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Mean-Variance Portfolio Selection with a Fixed Flow of Investment in a Continuous-Time Framework

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Abstract

We consider a mean-variance portfolio selection problem for a fixed flow of investment in a continuous time framework. We consider a market structure that is characterized by a cash account, an indexed bond and a stock. We obtain the expected optimal terminal wealth for the investor. We also obtain a closed-form expressions for the value functions, the optimal investment and mean-variance strategies using the stochastic maximum principle. We assume that the indexed bond and stock are governed by geometric Brownian motions with constant drifts and volatilities. The aim of the investor is to maximize the expected terminal wealth while minimizing the variance. We find that the higher the value of the control parameter used in minimizing the variance, the lower the variance. Furthermore, we find that if the parameter used to minimize the variance tends to zero, both the expected wealth and the variance tend to infinity. This means that if a risky asset becomes more risky, the wealth of the investor is expected to be "unmeasureable" otherwise the portfolio should remain only in the cash account.

Keywords: mean-variance, portfolio selection, investment, stochastic maximum principle.

1.0 Introduction

A mean-variance optimization is a quantitative tool used by asset managers, consultants and investment advisors to construct portfolios for the investors. When the market is less volatile, mean-variance model seems to be a better and more reasonable way of determining portfolio selection problem. One of the aims of mean-variance optimization is to find portfolio that optimally diversify risk without reducing expected return and to facilitate portfolio construction strategy. The method is based on the pioneering work of [5], the Nobel price-winning economist, widely recognized as the father of modern portfolio theory. The optimal investment allocation can be found by solving a mean-variance optimization problem. In solving a stochastic optimal control problem, one uses the "smoothing" property of the expectation operator, property that is failed to statisfied by the variance operator. Thus, a mult-period or continuous-time optimization problem with an objective function with variance is not immediate to solve.

Zhou and Li [8] shown in continuous-time how to transform the difficult problem into a tractable one. They embed the original problem into a stochastic linear-quadratic control problem, that can then be solved through standard methods. Bielecky *et al* [1], solved a mean-variance portfolio problem in the continuous-time with a constant against ruin. Delong *et al* [3] considered a mean-variance optimization problem in the accumulation phase of a defined benefit pension plan. Zhou and Li [8] studied a mean-variance portfolio selection problem in a defined contribution pension. They found the optimal policy and the efficient frontier of feasible portfolios in closed form. Dai *et al* [2] considered a continuous time Markowitz's mean-variance portfolio selection problems with proportional transaction costs. They obtained neccessry and sufficient conditions for existence of an optimal solution and optimal strategy. Xie *et al* [6] formulated a continuous-time mean-variance portfolio selection model with multiple risky assets and one liability in an incomplete market. They derived explicitly the optimal dynamic strategy and the mean-variance efficient frontier in closed forms by using the general stochastic linear-quadratic (LQ) control technique. Hϕjgaard and Vigna [4] considered a mean-variance portfolio selection problem in the accumulation phase of a defined contribution pension schemes. They compared the mean-variance approach with investment strategies adopted in a

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defined contribution pension schemes that is the target-based approach and lifestyling strategy. They shown that the corresponding mean and variance of the final fund belong to the efficient frontier and that each point on the efficient frontier corresponds to a target-based optimization problem. In this paper, we consider a mean-variance portfolio optimization problem for a fixed flow of investment in a continuous time.

The remainder of the paper is as follows. In section 2, we present the model of the wealth process and definitions. In section 3, we solve the mean-variance optimization problem of the wealth of the investor. We also determine the optimal portfolio and expected optimal wealth of the investor. Also, in section 3, we discuss the efficient frontier of our portfolio. In section 4, concludes the paper.

2.0 The Dynamics of the Model

We consider a financial market that consists of a cash account, an indexed bond and a stock with constant force of interest $r > 0$, the indexed bond and stock, whose prices follows a standard geometric Brownian motion with drift $\mu > r$ and $\lambda > r$ respectively. The constant rate of flow of investment asset $c > 0$ is paid continuously over time by prospective investor into the investment firm. The shares of portfolio invested in the indexed bond at time t is denoted by $\pi^I(t)$ and the shares of portfolio invested in the stock market at time t, is denoted by $\pi^S(t)$. The wealth, $F(t)$ at time t, grows according to the following stochastic differential equation (SDE):

