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Priority Queuing Models For Emergency Cases In Hospitals

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Abstract

Two priority queuing models have been identified as appropriate for emergency cases in University of Uyo Teaching Hospital in Nigeria. These are the preemptive priority model which treats patients on a FCFS basis and the nonpreemptive priority model which treats patients in the priority class. The arrival process of patients follows Poisson process with exponential interarrival time. Also, the service time distribution has been shown to follow exponential distribution. Performance measures of the queue models have also been obtained.

Keywords: Priority Queuing Models, Preemptive Model, Nonpreemtive Model, Emergency Cases, FCFS

Introduction

Emergency cases in hospitals are cases to be treated with priority. This causes delay in waiting line of the existing First- Come First- Serve (FCFS) queue discipline in the out-patients department. Two priority models – the Preemptive priorities (PRP) in which the lowest- priority patient with stable case where treatment can be delayed without adverse medical consequences being served is ejected back into the queue whenever a higher priority-patient with serious/critical case where prompt treatment is vital for survival enters the queuing system. And the Nonpreemptive priorities (NPRP) where a patient being served can not be ejected back into the queue even if a higher- priority patient enters the queuing system. Therefore, once a server (Doctor) has begun treatment the service must be completed without interruption.

Several works on the analysis of emergency waiting time and queuing systems with priority service discipline abound in the literature; see, for example [1], [2] and [9] among others. In this work, we consider the preemptive priorities model as an M/M/S/PRP/∞/∞ with the PRP as the FCFS queuing discipline and the nonpreemptive priorities as priority on the PRP.

1 Problem Formulation

A single queue with multi-server queing model exists in the emergency unit of University of Uyo Teaching Hospital in Nigeria. All patients arrive according to a Poisson input process and interarrivals follow exponential distribution as the service time. There is no limit on the number of waiting patients since the hospital policy does not stop patients from consulting Physicians on emergency at any point in time.

Patients are treated in order of priority, where those in the same category (the stable cases) are normally taken on a FCFS basis and are preemptive. A Doctor will interrupt treatment of a patient if a new case in the higher-priority category (serious/critical case) arrives, that is the NPRP. In this work, we use the priority-discipline queuing system, where the NPRP constitute the priority class and the PRP follows the usual FCFS discipline. Because treatment is interrupted by the arrival of higher-priority case, the PRP model is used. The corresponding results for the NPRP model will also be obtained to show the effect of preempting.

Data were collected between the hours of 8.00am to 5.00pm daily for seven days with two Physicians (as servers) and a single queue operating system of patients.

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2 Methodology

Using Kendal- Lee notation, see; [6], we denote a PRP model by Mi/M/S/PRP/∞/∞ Mi is a Poisson input process with the probability distribution of the ith interrarival times for the ith patient as exponential distribution with parameter λ and rate λ_i .M denotes that the probability distribution of service time follows exponential distribution with parameter μ and mean $\frac{1}{\mu}$

Analysis Of Interarrival Time

The mean interarrival time $E(t) = \frac{\sum ft}{\sum f}$

and the exponential mean interarrival time $\lambda_i = \frac{1}{E(ti)}$ of patients were calculated as follows:

Table 3.1: Calculated values of $E(t)$ and λ_i for the seven days period

Day	$E(t_i)$ min/patient	λ_i (patient/min)
1	0	0.049
2	17	0.059
3	17.5	0.057
4	19.9	0.05
5	21.58	0.046
6	20.3	0.049
7	17.7	0.058

Table3.2: Calculated values of μ and $\hat{\mu}$ for

Day	Server1		Server2	
	μ(min/pat)	$\hat{\mu}$ (pat/min)	μ(min/pat)	$\hat{\mu}$ (pat/min)
1	23.0	0.043	27.0	0.037
2	24.7	0.04	26.2	0.038
3	23.68	0.042	27.0	0.037
4	24.7	0.038	28.0	0.036
5	24.5	0.04	29.34	0.034
6	28.0	0.035	29.46	0.033
7	28.0	0.035	27.03	0.037

And the average interarrival for the seven days is;

$$E(t) = \frac{E(t_1) + E(t_2) + \ldots + E(t_7)}{7} = 19.52 \approx 20\text{min/pat and } \lambda = 0.05\text{pat/min}$$

Also, the χ^2 goodness-of-fit test was used to test that the interarrival times follow exponential distribution.

