity, let:

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$$\delta V(x,y,t) = \delta V,$$

$$V_x(x,y,t) = V_x,$$

$$\mathrm{V}_\mathrm{y}(x,y,t) = V\ ,$$

$$X^A = X.$$

From the HJB-equation (2.4), the first-order-condition with respect to consumption is:

$$c = V_x^{-1}. \qquad 2.5$$

The maximization of the RHS of (2.4) with respect to π gives the first-order-condition as:

$$V_x x\pi hy + V_{xx}x^2\pi^2\varepsilon^2 y + V_{xy}x\pi(\varepsilon q)y = 0,$$

$$\pi = -\frac{V_x h}{xV_{xx}\varepsilon^2} - \frac{V_{xy}\,q}{xV_{xx}\varepsilon}, \qquad 2.6$$

so that,

$$\pi_t^{\bullet} = -\frac{V_x h}{xV_{xx}\varepsilon^2} - \frac{V_{xy}\,q}{xV_{xx}\varepsilon}\ . \qquad 2.7$$

Putting back equations 2.5 and 2.6 into equation (2.4) and gathering terms gives a second order non-linear partial differential equation as:

$$\delta V = V_x\left[-\frac{V_x h^2}{V_{xx}\varepsilon^2} - \frac{V_{xy}qh}{V_{xx}\varepsilon}\right] + \frac{1}{2}V_{xx}\left[\frac{V_x^2h^2\varepsilon^2}{V_{xx}^2\varepsilon^4} + \frac{V_{xy}^2q^2\varepsilon^2}{V_{xx}\varepsilon^2} - 2\frac{V_xV_yhq}{V_{xx}\varepsilon}\right]$$

$$+ V_{xy}\left[-\frac{V_xhq\varepsilon}{V_{xx}\varepsilon^2} - \frac{V_{xy}q^2\varepsilon}{V_{xx}\varepsilon^2}\right] + \frac{\partial V}{\partial t} + V_y(b-Ky) + \frac{1}{2}V_{yy}q^2y \qquad 2.8$$

$$+ In\left((V_x^{-1}) - V_x(V_x^{-1})\right).$$

Simplifying the problem by assuming $u(c) = Inc$, and denoting the inverse of the marginal utility $u'(c)$ by $I_u(.)$, we have see for example [8]:

$$u'(c) = \frac{1}{c} = I_u(a). \qquad 2.9$$

Letting $a = V_x^{-1}$

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we have

$$\frac{1}{c} = I_u(V_x^{-1}). \qquad 2.10$$

So that (2.8) reduces to:

$$\delta V = \frac{\partial V}{\partial t} + \frac{1}{2}\left[\frac{V_x^2 h^2}{V_{xx}\varepsilon^2} + \frac{V_{xy}^2 q^2}{V_{xx}\varepsilon}\right] - \frac{V_x^2 h^2}{V_{xx}\varepsilon^2}$$

$$- \frac{V_x V_{xy} qh}{V_{xx}\varepsilon} - \frac{2V_x V_{xy} hq}{V_{xx}\varepsilon} + V_y(b - Ky)$$

$$+ \frac{1}{2}V_{yy}q^3 y + InV_x^{-1} - 1$$

Remark

We assume instead, in this case, that the solution of the HJB equation takes the form:

$$V(x,y,t) = A_1(t)Inx + A_2(y) + A_3(t) \qquad 2.11$$

for all non negative values of x, y, t.

The terminal condition on wealth is:

$$V(x,y,T) = Inx_T, \qquad 2.12$$

which implies that $A_1(T) = 1$ and $A_2(T) = A_3(T) = 0.$

With this guess, we have reduced the dimensionality of the unknown function $V(x,y,t)$ to three deterministic functions of time $A_1(t), A_2(t)$ *and* $A_3(t)$. We then find values of $A_1(t), A_2(t)$ *and* $A_3(t)$ such that $V(x,y,t)$ solves the HJB-equation.

