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Game Theory and its Relationship with Linear Programming Models

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Abstract

The emphasis in operations research in building models seeks to find optimal solutions to real life problems. The birth of operations research has placed a formidable tool in the hand of policy makers saddled with the duty of making decisions particularly in the business world. Game theory, a branch of operations research, has been successfully applied to solve various categories of problems arising from human decisions making characterized by the complexity of situations and the limits of individual processing abilities. This paper shows that game theory and linear programming problem are closely related subjects since any computing method devised for one of these theories can be used to solve problems arising in the other. It turns out that every game problem can be computed by converting it into a related linear programming problem, and every linear programming problem can be artificially converted into a game problem which as a result leads to a super linear programming problem.

INTRODUCTION

There are a numbers of reasons why gaming is relatively used in operations research. One obvious reason is the time and expense associated with the conduct of substantial gaming exercise. Another, reason is a general suspicion about the validity of the results and conclusion drawn from gaming exercise, [2]. However, according to [1], game theory attempts to mathematically capture strategies situations, in which an individual success in making choices depends on the choice of others, while is was initially developed to analyse competition in which an individual does better at another's expenses. Neumann and Morgestem [4], in their work on appropriate solution to game theory, treated many expositions and various applications of mathematical theory of games. The profound work contains the method for finding optimal solution for two-person zero sum game.

The theory of games is one of the important aspects of operations research. In the last few years a number of books and articles have appeared on the subject of game theory, but very little has been accomplished by ways of applying the theory. One can easily find materials in the literature on how to transform a game problem into a linear programming problem but the reverse has not been given a commensurate attention. Hence this paper presents how a linear programming problem can be transformed into a game problem with emphasis on the transformation of primal and symmetric dual linear programming problem in to a super linear programming problem resulting in optimal solution having probability values of basic variables which are also the optimal mixed strategies of a corresponding game problem, [5].

The economist is interested in the problem of game theory and the problem of linear programming each for its own sake. From his point of view, these are two special subjects, but the mathematician viewpoint about game theory and linear programming is that the two subjects are closely related. Since any computing methods devised for one of these theories can be used to solve problems arising in the other. It turns out that

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i. Every two-person zero-sum game problem can be computed by converting it into a related linear programming problem.
ii. Every linear programming problem can be artificially converted into a two-person zero-sum game, [3].

Formulation Of A Game Problem As A Linear Programming Problem.

Consider a constant-sum, two-person game (A,B,M) where A, the maximizing player, has m strategies, $\{\alpha_1,...\alpha_m\}$, B, the minimizing player, has n strategies $\{\beta_1,...\beta_n\}$ and where $aij = M(\alpha_i, \beta_j,)$ is positive for all i and j.

Without loss of generality, it can be assumed that every element of this matrix is greater than zero. If this is not true of the game as originally formulated, a sufficiently large constant can be added to the elements of the matrix to make them all positive. The addition of such a constant will leave the optimal strategies unchanged.
Player 1 can guarantee himself at least $V^*(V^* > 0)$, if there exists an

$$X = (x_1,...,x_m), \text{ where } x_i \geq 0 \text{ and } \sum_{i=1}^{m} x_i = 1 \quad (1)$$

Such that

$$M(X, \beta_j \geq V^*) \text{ for J=1,2,……..,n} \quad (2)$$

Which is equivalent to

$$\sum_{i=1}^{m} a_{ij} x_i \geq V^* \text{ for J=1,2,………..,n} \quad (3)$$

Equation 3 is equivalent to equation 2. Since multiplying by V^* and writing $u_i V^* = x_i$ yields equation 2. Consequently, we can view the problem confronting player 1 as follows.

Player 1's problem

Let U be the set of all m-tuples u= $(u_1, u_2, ..., u_m)$
Such that

$$U_i \geq 0, \text{ for i=1,2,……….m}$$

And

$$\sum_{i=1}^{m} a_{ij} U_i \geq 1 \text{ for J=1,2, ……..n}$$

To find those u belonging to U such that $\sum_{i=1}^{m} U_i$ is minimum

Remarks

(1a) if u= $(U_1.....U_m)$ belongs to U, we have seen that player 1 can guarantee himself at least $\frac{1}{\sum_1 U_i}$

In order to secure the maximum guarantee, player 1 should attempt to find a u in U which maximize $\frac{1}{\sum_1 U_i}$ or equivalently, minimizes $\sum_1 u_i$

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(1b) The problem of minimizing a linear form such as $\sum_1 u_i$ (or, more generally, $\sum_1 c_i u_i$) subject to restriction involving linear inequalities such as $\sum a_{ij} u_i \geq 1$, for j=1,2,……,n(or more generally, $\sum_i a_{ij} u_i \geq b_j$, for j=1,2,….,n) where $u_i \geq 0$, i=1,2,…,m is called a linear programming problem (of the minimizing form).

