Journal of the Nigerian Association of Mathematical Physics
***Volume* 17 (November, 2010), pp 367 -370**

Numerical Solution for M/G/1 Queueing System Using ETAQA

S.A. Ojobor

Department of mathematics and computer science,
Delta State University, Abraka

Abstract

In this paper we present the ETAQA (Efficient Technique for the Analysis of Quasi-birth and death processes by Aggregation) approach for the exact analysis of M/G/1 queueing processes. In contrast to other solution techniques for the M/G/1, the ETAQA exploit the repetitive structure of the infinite portion of the chain to derive a finite system of equations. The solutions presented here are exact and less expensive when compared to other methods.

Keywords: Markov chain, QBD, ETAQA and M/G/1 processes.

1.0 Introduction

M/G/1 processes mean a queueing system with exponential arrival with a general single server. Many methods are available for solving the M/G/1 processes. As a result of search for algorithm method suitable for computer implementation

[3] developed a matrix analytic method to solve the M/G/I and GI/M/1 queuing systems. Neuts argued that probabilistic method should be used to analyze stochastic model instead of theoretical approach by some researchers.

As a consequence of Neuts work various analytic methods have been developed by researchers for the solution of the M/G/I and GI/M/I queuing system. Some contributors in these area are [2], [4], [5] and many others. The matrix analytic algorithm provides a recursive function for the computation of the probability vector $\boldsymbol{\pi}^{(i)}$.

The recursive function is based on the computation of G (for the M/G/1 queuing system) or R (for the GI/M/1 QUEUING SYSTEM). For details work on stochastic complementation and application see [6].

ETAQA means efficient technique for the analysis of QBD processes by Aggregation. According to [8], Ciardo and Smirni introduced the ETAQA in 1999 for solution of limited class of QBD processes. The limited class allowed the return from level $J^{(i+1)}$ to $J^{(i)}$, $i \geq 1$, as a single state only i.e. it returns from higher level in markov's chain to lower level to be directed toward single state only.

In this paper, we shall apply the ETAQA technique to find the solution of M/G/I queueing system. We shall relax the above assumption of returns to a single state only and provide a solution approach that will work for any type of returns to a lower level. The ETAQA differs from the matrix analytic methods in the following ways:

- **i.** It constructs and solves finite linear system of m + 2n unknown (m numbers of states in boundary portion of the process and n numbers of states in each repetitive levels of the state space) to obtained an exact solution.
- **ii.** Instead of evaluating the stationary distribution of all states in each repetitive levels of the state space, we calculate the aggregate of the stationary probability distribution of the classes of the states appropriately defined.

Corresponding author: E-mail; ojoborsun@yahoo.com, Tel. +2348034029170

We shall show that the ETAQA produced results that are significantly more efficient than the traditional methods for the M/G/I queuing system.

2.0 BACKGROUND

The M/G/1 system as classified by Neuts (1984) have the infinitesimal generator block partition matrix as

$$Q_{M/G/1} = \begin{bmatrix} \hat{L} & \hat{F}^{(1)} & \hat{F}^{(2)} & \hat{F}^{(3)} & & \\ \hat{B} & L & F^{(1)} & F^{(2)} & \cdots & \\ 0 & B & L & F^{(1)} & & \\ & & \vdots & & \ddots & \vdots \\ & & & & \cdots & \end{bmatrix} \qquad (1)$$

Where L, F and B mean local, forward and backward transition rates respectively

To solve the system above we must compute G as the solution of the matrix equation

$$B + LG + \sum_{i=1}^{\infty} F^{(i)}\, G^{(i+1)} = 0 \qquad (2)$$

G is said to be stochastic if the process is recurrent and irreducible. The entry (K,L) in G is the conditional probability of the process entering $J^{(i-1)}$ through L given that its starts from K of $J^{(i)}$. G is obtained by solving equation (2) iteratively. There are ranges of iterative method available for obtaining G; the most efficient is the cyclic reduction algorithm in [2].To calculate the stationary probability vector we use the Ramaswani, recursive formula which is numerically stable since it entails only addition and multiplication [1].