Subsequently, we will derive the relevant derivatives of $V(x,y,t)$ from (2.11) and substitute them back in the HJB-equation of (2.8) to eliminate terms.

$$V_x = \frac{A_1(t)}{x}, \quad V_{xx} = -\frac{A_1(t)}{x^2},$$

$$V_y = A_2(t), \quad V_{yy} = 0,$$

$$V_{xy} = 0, \quad \frac{\partial V}{\partial t} = A_1'(t)Inx + A_2'(t)y + A_3'(t).$$

By substitution back into the HJB-equation of (2.8), and gathering terms together, simplifies to:

$$\delta[A_1(t)Inx + A_2(y) + A_3(t)] = A_1'(t)Inx + A_2'(t)y + A_3'(t)$$

$$+ A_1(t)\left[\frac{h^2}{2\varepsilon^2}\right] + A_2(t)(b - Ky) + Inx - In A_1(t) - 1 .$$

2.13

It is easy to verify that:

$$A_1(t) = \frac{1}{\delta}\left(1 - e^{-\delta(T-t)}\right) + e^{-\delta(T-t)}$$

(which is our consumption – wealth ratio),

$$A_2(t) = A_2(T)e^{-\delta(T-t)},$$

$$A_3(t) = \frac{1}{\delta}\left[A_1(t)\beta y - A_2(y)\gamma + In A_1(t) + 1 - \left(A_1(t)\beta y - A_2(y)\gamma + In A_1(t) + 1\right)e^{-\delta(T-t)}\right]$$

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2.3 Explicit Derivation of Optimal Consumption and Investment

It becomes pertinent that we derive the optimal consumption and investment strategies explicitly from the indirect utility function for a logarithmic investor see [8].

From equation (2.5), we have that

$$u'(c) = V_x = \frac{1}{c} \qquad 2.14$$

is the first order condition in our case for a logarithmic investor.

But
$$V_x = \frac{A_1(t)}{x}. \qquad 2.15$$

Therefore, by substitution,

$$\frac{1}{c} = \frac{A_1(t)}{x_t}, \qquad 2.16$$

which then implies that the optimal consumption rate is:

$$c_t^{\bullet} = \frac{1}{A_1(t)} X_t^{\bullet}. \qquad 2.17$$

Where $X_t^{\bullet}$ is the optimal wealth of the investor and $A_1(t)$ is the consumption-wealth ratio or savings ratio

$$0 \leq \frac{1}{A_1(t)} \leq 1$$

Consequently, we can derive the optimal investment strategy by maximization of the HJB equation with respect to π ,which gives:

$$\pi = -\frac{V_x h y}{x V_{xx} \varepsilon^2 y} - \frac{V_{xy}(\varepsilon q) y}{x V_{xx} \varepsilon^2 y},$$

where
$$V_x = \frac{A_1(t)}{x},\ V_{xx} = -\frac{A_1(t)}{x^2},\ V_{xy} = 0.$$

Substituting, we have that the optimal investment strategy is:

$$\pi_t^{\bullet} = \left(\frac{h}{\varepsilon^2}\right) y_t. \qquad 2.18$$

Which is the path of capital accumulation that maximizes discounted per capita consumption over a planning horizon. Thus, we have proved the following result.

Theorem:

The optimal investment strategy is:

$$\pi(x, y, t) = \left(\frac{h}{\varepsilon^2}\right) y_t.$$

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Where y_t is the state variable.

The optimal consumption strategy is:

$$C(x,y,t) = A_1(t)^{-1}x \ = \delta\left(1+[\delta-1]e^{-\delta(T-t)}\right)^{-1}x$$

The optimal consumption strategy is to consume a time-varying fraction of wealth.

Inserting the optimal strategies into the general expression for the dynamics of wealth, we have:

$$dx_t^{\bullet} = x_t^{\bullet}\left[\left(\left(\frac{h}{\varepsilon}\right)^2 y_t - A_1(t)^{-1}\right)dt + \frac{h}{\varepsilon}\sqrt{y_t}\,dz_t\right].$$

Therefore, optimal wealth evolves stochastically and depends on the state variable.

Conclusion

In this paper, we considered a single economic investor whose production plant value follows the mean reverting square root process. We derived explicit expressions for the optimal investment and consumption strategies that maximize the finite horizon expected logarithmic utility of consumption and terminal wealth and thus identified the path of capital accumulation that maximizes per capita consumption over a planning horizon.

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