3.2 Analysis Of Service Time

The mean service time μ and the exponential mean service time $\hat{\mu}$ for service times (ST) (1-30, 31-60 in minutes) were calculated and the results summarized on table 3.2 for servers 1 and 2.

Where $\mu = \frac{\sum ft}{\sum f}$ *and* $\hat{\mu} = 1/\mu$

And the average service time and the exponential mean service time for the seven days period for servers1 and 2 are:

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$$E(\bar{\mu}_1) = \frac{\mu_1 + \mu_2 + ... + \mu_7}{7} = 25.23 \min/\,pat \quad and \ \hat{\mu}_1 = 0.04$$

$$E(\bar{\mu}_2) = 27.52 \ and \ \hat{\mu}_2 = 0.036$$

Also, the service time was checked to have followed exponential distribution using the χ^2 goodness-of fit test.

4.0 PERFORMANCE MEASURES FOR PREEMPTIVE PRIORITY MODEL M/M/2: ∞/FCFS (where the preemptive patients are treated on FCFS)

NOTATIONS

Let P_n = probability of exactly n customers in the queuing system
L = expected number of customers in queuing system

L_q = expected queue length (excluding customers being served)
W = waiting time in the system (includes service time) for each individual customer, i.e. W = E(w)
W_q = waiting time in queue (excludes service time) for each individual customer i.e.
w_q = E(w_q)
N_t = number of servers in the queuing system at time t (t ≥)
Pn(t) = probability of exactly n customers in queuing system at time t
S = number of servers (parallel service channels) in queuing system
λn = mean arrival rate (expected number of arrival per unit time of new customers completing service per unit time n customers are in system
$\rho = \lambda/\mu$ = traffic intensity (utilization factor for the service facility)

From little's law, [6];

$$L_q = \lambda W_q \text{ or } W_q = W - \frac{1}{\mu} \tag{4.1}$$

$$W_q = E(w) = \frac{\lambda}{\mu(\mu-\lambda)} \tag{4.2}$$

When s>1
The utilization factor is:

$$U_k = \begin{cases} (\lambda/\mu)^n & for \ \ n = 1,2,...,s \\ \dfrac{(\lambda/\mu)^s}{s!} (\lambda/s\mu)^{n-s} = \dfrac{(\lambda/\mu)^n}{s!\,s^{n-s}} & for \ \ n = s, s_{n+1} \end{cases}$$

If N $\to \infty$ $and \ \lambda < s\mu, so \ that \ \rho = \lambda/s\mu < 1, then$

$$P_0 = \frac{1}{1 + \sum_{n=1}^{s-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!} \sum_{n=1}^{\infty} (\lambda/s\mu)^{n-s}}$$

$$= \frac{1}{\sum_{n=0}^{s-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!} \frac{1}{1-\lambda/s\mu}} = \left[\sum_{n=0}^{s-1} \frac{1}{n!}(\lambda/\mu)^n + \frac{1}{s!}(\lambda/\mu)^s \frac{s\mu}{s\mu-\lambda} \right]^{-1} \tag{4.3}$$

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and P_n = $$\left\{ \begin{array}{ll} \dfrac{(\lambda/\mu)^n}{n!} P_0 & if\ 0 \le n \le s \\ \dfrac{(\lambda/\mu)^n}{s! S^{n-s}} P_0 & if\ n \ge s \end{array} \right\}$$ (4.4)