The Ramaswani formula is given by

$$\boldsymbol{\pi}^{(i)} = -\left(\boldsymbol{\pi}^{(0)}\widehat{\boldsymbol{S}}^{(i)} + \sum_{k=1}^{i-1} \boldsymbol{\pi}^{(k)}\boldsymbol{S}^{(i-k)}\right)\boldsymbol{S}^{(0)-1} \quad \forall\, \boldsymbol{i} \geq \boldsymbol{1} \qquad (3)$$

Where $\boldsymbol{S}^{(i)} = \sum_{l=i}^{\propto} \boldsymbol{F}^{(l)}\boldsymbol{G}^{l-i}\,;\, \boldsymbol{i} \geq \boldsymbol{0}$ and $\widehat{\boldsymbol{S}}^{(i)} = \sum_{l=i}^{\propto} \widehat{\boldsymbol{F}}^{(l)}\boldsymbol{G}^{l-i}\,;\, \boldsymbol{i} \geq \boldsymbol{1}$.

Given $\boldsymbol{\pi}^{(i)}$ and the normalization condition, we can obtain $\boldsymbol{\pi}^{(0)}$ by solving the system of m linear equations

$$\boldsymbol{\pi}^{(0)}\left[\left(\widehat{\boldsymbol{L}} - \widehat{\boldsymbol{S}}^{(1)}\boldsymbol{S}^{(0)^{-1}}\widehat{\boldsymbol{B}}\right)^{\lozenge} \left| \boldsymbol{1}^{\boldsymbol{T}} - \left(\sum_{i=1}^{\infty}\widehat{\boldsymbol{S}}^{(i)}\right)\left(\sum_{i=0}^{\infty}\boldsymbol{S}^{(i)}\right)^{-1}\boldsymbol{1}^{\boldsymbol{T}}\right.\right] = [\boldsymbol{0}|\boldsymbol{1}] \qquad (4)$$

Where $\lozenge$ indicates that we replace one (any) column of the corresponding matrix since we add a column representing the normalization condition. Once $\boldsymbol{\pi}^{(0)}$ value is known we can then go on to compute $\boldsymbol{\pi}^{(i)}$ iteratively. We go on until the accumulated probability mass is close to one. Then measures of performance for the system can be computed. Because the relationship that exist within the $\boldsymbol{\pi}^{(i)}$ for $\boldsymbol{i} \geq \boldsymbol{1}$ is not straight forward, computing the measures of performance requires the generation of the whole stationary probability vector. This makes the recursion formula unstable. There are also other methods like Toeplitz matrices and fast Fourier transform for finding $\boldsymbol{\pi}$ [2]. We shall not discuss them because they have not attained the wide usage.

3.0 ETAQA Solution For M/G/I Queuing System

We present here the ETAQA approach that computes only $\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}$ and the aggregated probability vector $\boldsymbol{\pi}^{(*)} = \sum_{i=2}^{\infty} \boldsymbol{\pi}^{(i)}$. The approach is exact and very efficient with respect to both time and space complexity. From equation (1), we have the stationary probability vector $\boldsymbol{\pi}$ as $\boldsymbol{\pi} = [\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \ldots,]$ with $\boldsymbol{\pi}^{(0)} \in {}^{\boldsymbol{n}}\, \boldsymbol{for}\ \boldsymbol{i} \geq \boldsymbol{1}$. Let us rewrite the matrix equation

Corresponding author: E-mail; ojoborsun@yahoo.com, Tel. +2348034029170

$$\boldsymbol{\pi}^{(0)}\widehat{\boldsymbol{L}} + \boldsymbol{\pi}^{(0)}\widehat{\boldsymbol{B}} = \mathbf{0}$$

$$\boldsymbol{\pi}^{(0)}\widehat{\boldsymbol{F}}^{(1)} + \boldsymbol{\pi}^{(1)}\boldsymbol{L} + \boldsymbol{\pi}^{(2)}\boldsymbol{B} = \mathbf{0}$$

$$\boldsymbol{\pi}^{(0)}\widehat{\boldsymbol{F}}^{(2)} + \boldsymbol{\pi}^{(1)}\widehat{\boldsymbol{F}}^{(1)} + \boldsymbol{\pi}^{(2)}\boldsymbol{L} + \boldsymbol{\pi}^{(3)}\boldsymbol{B} = \mathbf{0} \quad (5)$$

To get the solution of M/G/I queueing system we first of all solve for G. G is computed using an efficient iterative method like cyclic reduction algorithm of Bini et al (2000) or from [8]. For detail analysis of the ETAQA technique, see [8].