$$\begin{cases} dF(t) = \left(F(t)\left[\pi^S(t)(\mu - r) + \pi^I(t)(\lambda + \rho - r) + r\right] + c\right)dt \\ + F(t)\left(\pi^S(t)\sigma^S dW^S(t) + \pi^I(t)\sigma_I dW^I(t)\right) \\ F(0) = F_0 > 0, \end{cases} \tag{2.1}$$

where $\rho > 0$, is the expected rate of inflation and F_0 is the intial wealth of the investor. $W^S(t)$ and $W^I(t)$ are standard Brownian motion defined on a complete filtered probability space $\left(\Omega, \Im, \left\{\Im_t^S\right\}, \left\{\Im_t^I\right\}, P\right)$ with $\Im_t^S = \sigma^S\left\{W^S(u) : u \le t\right\}$ and $\Im_t^I = \sigma^I\left\{W^I(u) : u \le t\right\}$ such that $\Im_t^S \cup \Im_t^I \in \Im$ and $\Im_t^S \cap \Im_t^I = \phi.$

We assume that the investor invest his resources from time 0 to time T. The aim of the investor is to maximize the expected final wealth and simultaneously minimize the variance of final wealth. Hence, the investor aim at minimizing the vector

$$\min_{\pi \in (\pi^S, \pi^I)} \left[-E(F(T,\pi)), Var(F(T,\pi))\right]$$

Definition 1: A portfolio strategy $\pi = \left(\pi^S, \pi^I\right)$ is said to be admissible if $\pi(.) \in L^2_{\Im}(0,T;\Re)$ such that $\pi^S(.) \in L^2_{\Im^S}(0,T;\Re)$ and $\pi^I(.) \in L^2_{\Im^I}(0,T;\Re)$.

Definition 2: The mean-variance optimization problem is defined as

$$\min_{\pi \in (\pi^S, \pi^I)} \left[-E(F(T,\pi)), Var(F(T,\pi))\right] \tag{2.2}$$

Subject to: $\begin{cases} \pi(.) \ \ admissible \\ F(.), \pi(.) \ satisfy \ (1). \end{cases}$

Solving Eq.(2.2) is equivalent to solving the following equation

$$\min_{\pi(.)} \left[-E\left(F(T,\pi(.))\right) + \psi Var\left(F(T,\pi(.))\right)\right], \tag{2.3}$$

where $\psi > 0.$

By definition, in elementary statistics,

$$Var\left(F(T,\pi(.))\right) = E\left[F(T,\pi(.))^2\right] - \left(E\left[F(T,\pi(.))\right]\right)^2 \tag{2.4}$$

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Substituting Eq.(4) into Eq.(2.3), we obtain

$$\min_{\pi(.)} E\left[F(T,\pi(.))^2 - \beta F(T,\pi(.))\right] \tag{2.5}$$

where, $\beta = 1 + \psi E(F(T,\pi(.)))$.

Eq.(2.5) is known as a linear-quadratic control problem. Hence, instead of solving Eq.(2.2), we now solve the following:

$$\min(V(\pi(.)),\psi,\beta) = E\left[F(T,\pi(.))^2 - \beta F(T,\pi(.))\right] \tag{2.6}$$

Subject to: $\begin{cases} \pi(.) \;\; admissible \\ F(.),\pi(.) \;\; satisfy \; (2.1). \end{cases}$

3.0 The Optimization of our Problem

In solving Eq.(2.6), we set $\omega = \dfrac{\beta}{\psi}$ and $H(t) = F(t) - \omega$ (see [4], [7]). It implies that

$$\min E\left[F(t,\pi(t))^2 - \beta F(t,\pi(t))\right] = E\left[\psi F(t)H(t)\right].$$

It turns out that our problem is equivalent to solving

$$\min_{\pi(.)} E\left[\frac{\psi H(T)}{2}^2 - \frac{\beta^2}{2\psi}\right] = \min_{\pi(.)} V(\pi(.),\psi,\beta),$$

where the process $H(t)$ follows the SDE

$$\begin{cases} dH(t) = \left\{(H(t)+\omega)\left[\pi^S(t)(\mu - r) + \pi^I(t)(\lambda + \rho - r) + r\right] + c\right\}dt + \\ (H(t)+\omega)\left(\sigma^S \pi^S(t) dW^S(t) + \sigma_I \pi^I(t) dW^I(t)\right) \\ H(0) = h - \omega \end{cases} \tag{2.7}$$