Furthermore,

$$L_q = \sum_{n=s}^{\infty} (n-s) P_n$$

$$= \sum_{j=0}^{\infty} j P_{s+j} = \sum_{j=0}^{\infty} j \frac{(\lambda/\mu)^s}{s!} \rho^j P_0 \quad = P_0 \frac{\left(\lambda/\mu\right)^s}{s!} \rho \frac{d}{d\rho} \left(\sum_{j=0}^{\infty} \rho^j \right)$$

$$= P_0 \frac{\left(\lambda/\mu\right)^s}{s!} \rho \frac{d\left(\frac{1}{1-\rho}\right)}{d\rho} = \quad P_0 \frac{\left(\lambda/\mu\right)^s \rho}{s!(1-\rho)^2}$$

$$W_q = \frac{Lq}{\lambda} \tag{4.5}$$

$$W_s = W_q + 1/\lambda \quad = \frac{Lq}{\lambda} + 1/\mu \tag{4.6}$$

$$L_s = \lambda\left(W_q + 1/\mu\right) = L_q + \lambda/\mu \tag{4.7}$$

The probability distribution of waiting times for $t \ge 0$ is;

$$P\{\omega > t\} = e^{-\mu t} \left[1 + \frac{P_0 \left(\frac{\lambda}{\mu}\right)^s}{s!(1-\rho)} \left(\frac{1 - e^{-\mu t\left(s-1-\lambda/\mu\right)}}{s-1-\lambda/\mu} \right) \right]$$

and $P\{\omega_q > t\} = (1 - P\{\omega_q = 0\})e^{-s\mu(1-\rho)t}$ where $P\{\omega_q = 0\} = \sum_{n=0}^{s-1} P_n$

4.3 Nonpremptive Priority Model

Let S = number of servers

μ = mean service rate per busy server

k = the number of priority classes

$$\lambda = \sum_{i=1}^{N} \lambda_i \ and\ r = \lambda/\mu \tag{4.8}$$

This result assumes that $\sum_{i=1}^{k} \lambda_i$ < sμ; see [6]

Let W_k be the steady state expected waiting time in the system (including service time) for a member of priority class k. Then

$$W_k = \frac{1}{AB_{K-1}B_K} + \frac{1}{\mu}, \text{ k = 1, 2,…,N} \tag{4.9}$$

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where $A = S!\frac{S\mu - \lambda}{r^s}\sum_{j=0}^{s-1} r^j + s\mu$ (4.10)

$$B_0 = 1$$

$$B_k = 1 - \frac{\sum_{i=1}^{k}\lambda_i}{s\mu} \quad \text{for k = 1,2,…,N} \quad (4.11)$$

4.4 The Preemptive Priority Model

Having preemption under the conditions in NPRP changes the total expected waiting time in the system (including the total service time) to:

$$W_k = \frac{1/\mu}{B_{k-1}B_k} \quad \text{for k = 1,2,…,N and s = 1} \qquad (4.12)$$

and $L_k = \lambda_k W_k$ for k = 1,2,…,N

when s > 1, W_k can be calculated by an iterative procedure.

To determine the expected waiting time in the queue (excluding service time) for priority class k;

$W_q = W_k - 1/\mu$ (4.13)

Also, if s > 1, W_k can be calculated by an iterative procedure

5.0 Computational Results

5.1 Performance Measures Of M/M/2:∞/Fcfs

We first obtain the traffic intensity $\rho = \lambda / s\mu$ where μ = 0.034patience/hr

= 0.034x60 = 2pat/min

$\rho = 3/2x2 = 0.75 = 75\%$ of busy time

And the expected idle time for each Doctor is 1 - ρ = 0.25 = 25%

$$L_q = \left[\frac{1}{(s-1)!}\left(\lambda/\mu\right)^s \frac{\lambda\mu}{(s\mu - \lambda)^2}\right]P_0$$

where P_0 in (4.3) = $\left[\frac{1}{0!}\left(\frac{3}{2}\right)^0 + \frac{1}{1!}\left(\frac{3}{2}\right)^1 + \frac{1}{2!}\left(\frac{3}{2}\right)^2 \frac{4}{4-3}\right]^{-1} = 0.1429$

This implies 14.2% of idleness in the system.

$$\therefore L_q = \left[\frac{1}{1!}\left(\frac{3}{2}\right)^2 \frac{2x3}{(4-3)^2}\right]0.1429 = 1.91 = 2 \textit{ patients} .$$

This means that 2 patients are expected to wait in the queue at every point in time.