We next investigate the game problem from player 2 point of view which is reduced to a maximization problem involving linear inequalities.

Player 2 can guarantee that player 1 gets at most $V^*(V^* > 0)$, by using $y = (y_1, y_2, ..., y_n)$ where $\sum y_j = 1$ and $y_j \geq 0$ for J=1,2, ……..,n

Provides that

$M(\alpha_i, y) \leq V^*$, for i=1,2, …….,m

or equivalently, provided that

$$\sum_i a_{ij} y_i \leq V*, \text{ for i=1,2, …..,m}$$

Equivalently, it is easily seen that player 1 can get at most V^* if there exist a $w = (w_1, w_2, ..., w_n)$ where $w_j \geq 0$ for j=1,2, …,n.

Consequently, we can view the problem confronting player 2 as follows

Players 2'S problem

Let W be the set of all n-tuples $w = (w_1, w_2, ..., w_n)$

Such that

$$w_j \geq 0 \text{ for j=1,2, …,n.}$$

And

$$\sum_{j=1}^{n} a_{ij} w_j \leq 1, \text{ for i=1,2,…….,m}$$

To find those w belonging to W such that

$$\sum_{j=1}^{n} w_j \text{ is a maximum.}$$

Remark (2a): If $w = (w_1, w_2, ..., w_n)$ belongs to W, we have see that player 2 can hold player 1 down to at most $\frac{1}{\sum_j w_j}$. In order to hold player 1 down as much as possible, player 2 should attempt to find a w in W which minimizes $\frac{1}{\sum_j w_j}$ or equivalently maximize $\sum_j w_j$

(2b) The problem of maximizing a linear form such as $\sum_j w_j$ (or more generally, $\sum_j b_j w_j$) subject to restriction involving linear inequalities as $\sum a_{ij} w_j \leq 1$, for i=1,2,….,m (or, more generally, $\sum a_{ij} w_j \leq c_i$ for i=1,2,……,m), where $w_j \geq 0$ for j=1,2,……n is called a linear programming problem (of the maximizing form).

Remark

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3. Player 1's problem and player 2's problem are said to be dual-programming problem.

Any u in U guarantee player 1 at least $\frac{1}{\sum u_i}$. Any w in W guarantee player 1 at most $\frac{1}{\sum w_j}$. Since player 1 get at least $\frac{1}{\sum u_i}$ and at most $\frac{1}{\sum wj}$, we must have $\frac{1}{\sum_i u_i} \le \frac{1}{\sum_j w_j}$

$$\sum_j w_j \le \sum_i u_i$$

But we know that all zero sum games have a value which can be interpreted as follows; player 1 can get at least V (i.e. there is a $u^{(0)}$ in U such that $\frac{1}{\sum_i u_i^{(0)}} = V$)

And player 2 can hold player 1 down to at most V (i.e. there is a $w^{(0)}$ in W such that $\frac{1}{\sum_j w_j^{(0)}} = V$)

Summarizing we have the symmetric problem.

SYMMETRIC PROBLEM

To find $u^{(0)}$ in U and $w^{(0)}$ in W such that

$$\sum_{J=1}^{n} w_j^{(0)} = \sum_{i=1}^{m} u_i^{(0)}$$

Remarks (4). If $u^{(0)}$, $w^{(0)}$ solve the symmetric problem, then

$$V = \frac{1}{\sum w_j^{(0)}} = \frac{1}{\sum u_i^{(0)}}$$

And $x^{(0)} = \left(x^{(0)} = x^{(0)}, \ldots\ldots, x_m^{(0)}\right)$, where $x_i^{(0)} = V.u_l^{(0)}$ for i=1,2,…… is maximin for player 1; and $y = \left(y_1^{(0)}, \ldots.., y_n^{(0)}\right)$, where $y_j^{(0)} = V.w_j$ for J=1,2,…..,m is minimax for player 2. Conversely, if $\left(x^{(0)}, Y^{(0)}, V\right)$ constitutes a solution of the game, defining

$u_i^{(0)} = \frac{x_i^{(0)}}{V}, w_j^{(0)} = \frac{y_j^{(0)}}{V}$ yield a solution of the symmetric problem. Furthermore, $U^{(0)}$ is a solution of player 1's problem and $w^{(0)}$ is a solution of player 2's problem.

