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A Robust Mathematical Model On Infectious Diseases

Omorogbe Dickson E.A. and **Omoregie Nosa E.A.
***Institute Of Education(Ekewan Campus)**
University Of Benin, Nigeria.
****Faculty Of Management Sciences**
Ambrose Alli University
Ekpoma, Nigeria

Abstract

The paper presents a robust epidemiological compartment model on infectious diseases. The model obviates the limitations of the classical epidemiological model by accommodating different levels of vulnerability and susceptibility to infections within different social class and spatial structures. Unlike the classical model which considers every member from different compartments to be geographically homogenous, implying the chances of infection by communicable diseases to be the same. This is unrealistic in real life situation. The Robust model in this paper is highly realistic and suitable in real life situation. This paper also analyses the actuarial implications of infectious disease plan and it is recommended that the annuity for hospital and lump sum for hospital plans would fairly reduce the cost that could cushion blow of any possible epidemic to our health care system.

Keywords *:* Epidemiological model, Annuity for hospitalization, infectious diseases and infectious virus-carrier.

1.0 Introduction

In the classical epidemiological model a whole population is usually separated into compartments with label S, I, and R. there acronyms are used in different patterns according to transmission dynamics of the studied disease. In a nutshell, class S denotes the group of individual without immunity or susceptible to a certain disease, in an environment exposed to communicable disease infection like the rural community of the Niger – Delta region of Nigeria with a lot of gas flaring and disease carrying germs. Some individuals come into contact with the germs. Those infected and are able to transmit the disease are considered in class I. Due to medical therapy, individuals, removed from the epidemic as a result of recovery from the ailment are counted in class R. This is illustrated in [8], [9] and [10]. The major limitation of the classical epidemiological compartment model is that it considers every member from different compartments to be geographically homogenous which is totally unrealistic in real life situation. The susceptible people in the geographical neighbourhoods of an infectious virus-carrier are likely to be infected than those remote from the carrier/environment. For example the health workers' environment is more susceptible to infection than workers in the clean environment of the school or office environment of the ministries. In order to address the limitation of the classical epidemiological model, we present a robust mathematical model which obviate this deficiency and it is highly realistic in real life situations.

According to [1], over the last century, many contributions to the mathematical modelling of epidemiological and communicable disease have been made by a great number of public health physicians, epidemiological mathematicians and statisticians, their brilliant work ranges from empirical data analysis to differential equation theory. But [2] claimed that some have achieved success in clinical data analysis and effective predictions. Barnes and Fulford [3] considered mathematical modelling with case studies. And Brauer [4] studies the deterministic

*Corresponding author: E-mail: dickomorogbe@yahoo.com, Tel.+23480333128330

compartment models in epidemiology for a complete review of a variety of mathematical and statistical models. Interested readers are referred to [6] and [7]. According to [5] in their paper on epidemiological model in actuarial mathematics, they opined from a social point of view that an effective protection – against diseases depends not only on the development of medical technology to identify viruses and to treat infected patients, but also on a well-designed health –care system. In order to get a better understanding of this paper we give a fairly detailed treatment of the classical epidemiological compartment model in the next section.

2.0 The Classical Epidemiological Compartment Model

The SIR model is expressed by the following system of differential equations.(Where the acronyms S, I,R, represent The susceptible, Infected and Recover classes from the disease respectively). See [8].

$$S'(t) = -\beta S(t)I(t)/N,\ t\geq 0, \tag{2.1}$$

$$I'(t) = \beta S(t)I(t)/N - \alpha I(t),\ t\geq 0, \tag{2.2}$$

$$R'(t) = \alpha\, I(t),\ t\geq 0, \tag{2.3}$$

With given initial values $S(0) = S_0$, $I(0) = I_0$ and $S_0 + I_0 = N$

The assumption for the model is as follows:

(a) The total number of individuals keeps constant, N= S(t) + I(t) + R(t) , representing the total population size.

(b) An average person makes an average number β of adequate contacts (i.e. contacts sufficient to transmit infection) with others per unit time.

(c) At any time a fraction α of the infected leave class I instantaneously. α is also considered to be constant.

(d) There is no entry into or departure from the population, except possibly through death from the disease. For our purpose of setting up an insurance model, the demographic factors like natural births and deaths are negligible, as the time scale of an epidemic is generally shorter than the demographic time scale.

Since the probability of a random contact by an infected person with a susceptible individual is S/N then the instantaneous increase of new infected individuals is β(S/N)I=βSI/N. The third assumption implies that the instantaneous number of people flowing out of the infected class I in to the removal class *R* is *α*I.