$L_s = L_q + \lambda/\mu = 1.9 + 1.5 = 3.4$

This means that about 4 patients are expected in the system at every point in time.

$W_q = \frac{L_q}{\lambda} = \frac{1.9}{3} = 0.63hr = 38 \min s.$

$W_s = W_q + 1/\mu = 0.63 + 0.5 = 1.13hrs = 67.8 \min s.$

The probability that an arriving patient has to wait (busy period) is:

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$$P_w(n \geq 2) = \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{s\mu}{s\mu - \lambda} P_0 \quad = \frac{1}{2!}\left(\frac{3}{2}\right)^2 \frac{4}{4-3} 0.1429 \quad = 0.6431$$

5.2 Expected waiting time for priority classes

Table 5.1(a) shows the arrival time and the respective frequencies for priority classes for day 1-7. Hence;

$\lambda_{NPRP} = 0.66\, per\ hr \qquad \lambda_{PRP} = \ 2.1 per\ hr$

(a) For the Nonpreemptive class, where k = 1; (4.9) becomes

$$W_k = \frac{1}{AB_0 B_1} + \frac{1}{\mu} \quad where \quad B_0 = 1 \quad and \quad B_1 = 1 - \frac{0.66}{4} = 0.835$$

From (4.8); $\lambda = 2.76 \ and \ r = \frac{2.76}{2} = 1.38$

And A in (4.10) = $2!\frac{(4-3)}{(1.38)^2}\left[\frac{(1.38)^0}{0!} + \frac{(1.38)^1}{1!}\right] + 4 = 6.5$

Then $W_k = \frac{1}{6.5 \times 0.835 \times 1} + \frac{1}{2} = 0.68 hrs \equiv 41 \min/ \ pat$. Similar computations for Day 2- 7 yield the results in Table 5.1b for NPRP.

(b) For the Preemptive class, we have k = 2 and (4.12) becomes;

$$W_k = \frac{1/\mu}{B_1 B_2} = \frac{1/2}{0.835 \times 0.31} = 1.932 \ hrs \ \equiv 115.9 \approx 116 \ \min \ s/pat$$

Similar computations for Day 2- 7 yield the following results in Table 5.1b

Arrival Time (min)	nonpreemtive Patient N_{pp} (f_1)	Preeemtive patient (f_2)
8-9	2	5
9-10	1	3
10-11	0	2
11-12	2	3
12-1	0	0
1-2	1	2
2-3	0	2
3-4	0	1
4-5	0	1

Table 5.1(a)

Waiting Time		
Day	PRP	NPRP
1	116mins/pat	41mins/pat
2	131mins/pat	41mins/pat
3	101mins/pat	39mins/pat
4	107mins/pat	39mins/pat
5	88mins/pat	38mins/pat
6	138mins/pat	41mins/pat
7	101mins/pat	39mins/pat

Table 5.1(b)

From Table 5.1(b), the average expected waiting time for NPRP = 40 minutes/patient while the average expected waiting time for PRP = 112mins/patient. This difference is caused by the interruption (bumps) during service (treatment) for PRP by NPRP patients.

5.3 Conlusion

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The results show that the distribution of interarrival time and service time for patients in the emergency unit of the hospital under consideration followed exponential distribution having the queue model $M_i/M//2: \infty/PRP$. Performance measures of the identified $M/M/2: \infty/PRP$ queuing model were obtained with $\rho = 0.75$ showing 75% busy time of the service facility (Doctors) with $L_q = 2$ patients, $L_s = 4$ patients, $W_q = 38$mins and Ws = 67.8 mins and the probability that an arriving patient has to wait is 0.6431.
The higher priority cases (NPRP) have no effect on the interarrival and service times but causes about 180% increase in the waiting time of the lower priority patients (PRP). Hence, additional services facility (Doctor) should be employed to service the NPRP patients to avoid undue delay and its attendant cost on the PRP patients.

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