Transformation of Primal and Symmetric Dual Linear Programming Problems into a Super Linear Programming Problem

There is more than one way of performing this conversion. However, the most convenient method seems to be that of converting our programming problem into a skew-symmetric or symmetric game. Though the symmetric game appears to have more rows and columns. It has the advantages of automatically yielding problem as well and it contains so many zeros that its larger size is not much of a problem.

Consider the standard linear programming problem

$$\text{Max } z=b^Ty$$

Subject to

$$AY \le C, \quad Y \ge 0$$

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The dual is given as

$$\text{Min } Z^* = c^T X$$

Subject to

$$A^T X \geq b, \quad X \geq 0$$

i.e. the primal is given as

$$\max \; Z = b_1 y_1 + b_2 y_2 + \dots\dots + b_m y_m$$

subject to

$$a_{11} y_1 + a_{12} y_2 + \dots\dots\dots\dots + a_{1n} y_m \leq C_1$$
$$a_{21} y_1 + a_{22} y_2 + \dots\dots\dots\dots + a_{2n} y_m \leq C_2$$
$$\text{“} \qquad \text{“} \qquad + \quad \dots \quad + \qquad \text{“}$$
$$a_{m1} y_1 + a_{m2} y_2 + \dots\dots\dots + a_{mn} y_m \leq C_n$$

Taking the dual

$$\text{Min } Z^* = c_1 x_1 + c_2 x_2 + \dots\dots\dots + c_n x_n$$

Subject to

$$a_{11} x_1 + a_{21} x_1 + \dots\dots\dots\dots + a_{m1} x_1 \geq b_1$$
$$a_{12} x_2 + a_{22} x_2 + \dots\dots\dots\dots + a_{m2} x_2 \geq b_2$$
$$\text{“} \qquad \text{“} \qquad + \quad \dots \quad + \qquad \text{“}$$
$$a_{1n} x_1 + a_{2n} x_2 + \dots\dots\dots + a_{mn} x_n \geq b_m$$

Where

$$x_1, x_2, \dots\dots\dots\dots, x_n \geq 0$$
$$y_1, y_2, \dots\dots\dots\dots, y_m \geq 0$$

Converting the dual to a maximization case, we have

$$-a_{11} x_1 - a_{21} x_1 - \dots\dots\dots\dots - a_{m1} x_1 \leq -b_1$$
$$-a_{12} x_2 - a_{22} x_2 - \dots\dots\dots\dots - a_{m2} x_2 \leq -b_2$$
$$\text{“} \qquad \text{“} \qquad - \quad \dots \quad - \qquad \text{“}$$
$$-a_{1n} x_n - a_{2n} x_n - \dots\dots\dots - a_{mn} x_n \leq -b_n$$

Combining these two m variable and n variable problem into a third super (n+m)X(n+m) variable problem. We select x's and y's that maximize $z - z^*$ subject to all the inequalities of both problems. We seen that the basic duality principle guarantee that for optimal x's and y's. The z of our original problem and the z^* of its dual must exactly be equal. The third problem can be decomposed into its two independent parts, since one set of constraints involves only x's and the other involves only y's.

The maximum-value answer to our new super problem is zero. Our real point of formulating the problem, however, is to introduce a special kind of symmetry into the problem. The skewed symmetry results from adjoining the dual problem in the original problem.