Eq.(2.7) is a standard stochastic optimal control problem. Let

$$U(t,h) = \inf_{\pi(.)} E_{t,h}\left[\frac{\psi H(T)^2}{2} - \frac{\beta^2}{2\psi}\right] = \inf_{\pi(.)} V(\pi(.),\psi,\beta).$$

Then the value function U satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{cases} U_t + \left((h+\omega)\left[\pi^S(t)(\mu - r) + \pi^I(t)(\lambda + \rho - r) + r\right] + c\right)U_h + \\ \dfrac{(h+\omega)^2}{2}\left[{\sigma^S}^2 \pi^S(t)^2 + {\sigma_I}^2 \pi^I(t)^2\right]U_{hh} = 0 \\ U(T,h) = \dfrac{1}{2}\psi h^2 - \dfrac{\beta^2}{2\psi}. \end{cases} \tag{2.8}$$

Assuming U to be a convex function of h, then first order conditions lead to the optimal proportion of portfolio to be invested in stock and indexed bond at time t which is obtained by solving the Hamiltonian J:

$$J = \left((h+\omega)\left[\pi^S(t)(\mu - r) + \pi^I(t)(\lambda + \rho - r)\right]\right)U_h + \frac{(h+\omega)^2}{2}\left[{\sigma^S}^2 \pi^S(t)^2 + {\sigma_I}^2 \pi^I(t)^2\right]U_{hh}$$

Finding the partial derivative of J with respect to $\pi^S(t)$ and $\pi^I(t)$ and set to zero, we obtained respectively:

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$$\pi^{S*}(t) = -\frac{(\mu - r)U_h}{(h+\omega)\sigma^{S^2} U_{hh}} \tag{2.9}$$

$$\pi^{I*}(t) = -\frac{(\lambda + \rho - r)U_h}{(h+\omega)\sigma_I^2 U_{hh}} \tag{2.10}$$

These are the optimal control of the problem. Substituting Eq.(2.9) and Eq.(2.10) into Eq.(2.8), we obtain the following non-linear partial differential equation for the value function

$$U_t + ((h+\omega)r + c)U_h - \frac{1}{2}\frac{(\lambda+\rho-r)^2 U_h{}^2}{\sigma_I^2 U_{hh}} - \frac{1}{2}\frac{(\mu-r)^2 U_h{}^2}{\sigma^{S^2} U_{hh}} = 0$$

$$U_t + ((h+\omega)r + c)U_h - \frac{\theta}{2}\frac{U_h{}^2}{U_{hh}} = 0 \tag{2.11}$$

where, $\theta = \frac{(\lambda+\rho-r)^2}{\sigma_I^2} + \frac{(\mu-r)^2}{\sigma^{S^2}}$

Let us adopt a quadratic utility of the form

$$U(t,h) = P(t)h^2 + Q(t)h + R(t) \tag{2.12}$$

Now, from Eq.(12), we obtain

$$U_t = P'(t)h^2 + Q'(t)h + R'(t) \tag{2.13}$$

$$U_h = 2P(t)h + Q(t) \tag{2.14}$$

$$U_{hh} = 2P(t) \tag{2.15}$$

Substituting Eq.(2.13)-Eq.(2.15) into Eq.(2.11), we obtain

$$\begin{aligned} &P'(t)h^2 + Q'(t)h + R'(t) + [(h+\omega)r + c](2P(t)h + Q(t)) - \\ &\frac{\theta(P(t)^2 h^2 + P(t)Q(t)h)}{P(t)} - \frac{\theta Q(t)^2}{4P(t)} = 0 \end{aligned} \tag{2.16}$$

From Eq.(2.16), we have

$$P'(t)h^2 - \theta P(t)h^2 + 2P(t)rh^2 = 0$$

$$Q'(t)h + hrQ(t) + 2P(t)rh\omega + 2P(t)ch - \theta Q(t)h = 0$$

$$R'(t) + \omega r Q(t) + cQ(t) - \frac{\theta Q(t)^2}{4P(t)} = 0$$

We now have the following ordinary differential equations:

$$\begin{cases} P'(t) = (\theta - 2r)P(t) \\ Q'(t) = (\theta - r)Q(t) - 2(\omega r + c)P(t) \\ R'(t) = \dfrac{\theta Q(t)^2}{4P(t)} - (c + \omega r)Q(t) \end{cases} \tag{2.17}$$