3.0 Actuarial Analysis

Since mortality analysis is based on ratios instead of absolute counts, we now introduce s(t), i(t) and r(t) respectively as fraction of the whole population, in each of class *S*, I and *R*. Dividing equations (2.1)-(2.3)by the constant total population size N yields.

$$s'(t) = -\beta\, i(t)\, s(t),\ t \geq 0, \tag{3.1}$$

$$i'(t) = \beta\, i(t)\, s(t) - \alpha\, i(t),\ t \geq 0, \tag{3.2}$$

$$r(t) = 1 - s(t) - i(t),\ t \geq 0, \tag{3.3}$$

One could actually interpret the ratio function s(t), i(t) and r(t) as the probability of an individual being susceptible, infected or removed from infected class respectively at the time spot t.

Since all these ratio functions lie in the interval [0, 1] we could easily interpret them as the probability of an individual remaining susceptible, infected or removed at the time point t. however, it should be noted that due to the laws of mass action, movements among the compartments rely on the sizes of each other. Thus these probabilities represent mutually dependent risks as opposed to the independent hazards one always sees in multiple decrement life insurance models. With these probability functions s(i), i(t) and r(t), we now incorporate actuarial methods to formulate the quantities of interest for an infectious disease insurance.

4.0 Annuity for Premium Payments and Annuity for Hospitalization

We assume that the infection plan works in a simple annuity fashion. Individual premiums are collected continuously as long as the covered person remains susceptible, whereas medical expenses are continuously reimbursed to each infected policyholder during the whole period of treatments. Once the individual recovers from the disease, the protection ends right away.

Following the international Actuarial Notation, the actuarial present value (APV) of premium payments from an insured person for the whole epidemic is denoted by $\overline{a}_0^s$ with the superscript indicating payments from class s, and APV of benefit payments from the insurer is denoted by $\overline{a}_0^s$ with the superscript indicating payments to class I.

*Corresponding author: E-mail: dickomorogbe@yahoo.com, Tel.+23480333128330

On the debit side of the insurance product, the total discounted future claim is given by

$$\bar{a}_0^i = \int_0^\infty e^{-\delta t} s(t)dt, \tag{4.1}$$

while on the revenue side, the total discounted future premiums is

$$\bar{a}_0^s = \int_0^\infty e^{-\delta t} s(t)dt, \tag{4.2}$$

where δ is the force of interest. Our study in this paper is based on the fundamental *Equivalence principle* in Actuarial Mathematics for the determination of level premiums, which requires;
E[present value of benefits]=E[present value of benefit premiums].

Therefore, the level premium for the unit annuity for hospitalisation plan is given as.

$$\bar{P}\left(\bar{a}_0^i\right) = \frac{\bar{a}_0^i}{a_0^{-s}} \tag{4.3}$$

Just like in life insurance, where the force of mortality is defined as the additive inverse of the ratio of the derivative of the survival function to the survival function itself, we define here the force of infection as

$$\mu_t^s = \frac{s'(t)}{s(t)}, \qquad t \geq 0,$$

and the force of removal as

$$\mu_t^i = \frac{i'(t)}{i(t)}, \qquad t \geq 0,$$

specifically from (3.1)-(3.2), we see that $\mu_t^i = -\beta i(t) \; and \; \mu_t^i = -\beta s(t) + \alpha.$
Note that the above definitions imply that

$$s(t) = \exp\left\{-\int_0^t \mu_r^s dr\right\} = \exp\left\{-\beta\int_0^t i(r)dr\right\}, \; t \geq 0, \tag{4.4}$$

and

$$i(t) = \exp\left\{-\int_0^t \mu_r^i dr\right\} = \exp\left\{-\beta\int_0^t s(r)dr + \alpha t\right\}, \; t \geq 0, \tag{4.5}$$

Proposition 4.1 in the *SIR* model in (3.1)-(3.2),

$$\left(1 + \frac{\alpha}{\delta}\right)\bar{a}_0^i + \bar{a}_0^s = \frac{1}{\delta}. \tag{4.6}$$

Proof. From (3.1) and (3.2), we obtain that

S'(t)+i'(t)=-αi(t), $t \geq 0$.

Integrating from 0 to a fixed t gives

$$s(t) + i(t) - 1 = -\alpha\int_0^t i(r)dr, \; t \geq 0.$$

Multiplying both sides by $e^{-\delta t}$ and integrating with respect to t from 0 to ∞ yields

$$\bar{a}_0^s + \bar{a}_0^i - \frac{1}{\delta} = -\frac{\alpha}{\delta}\bar{a}_0^i.$$

where the right hand side comes from exchanging the order of integrals,

$$\int_0^\infty \exp(-\delta t)\int_0^t i(r)drdt = \frac{1}{\delta}\int_0^\infty\int_0^t i(r)drd(\exp(-\delta t)) = \frac{1}{\delta}\int_0^\infty \exp(-\delta r)i(r)dr = \frac{1}{\delta}\bar{a}_0^i.$$