The super-problem becomes

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_{11} & -a_{21} & \cdots & -a_{m1} & b_1 \\ 0 & 0 & \cdots & 0 & -a_{12} & -a_{22} & \cdots & -a_{m2} & b_2 \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & . & \\ 0 & 0 & \cdots & 0 & -a_{1n} & -a_{2n} & \cdots & -a_{mn} & b_m \\ a_{11} & a_{12} & \cdots & a_{1n} & 0 & 0 & \cdots & 0 & -c_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 0 & \cdots & 0 & -c_2 \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots & . & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 0 & -c_n \\ -b_1 & -b_2 & \cdots & -b_m & c_1 & c_2 & \cdots & c_m & 0 \end{bmatrix}$$

$$y_1 \geq 0 \; \cdots \; y_m \geq 0, \; x_1 \geq 0 \; \cdots \; x_n \geq 0$$

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Maximize

$$b_1 y_1 ......+ b_m y_m - c_1 x_1 - ....- c_n x_n = z - z^{\times}$$

Replacing the maximum statement of the last line by the equivalent requirement.

$$b_1 y_n + ...+ b_m y_m - c_1 x_1 ....-, c_n x_n \geq 0$$

Where the > sign has been redundantly inserted in the full knowledge that no feasible solution will ever require its presence.

Considering the linear programming problem of the maximizing or minimizing variety described earlier. We shall now exhibit a two-person zero-sum game whose solutions provide solution to the linear programming problem provided solution exist at all. The appropriate game matrix is

$$\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \\ \alpha_{n+1} \\ \alpha_{n+2} \\ \vdots \\ \alpha_{n+m} \\ \alpha_{n+m+1} \end{array} \begin{array}{c} \begin{array}{cccccccccc} \beta_1 & \beta_2 & \cdots & \beta_n & \beta_{n+1} & \beta_{n+2} & \cdots & \beta_{n+m} & \beta_{n+m+1} \end{array} \\ \left[\begin{array}{ccccccccc} 0 & 0 & \cdots & 0 & -a_{11} & -a_{21} & \cdots & -a_{m1} & b_1 \\ 0 & 0 & \cdots & 0 & -a_{12} & -a_{22} & \cdots & -a_{m2} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_m & -a_{2n} & \cdots & -a_{mn} & b_m \\ a_{11} & a_{12} & \cdots & a_{1n} & 0 & 0 & \cdots & 0 & -c_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 0 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 0 & -c_n \\ -b_1 & -b_2 & \cdots & -b_m & c_1 & c_2 & \cdots & c_n & 0 \end{array}\right] \end{array}$$

Because the game matrix is skewed symmetric, one conjectures that the value of the game must be zero. This is easily shown if both players uses identical strategies (i.e., put the same probability weight on their j^{th} pure strategies for j=1,2,….……,n+m+1), the payoff to each is zero. Thus, there is no strategy which will guarantee player 1 a positive return.

Since the value is zero, the mixed strategy

$\left[z_1^{(0)} \beta_1, ..., z_j^{(0)} \beta_j, ... z_n^{(0)} \beta_n, ... z_{n+1}^{(0)} \beta_{n+1}, ..., z_{n+1} \beta_{n+1}, ..., z_{n+1}^{(0)} \beta_{n+m}, z_{n+m+1} \beta_{n+m+1} \right]$ is minimax for player 2 if and only if it hold player 1 down to 0, that is if and only if $z_k^{(0)} \geq 0,$ for k=1,2,………. n+m+1, $\sum_{k=1}^{n+m+1} z_k^{(0)} = 1$

And

i. $\left[Z_{n+1} a_{ij} + ...+ z_{n+1}^{(0)} a_{ij} + ...+ Z_{n+m} a_{mj} \right] + Z_{n+m+1} b_j \leq 0,$ for j=1,………,n

ii. $\left[z_i^{(0)} a_{11} + ... z_j^{(0)} a_{ij} + ...+ z_n^{(0)} a_m \right] - z_{n+m+1} c_i \leq 0$ i =1…….,m

iii. $-\left[z_1^{(0)} b_1 + ... z_n^{(0)} b_n \right] + \left[z_{n+1}^{(0)} c_1 + ...+ z_{n+m} c_m \right] \leq 0$

Conclusion

Game theory analysis serves as a framework for all quantitative methods to analyze decision in conflicts of interest. Game theory and linear programming are closely related, and every linear programming problem can be artificially converted into a game problem which as a result leads to a super linear programming problem. The solution to such super linear programming problem resulting in optimal solution having probabilities values of basic variables which are also the optimal mixed strategies of a corresponding game problem can then be solved using manual computation or computer implementation of its solution procedures.

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