with boundary conditions

$$P(T) = \frac{\psi}{2},\ Q(T) = 0,\ R(T) = -\frac{\beta^2}{2\psi}. \tag{2.18}$$

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Solving Eq.(17) and Eq.(18), we obtain

$$\begin{cases} P(t)=\frac{1}{2}\psi\exp\left[-(\theta-2r)(T-t)\right] \\ Q(t)=\frac{\psi(c+\omega r)}{r}\exp\left[-(\theta-2r)(T-t)\right]\left(1-\exp\left[-r(T-t)\right]\right) \\ R(t)=-\int_t^T\frac{\psi(c+\omega r)}{r}\exp\left[-(\theta-2r)(T-u)\right]\left(1-\exp\left[-r(T-u)\right]\right)\left\{\frac{\theta}{2r}\left(1-\exp\left[-r(T-u)\right]\right)-1\right\}du \end{cases} \tag{2.19}$$

We observe that our assumption of convexity of U holds as

$$U_{hh}=2P(t)>0,\ \psi>0.$$

Substituting the Eq.(2.13)-Eq.(2.15) into Eq.(2.9) and Eq.(2.10) and replacing $h+\omega$ with F, we obtain the following

$$\pi^{S*}(t)=-\frac{(\mu-r)}{F\sigma^{S\,2}}\left[F-\omega\exp[-r(T-t)+\frac{c}{r}(1-\exp[-r(T-t)])\right] \tag{2.20}$$

$$\pi^{I*}(t)=-\frac{(\lambda+\rho-r)}{F\sigma_I^2}\left[F-\omega\exp[-r(T-t)+\frac{c}{r}(1-\exp[-r(T-t)])\right] \tag{2.21}$$

Hence, the wealth dynamics under the optimal controls Eq.(2.20) and Eq.(2.21) is given as

$$\begin{aligned} dF^*(t)=&\left[(r-\gamma)F^*(t)+\gamma\omega\exp[-r(T-t)]+\frac{c\gamma}{r}\exp[-r(T-t)]+c\left(1-\frac{\gamma}{r}\right)\right]dt+ \\ &\left[\left(\gamma\omega+\frac{\gamma c}{r}\right)\exp[-r(T-t)]-\frac{\gamma c}{r}-\gamma F^*(t)\right]\sigma dW(t) \end{aligned} \tag{2.22}$$

where, $\sigma dW(t)=\sigma^S dW^S(t)+\sigma_I dW^I(t)$, $\gamma=\frac{(\lambda+\rho-r)}{\sigma_I^2}+\frac{(\mu-r)}{\sigma^{S\,2}}$

Applying *Ito* Lemma to Eq.(2.22), we obtain the SDE that governs the evolution of $F^*(t)^2$:

$$\begin{aligned} dF^*(t)^2=&\left(\begin{array}{l}\left(2(r-\theta)+\gamma^2\sigma^2\right)F^*(t)^2+ \\ 2\left(\left(\omega+\frac{c}{r}\right)(1+\gamma\sigma^2)\exp[-r(T-t)]+c\left(1-\frac{\gamma}{r}(1-\gamma\sigma^2)\right)\right)F^*(t) \\ +\gamma^2\sigma^2\left(\left(\omega+\frac{c}{r}\right)\exp[-r(T-t)]-\frac{c}{r}\right)^2\end{array}\right)dt+ \\ &\left(2F^*(t)\left(\gamma\omega+\frac{\gamma c}{r}\right)\exp[-r(T-t)]-\frac{2\gamma c}{r}F^*(t)-\gamma F^*(t)\right)\sigma dW(t) \end{aligned} \tag{2.23}$$

Taking expectation of bothsides of Eq.(2.22) and Eq.(2.23), we obtained the expected value of the optimal wealth and the expected value of its square, given as