Notice that the right hand side represents the perpetual annuity. The intuitive interpretation of the left hand side is that if every one in the insured group is rewarded with a perpetual annuity, the APV of expenses from class S accounts for $\bar{a}_0^s$ and similarly that of expenses from class I adds $\bar{a}_0^i$ to the cost. Recall that at any time a fraction α of the infected subgroup move forwards to class R. each of them would receive a perpetual of value 1/δ as well at the time of transition. Therefore, the APV of expenses from this compartment would be $(\alpha/\delta)\bar{a}_0^i$ .It is reasonable

*Corresponding author: E-mail: dickomorogbe@yahoo.com, Tel.+23480333128330

that it should sum up to the value of a unit perpetual annuity regardless of the policyholder's location among compartments.
With this relation in mind, we could easily find the net level premium for the unit annuity for hospitalization plan, as follows:

$$\bar{P}(\bar{a}_0^i) = \frac{\bar{a}_0^i}{\bar{a}_0^s} = \frac{\delta \bar{a}_0^i}{1-(\delta+\alpha)\bar{a}_0^i}. \qquad (4.7)$$

5.0 Lump Sum for Hospitalization

The analogy of the plan is with a whole life insurance in actuarial mathematics. When a covered person is diagnosed being infected with the disease and hospitalized, the medical expenses is to be paid immediately in lump sum and insurance protection ends. Then the APV of benefit payments to the infected denoted by $\bar{A}_0^i$ can be obtained as

$$\bar{A}_0^i \triangleq \beta \int_0^\infty e^{-\delta t} s(t) i(t) dt, \qquad (5.1)$$

since the probability of being newly infected at time t is $\beta s(t) i(t)$
proposition 5.1

$$\frac{1}{\delta}\bar{A}_0^i + a_0^s = \frac{1}{\delta} s_0, \qquad (5.2)$$

and

$$\frac{1}{\delta} i_0 + \frac{1}{\delta}\bar{A}_0^i = \frac{\alpha}{\delta}\bar{a}_0^1 + \alpha_0^{-i} \qquad (5.3)$$

(see Proof in [8])
The above proposition also provides an interesting insight into the break-down of expenses in each class. To understand (5.2), we suppose every susceptible individual claims one unit perpetual annuity. The APV of the total cost is s_0/δ. From an another perspective, it is equivalent to give every one a unit annuity as long as person remains healthy in the group and then grant them each a unit perpetual immediately as he or she becomes infected. The APV of these two payments is exactly $(1/\delta)\bar{A}_0^i + \bar{a}_0^s$. If one thinks of class I as a transit, the left hand side of (5.3) count the expenses at the point of entry. Since expenses for initial members is i_0/δ and other individuals from class S each add $1/\delta$. Hence the total expenses add up to $i_0/\delta + (1/\delta)\bar{A}_0^i$. The class of the infected persons accounts for $\bar{a}_0^i$, and any one leaving the class takes away a perpetual of value of $1/\delta$. Thus the right hand side sums up to $(\alpha/\delta)\bar{a}_0^i + \bar{a}_0^i$.
Therefore for the lump sum payment plan with a unit benefit, the equivalence principle gives the net level premium $\bar{P}(\bar{A}_0^i)$:

$$\bar{P}(\bar{A}_0^i) = \frac{\bar{A}_0^i}{\bar{a}_0^s} = \frac{(\alpha+\delta)\bar{a}_0^i - i0}{1-(\alpha+\delta)\bar{a}_0^i}.$$

The major limitation of the classical SIR model is that it considers the chances of infection in a whole populations as homogenous, which is unrealistic and untrue in real life situation. We therefore present a Robust epidemiological model which obviate this limitation in the next section.

6.0 The Robust Epidemiological Compartment Model.

The Robust epidemiological model that distinguished different levels of vulnerability or infectiousness within different social groups, spatial structures is giving by the ODE systems below:

*Corresponding author: E-mail: dickomorogbe@yahoo.com, Tel.+23480333128330

$$s_1^1(t) = -\beta S_1(t)\frac{I(t)+qE(t)+\iota J(t)}{N}, \qquad t \geq 0, \qquad (6.1)$$

$$s_2^1(t) = -\beta S_2(t)\frac{I(t)+qE(t)+\iota J(t)}{N}, \qquad t \geq 0, \qquad (6.2)$$

$$E'(t) = \beta(S_1(t)+pS_2(t)\frac{I(t)+qE(t)+\iota J(t)}{N} - kE(t), \quad t \geq 0, \qquad (6.3)$$