4.0 Computing Performance Measures for the M/G/1 System

To compute the performance measures for the system we shall considered methods that can be expressed as expected reward rate. The reward rate is given as

$$r = \sum_{j=0}^{\infty} \sum_{i \in J^{(i)}} \rho_j^{(i)} \quad \pi_j^{(i)} \quad (6)$$

Where $\rho_j^{(i)}$ is the reward rate of state $S_j^{(i)}$.

For example, to compute the expected queue length in steady state, let $J^{(i)}$represent the system state with i customers in the queue, we have $\rho_j^{(i)} = i$. Similarly the second moment we be $\rho_j^{(i)} = i^2$ and so on. Since the ETAQA compute only $\pi^{(0)}, \pi^{(1)}$ and $\sum_{i=2}^{\infty} \pi^{(i)}$ we rewrite the rate r as

$$r = \pi^{(0)}\rho^{(0)\tau} + \pi^{(1)}\rho^{(1)\tau} + \sum_{i=2}^{\infty} \pi^{(i)}\rho^{(i)\tau} \quad (7)$$

Where $\rho^{(0)} = \left[\rho_1^{(0)}, \ldots, \rho_m^{(0)}\right]$ and $\rho^{(i)} = \left[\rho_1^{(i)}, \ldots, \rho_n^{(i)}\right] \, for \; i \geq 1.$

It is also possible to compute (6) without using the values of $\pi^{(i)}\,; i \geq 2$ if the reward rate of the state $S_j^{(i)}, i \geq 2 \; and \; j = 1,2, \ldots, n$ is a polynomial of degree k in i with arbitrary coefficient$a_j^{[0]}, a_j^{[1]} \ldots, a_j^{[k]} \;\; \forall \, i \geq 2 \; and \; j \in [1,2, \ldots, n]$. We have

$$\rho_j^{(i)} = a_j^{[0]} + a_j^{[1]} i + \cdots + a_j^{[k]} i^k \quad (8)$$

Next we compute $\sum_{i=2}^{\infty} \pi^{(i)}\rho^{(i)\tau}$ which is reduce to $r^{[0]}a^{[0]T} + r^{[1]}a^{[1]T} + \cdots + r^{[k]}a^{[k]T}$

Once $r^{[0]}$ is known, $r^{[k]}$ Can then be computed.

Multiplying (5) from the second line on factor i^k, summing, rearranging and exchanging the order of summation we have

$$r^{[k]}(B + L + \sum_{i=1}^{\infty} F^{(i)} = (b^{[k]}) \quad (9)$$

$r^{[k]}$ is then computed using (8). Where $b^{[k]}$ is an expression that can be computed from $\pi^{(0)}, \pi^{(1)}$ and the vector $r^{[0]}$ and

$r^{[k-1]}$.

5.0 Conclusion

We have been able to show that the ETAQA provide an efficient and exact result for the M/G/I system.

In future works, we shall look at the numerical stability of the ETAQA.

Corresponding author: E-mail; ojoborsun@yahoo.com, Tel. +2348034029170

REFERENCES

[1] Grassmann, W. K. (2000): *Computational Probability*. Kluwer's Academic Publisher. London.

[2] Grassman, W.K. and Stanford, D.A. (2000): Matrix Analytic Methods. *Computational Probability edited by Grassman.* Kluwer's Academic Publisher. London.

[3] Gross D., Shortly J.F., Thompson J.M and Harris G.M. (2008): Fundamentals of Queuing Theory. John Wiley & Son New Jersey.

[4] Latouche G. and Stewart G.W. (1995): Numerical methods for M/G/1 type queues. W. J. Stewart ed. Computations with Markov Chains. Kluwer's Academic Publisher. Boston, MA, 571-581.

[5] Meini, B. (1998): Solving M/G/1 type Markov Chains: Recent advances and applications. Comm. Statist. Stochastic Models Vol 14 479-496.

[6] Neuts, M.F. (1984): Matrix-analytic methods in queueing theory. European journal of operational research. Vol. 15 pp 2-12.

[7] Ojobor, S.A. (2009): Numerical Solution for Discrete Time Markov Chains. International Journal of Numerical Mathematics vol 4 No 2 pp 328-340.

[8] Riska A. and Smirni E. (2007): ETAQA solutions for infinite Markov processes with repetitive structure. Informs Journal of computing, vol 19 pp 215-228.

Corresponding author: E-mail; ojoborsun@yahoo.com, Tel. +2348034029170