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$$\begin{cases} dE\left(F^*(t)\right)=\left[(r-\gamma)E\left(F^*(t)\right)+\gamma\omega\exp[-r(T-t)]+\dfrac{c\gamma}{r}\exp[-r(T-t)]+c\left(1-\dfrac{\gamma}{r}\right)\right]dt \\ E\left(F^*(0)\right)=F_0 \end{cases} \quad (2.24)$$

$$\begin{cases} dE\left(F^*(t)^2\right)=\left(\begin{matrix}\left(2(r-\theta)+\gamma^2\sigma^2\right)E\left(F^*(t)^2\right)+ \\ 2\left(\left(\omega+\dfrac{c}{r}\right)\left(1+\gamma\sigma^2\right)\exp[-r(T-t)]+c\left(1-\dfrac{\gamma}{r}\left(1-\gamma\sigma^2\right)\right)\right)E\left(F^*(t)\right) \\ +\gamma^2\sigma^2\left(\left(\omega+\dfrac{c}{r}\right)\exp[-r(T-t)]-\dfrac{c}{r}\right)^2\end{matrix}\right)dt \\ E\left(F^*(0)\right)^2=F_0^2. \end{cases} \quad (2.25)$$

By solving Eq.(2.24) and Eq.(2.25), we find that the expected value of the wealth under optimal control at time t is

$$E\left(F^*(t)\right)=\left(F_0+\frac{c}{r}\right)\exp[-(\gamma-r)t]+\left(1-\exp[-\gamma t]\right)\left(\omega+\frac{c}{r}\right)\exp[-r(T-t)]-\frac{c}{r} \quad (2.26)$$

and the expected value of the square of the wealth under optimal control at time t is:

$$\begin{aligned} E\left(F(t)^2\right)&=\Gamma_1\exp\left(r(2t-T)+\gamma\left(\gamma\sigma^2-2\right)t\right)-\Gamma_2\exp\left((2r-\gamma)t-rT\right) \\ &\Gamma_3\exp\left(-2r(T-t)-\gamma t\right)-\Gamma_4\exp\left(-2r(T-t)\right)+\Gamma_5\exp\left(-2r(T-t)+\gamma\left(\gamma\sigma^2-2\right)\right)+ \\ &\Gamma_6\exp\left(3(1-r)T+\left(2+\gamma\left(\gamma\sigma^2-2\right)\right)t\right)+\Gamma_7\exp\left(2r+\left(2+\gamma\left(\gamma\sigma^2-2\right)\right)t\right)+\Gamma_8\exp\left(-r(T-t)\right)+ \\ &\Gamma_9\exp\left(-r(T-t)-\gamma t\right)-\Gamma_{10}\exp\left((r-\gamma)t\right)+\Gamma_{11} \end{aligned} \quad (27)$$

where,

$$\Gamma_1=\frac{2\left(\gamma\sigma^2-2\right)\left(\begin{matrix}F_0\omega r^2\left(1+\gamma\sigma^2\right)\left(r+\gamma\left(\gamma\sigma^2-2\right)\right)-\left(\gamma^2+\gamma-r+\gamma\left(\gamma-r-2\gamma^2\right)\sigma^2+\gamma^4\sigma^4\right) \\ +2cr\left(\begin{matrix}F_0\left(r+\gamma\left(\gamma\sigma^2-2\right)\right)+r+r\gamma\sigma^2- \\ \gamma\left(1+\gamma+(1-2\gamma)\gamma\sigma^2+\gamma^3\sigma^4\right)\end{matrix}\right)\omega\end{matrix}\right)}{r^2\gamma\left(2+\gamma\sigma^2\left(\gamma\sigma^2-3\right)\right)\left(r+\gamma\left(\gamma\sigma^2-3\right)\right)}$$

$$\Gamma_2=-\frac{2\left(\gamma\sigma^2-2\right)\left(\gamma\sigma^2-1\right)\left(r+\gamma\left(\gamma\sigma^2-2\right)\right)\left(c^2+F_0\omega r^2+cr\left(F_0+\omega\right)\right)}{r^2\gamma\left(2+\gamma\sigma^2\left(\gamma\sigma^2-3\right)\right)\left(r+\gamma\left(\gamma\sigma^2-3\right)\right)}$$

$$\Gamma_3=\frac{2\left(\gamma\sigma^2-2\right)\left(\gamma\sigma^2+1\right)\left(r+\gamma\left(\gamma\sigma^2-2\right)\right)\left(c^2+4\omega rc+\omega^2r^2\right)}{r^2\gamma\left(2+\gamma\sigma^2\left(\gamma\sigma^2-3\right)\right)\left(r+\gamma\left(\gamma\sigma^2-3\right)\right)}$$