$$I\ (t) = kE(t)-(\alpha+\lambda_1+\delta)I(t), \quad t \geq 0, \qquad (6.4)$$

$$J'(t) = \alpha I(t)-(\lambda_2+\delta)J(t), \quad t \geq 0, \qquad (6.5)$$

$$R'(t) = \gamma_1 I(t)+\gamma_2 J(t), \qquad t \geq 0, \qquad (6.6)$$

In this model, there are two distinct susceptible compartments with different levels of exposure to the communicable diseases infection e.g. HIV/AIDS, namely S_1 for the most susceptible Health care community and S_2 for the other less susceptible members of the population. Initially, $S_1(0) = pN$ and S_2 (0) = (1 – p)N, where p is the proportion of Health Care inhabitants is total population. An average highly susceptible person (in the Class S_1) makes an average number of β adequate contacts (i. e. contacts sufficient to transmit infection) with others per unit time. As a result of frequent visits to Health Care areas which is more prone to infection, an average lower susceptible person (in the Class S_2) would only be exposed to an average number of $p\beta$ adequate contacts with others per unit time. Therefore an individual infected with HIV/AID virus may experience an incubation period of 3 – 6 months before the onset of any visible symptom. An infectious class is set up for those infected but not yet manifesting the symptoms. The parameter q is used to measure the lower level of infectivity during the incubation. As the time elapses, the infected individual would develop observable symptoms and become fully infectious in Class I with q = 1. In order to distinguish their potential disease transmission to general public, the Class I is separated for those infectious individuals that are undiagnosed. Since almost all diagnosed cases are quarantined in hospitals, the Class J has a lower infectivity level reflected by a reduction factor l.

The rates of population transferring from E, I and J to their chronologically adjacent compartments I, J and the recovered class R are respectively κ, α and γ_2. Considering that even before being diagnosed HIV/AIDS patients many either recover naturally at the rate of γ_1 or die at the force of fatality δ, we also have the class D keeping track of deaths as a result of the HIV/AIDS from two sources I and J. The patients under medical treatments in Class J suffer death at the rate assumed to be same as the mortality in Class I.

Notice that both E and I are undiagnosed phases, there is literally no statistical data for estimating their parameters. Therefore, another compartment C for reported probable cases is set aside to trace back the original time of incidences by a time series. Figure 1 gives transfer directions among the different compartments.

From an insurer's point of view, this model presents many business opportunities. On the one hand, individuals in Classes S_1 and S_2 are potential buyers facing the risk of infection with HIV/AIDS. On the other hand, there is an evident need for insurance covering costs in both S_1 and S_2, medical examination expenses for probable cases in Class I, hospitalization and guarantee expenses for Class J and death benefit for Class D. since a number of parties are involved in the health care system, such as insurance companies, policy – holders, government health agencies, and hospitals. Numerous business models could be designed to bring them together in order to reduce the financial impact to the lowest level. In this paper we recommend the annuity for hospital plan and lump sum for hospital plan because these insurance plans would fairly reduce cost that could cushion the blow of any possible epidemic to health care system.

*Corresponding author: E-mail: dickomorogbe@yahoo.com, Tel.+23480333128330

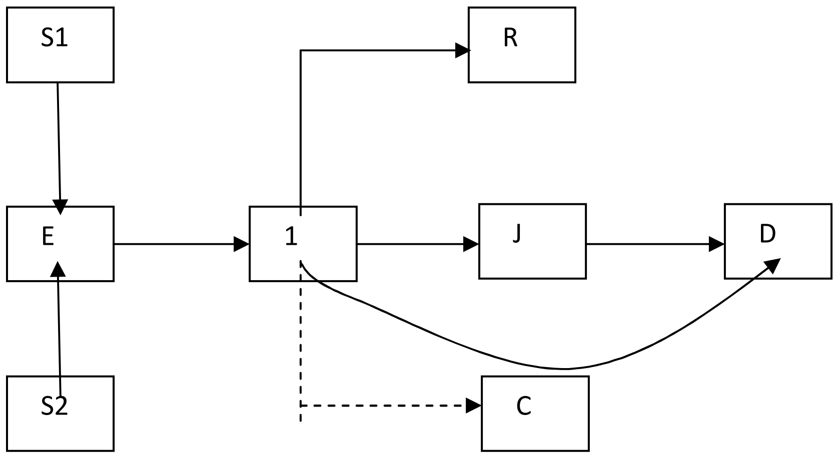

Figure 1: Transfer diagram of the communicable disease (e.g. Hiv/Aids) epidemic dynamics

7.0 Conclusion

In this paper, we present a Robust epidemiological model which obviates the limitation of the classical model (in [8], [9] and [10]). In this Robust model the level of vulnerability and susceptibility varies and depends on the geographical neighbourhood/environment of the individual to the infection virus-carrier. Therefore the closer the individual to the virus-carrier/environment, the more susceptible to infection. This model is highly realistic and true in real life situation.

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*Corresponding author: E-mail: dickomorogbe@yahoo.com, Tel.+23480333128330