$$\Gamma_4=-\frac{\left(r+\gamma\left(\gamma\sigma^2-2\right)\right)\left(-2+\gamma^2\sigma^2\left(-1+(2+\gamma)\sigma^2\right)\right)\left(c^2+2\omega rc+\omega^2r^2\right)}{r^2\gamma\left(2+\gamma\sigma^2\left(\gamma\sigma^2-3\right)\right)\left(r+\gamma\left(\gamma\sigma^2-3\right)\right)}$$

$$\Gamma_5=\frac{\left(r+\gamma\left(\gamma\sigma^2-2\right)\right)\left(2+\gamma\sigma^2\left(2+\gamma\left(\gamma\sigma^2-1\right)\right)\right)\left(c^2+2\omega rc+\omega^2r^2\right)}{r^2\gamma\left(2+\gamma\sigma^2\left(\gamma\sigma^2-3\right)\right)\left(r+\gamma\left(\gamma\sigma^2-3\right)\right)}$$

$$\Gamma_6=\frac{c^2\gamma\left(2\gamma\sigma^2\left(\gamma\sigma^2-3\right)\right)\left(r+\gamma\left(\gamma\sigma^2-2\right)\right)}{r^2\gamma\left(2+\gamma\sigma^2\left(\gamma\sigma^2-3\right)\right)\left(r+\gamma\left(\gamma\sigma^2-3\right)\right)}$$

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$$\Gamma_7 = \frac{F_0\left(r^2F_0 + 2cr\right)\left(r + \gamma\left(\gamma\sigma^2 - 2\right)\right)}{r^2\left(r + \gamma\left(\gamma\sigma^2 - 3\right)\right)}$$

$$\Gamma_8 = \frac{2c\gamma\left(c + r\omega\right)\left(1 - r + \gamma + \gamma\sigma^2\right)}{\gamma r^2\left(r + \gamma\left(\gamma\sigma^2 - 3\right)\right)}$$

$$\Gamma_9 = \frac{2c\left(c + r\omega\right)\left(r + \gamma\left(\gamma\sigma^2 - 2\right)\right)}{r^2\left(r + \gamma\left(\gamma\sigma^2 - 3\right)\right)}$$

$$\Gamma_{10} = -\frac{2c\left(c + rF_0\right)\left(r + \gamma\left(\gamma\sigma^2 - 2\right)\right)}{r^2\left(r + \gamma\left(\gamma\sigma^2 - 3\right)\right)}$$

$$\Gamma_{11} = \frac{c^2\left(r + \gamma\left(\gamma\sigma^2 - 2\right)\right)}{r^2\left(r + \gamma\left(\gamma\sigma^2 - 3\right)\right)}$$

And by defintion of ω, we have that

$$\omega = \frac{\left(\frac{1}{\psi} - \frac{c}{r}\right) + \left(F_0 + \frac{c}{r}\right)\exp\left(-\left(\gamma - r\right)t\right) + \frac{c}{r}\left(1 - \exp\left[-\gamma t\right]\right)\exp\left(-r\left(T - t\right)\right)}{1 - \exp\left(-r\left(T - t\right)\right) + \exp\left(-r\left(T - t\right) - \gamma t\right)}$$

At terminal time T, we have:

$$\omega = \frac{\left(\frac{1}{\psi} - \frac{c}{r}\right) + \left(F_0 + \frac{c}{r}\right)\exp\left(-\left(\gamma - r\right)T\right) + \frac{c}{r}\left(1 - \exp\left[-\gamma T\right]\right)}{\exp\left(-\gamma T\right)}$$

Hence, the optimal wealth can be expressed in terms of ψ as follows:

$$E\left(F^*(T)\right) = F_0\exp\left(rT\right) + \frac{c\left(\exp\left(rT\right) - 1\right)}{r} + \frac{\exp\left(\gamma T\right) - 1}{\psi} \qquad (2.28)$$

Therefore, the expected optimal wealth is the sum of wealth an investor will get by investing the entire portfolio in the cash account plus a term, $\frac{\exp\left(\gamma T\right) - 1}{\psi}$ that depends on the goodness of the stock and the indexed bond with respect to the cash account and on the weight given to the minimization of the variance. Hence, the higher the value of γ, the higher the expected wealth of the investor, for all other parameter held constant. Again, the higher the value of ψ, the lower the expected wealth of the investor.

$$E\left(F(T)^2\right) = \Gamma_1\exp\left(rT + \gamma\left(\gamma\sigma^2 - 2\right)T\right) - \Gamma_2\exp\left(\left(r - \gamma\right)T\right) + \Gamma_3\exp\left(-\gamma T\right) - \Gamma_4 +$$
$$\Gamma_5\exp\left(\gamma\left(\gamma\sigma^2 - 2\right)\right) + \Gamma_6\exp\left(3\left(1 - r\right)T + \left(2 + \gamma\left(\gamma\sigma^2 - 2\right)\right)T\right) + \Gamma_7\exp\left(2r + \left(2 + \gamma\left(\gamma\sigma^2 - 2\right)\right)T\right) \qquad (2.29)$$
$$+ \Gamma_8 + \Gamma_9\exp\left(-\gamma T\right) - \Gamma_{10}\exp\left(\left(r - \gamma\right)T\right) + \Gamma_{11}$$

We may observe that the higher the weight given to the minimization of the variance, the higher the expected wealth and vice versa. It is obvious that a necessary and sufficient condition for the wealth to be invested at any time t in the stock and indexed bond is $\psi = +\infty$. Hence, the choice of investing the entire portfolio in the stock and indexed bond is optimal if and only if the weight given to the minimization of the variance is infinite.

Let $\tilde{\pi}^*(t,F) = \pi^{S*}(t,F) + \pi^{I*}(t,F)$, then the optimal proportion to be invested in stock and indexed bond in terms of ψ is given as

$$\tilde{\pi}^*(t,F) = -\gamma\left(F - \left\{F_0\exp\left[rt\right] - \frac{c}{r}\left(1 - \exp\left[rt\right]\right) - \frac{\exp\left[-r\left(T - t\right) + \gamma T\right]}{\psi}\right\}\right)$$

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The amount $F\tilde{\pi}^*(t,F)$ invested in the stock and indexed bond at time t is proportional to difference between the wealth F at time t and the wealth available for the cash account minus a the term that depends on the evolution of the market over time and time to retirement.

We consider the efficient frontier of portfolios. By definition,

$$
\begin{aligned}
&Var\left(F^*(T)\right)=E\left(F^*(T)^2\right)-\left(E\left(F^*(T)\right)\right)^2 \\
&=\Gamma_1\exp\left(rT+\gamma\left(\gamma\sigma^2-2\right)T\right)-\Gamma_2\exp\left((r-\gamma)T\right)+\Gamma_3\exp\left(-\gamma T\right)-\Gamma_4+ \\
&\Gamma_5\exp\left(\gamma\left(\gamma\sigma^2-2\right)\right)+\Gamma_6\exp\left(3(1-r)T+\left(2+\gamma\left(\gamma\sigma^2-2\right)\right)T\right)+\Gamma_7\exp\left(2r+\left(2+\gamma\left(\gamma\sigma^2-2\right)\right)T\right) \\
&+\Gamma_8+\Gamma_9\exp\left(-\gamma T\right)-\Gamma_{10}\exp\left((r-\gamma)T\right)+\Gamma_{11}-\left(F_0\exp\left(rT\right)+\frac{c\left(\exp\left(rT\right)-1\right)}{r}+\frac{\exp\left(\gamma T\right)-1}{\psi}\right)^2
\end{aligned}
$$

We observed that as $\psi\to\infty$, we have a minimum variance. Hence, the higher the the value of ψ the lower the variance. At this point we obtained the optimal expected wealth. Observe that when the variance is high the expected wealth will be high. This make sense since the higher the risk in an investment, the higher the expected wealth , otherwise there is no need engaging in such investment, or the whole portfolio should then be invested in cash account. We also observed that as $\psi\to 0$, the expected wealth tends to infinity. This means that if a risky asset becomes more risky, the wealth of the investor is expected to be very high. This also shows that as $\psi\to 0$, the variance tends to infinity.

4. Conclusion

We considered a mean-variance portfolio selection problem for a fixed flow of investment in a continuous-time. We obtained explicitly the expected optimal terminal wealth for the investor. We also obtained closed-form expressions for the value functions, the optimal investment and mean-variance strategies. We then maximized the expected terminal wealth while minimizing its variance. We found that the higher the value of parameter, ψ (variance minimizer) the lower the variance. We also found that when the variance is high the expected wealth will be higher. We further found that if the parameter used to minimize the variance tends to zero, both the expected wealth and the variance tend to infinity.

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