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On The Travelling Wave Solution For An SEIR Epidemic Disease Model.

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Abstract

We present the travelling wave solution for a Susceptible, Exposed, Infective and Removed (SEIR) epidemic disease model. For this SEIR model, the disease is driven by both the latent and infective class (the diffusion term is included in both classes). The population is closed.

Keywords and phrases: Epidemic model, spatial spread, travelling wave Solution.

1.0 Introduction

There has been a recent impetus for the study of spatial epidemiology, following the planned investment of billions of dollars by the United States government in biological weapons defence [18] and concerns over the potential use of small pox and other diseases as biological weapons [18].

However, a fundamental challenge in Mathematical epidemiology is determining how the structure of a population influences disease transmission. For instance, the Severe Acute Respiratory Syndrome (SARS) epidemic spread through twelve countries within a few weeks and the swine flu has spread through all the continents (including some parts of Africa, e.g. Badagry in Lagos, Nigeria) within a few months of its outbreak.
Several scientists including [4] have made significant contribution in the area of travelling wave solution for disease model. He considered the travelling wave solution for an SIR epidemic model when the population is closed.

Abual-Rub [3] used the idea of travelling wave to introduce vaccination/control for a susceptible and infective (SI) disease model which in turn keeps the number of infective and susceptible unchanged and specific in the long run.

Jing Li and Xingfu Zou [12] formulated a susceptible infective and removed (SIR) model with a simple demographic structure for the population living in a spatially continuous environment with the assumptions that an infectious disease in a population has a fixed latent period and the latent individuals of the population may diffuse.

Abramson and Kenkre [1] analyzed the propagation of travelling wave fronts in a simple one-dimensional model of the ecology and epidemiology of the Hantavirus in deer mouse.

Lewis et al [14] developed and analyzed a reaction-diffusion model for the spatial spread of the West Nile Virus.

Burie et al [5] investigated the structure of travelling waves for a model of a fungal disease propagating over a vineyard.

Reluga et al [19] formulated the restricted-movement model to describe spatial patterns of disease transmission.

However, in all the works available for review, the travelling wave solution for an SEIR epidemic model was not investigated. we will consider model formulation and analysis of an SEIR epidemic model. This formulation shall take into consideration the spatial spread of disease. In this model "exposed (latent) but not yet infections" individuals are denoted by E.

Model Formulation And Analysis

The model shall include diffusion of both the latent and infective only. The susceptible class does not include the diffusion term because the exposed and infective population are very active in infesting other individual in the total population and they are capable of moving more. S, E, I and R must be thought of respectively as the population densities of susceptible, exposed, infective and removed individuals depending on position as well as time.

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The SEIR model is given by

$$\frac{\partial S}{\partial \tau} = -\beta IS\,, \quad \frac{\partial E}{\partial \tau} = \beta TS - SE + D_E \frac{\partial^2 E}{\partial \xi^2}\,, \quad \frac{\partial I}{\partial \tau} = \delta E - \gamma I + DI \frac{\partial^2 I}{\partial \xi^2}\,, \quad \frac{\partial R}{\partial \tau} = \gamma I \tag{2.1}$$

In (2.1), the population is closed where birth and non-disease related deaths are neglected.

βIS represents the incidence of the disease, δ is the rate at which individuals leave the latent class and enters the infectious class. Also γ is the rate of recovery, D_E *and* D_I represent the diffusion coefficient in the latent and infectious class respectively.

The initial condition for (2.1) is

$$S \to N,\ E \to 0,\ I \to 0 \ \text{and}\ R \to 0 \ \text{as}\ \xi \to 0 \tag{2.2}$$

We shall non-dimensionalize (2.1), rescale the time variable, τ and spatial variable, ξ in (2.1) using the following substitutions.

$$\bar{u} = \frac{S}{N},\ \bar{v} = \frac{E}{N},\ \bar{w} = \frac{I}{N},\ \bar{z} = \frac{R}{N} \tag{2.3a}$$

$$t = \gamma\tau,\ x = \xi\sqrt{\frac{\gamma}{D}} \tag{2.3b}$$

Differentiating $\bar{u}, \bar{v}, \bar{w}$ *and* $\bar{z}$ of (2.3a) respectively with respect to $S, E, I,$ *and* R, we have

$$\partial S = N\partial\bar{u},\ \partial E = N\partial\bar{v},\ \partial I = N\partial\bar{w} \ \text{and}\ \partial R = N\partial\bar{z} \tag{2.4}$$

Also, differentiating each of (2.3b) with respect to t *and* ξ respectively gives

$$\partial\tau = \frac{1}{\gamma}\partial t \ \text{and}\ \partial\xi = \sqrt{\frac{D}{\gamma}}\,\partial x \tag{2.5}$$

Substituting (2.3a), (2.3b), (2.4) and (2.5) into the system (2.1), we obtain after simplication

$$\frac{\partial\bar{u}}{\partial t} = -R_0\bar{w}\bar{u}\,, \quad \frac{\partial\bar{v}}{\partial t} = R_0\bar{w}\bar{u} - \varepsilon\bar{v} + \frac{\partial^2\bar{v}}{\partial x^2},$$
$$\frac{\partial\bar{w}}{\partial t} = \varepsilon\bar{v} - \bar{w} + \frac{\partial^2\bar{w}}{\partial x^2} \quad \text{and} \quad \frac{\partial\bar{z}}{\partial t} = \bar{w} \tag{2.6}$$

Where $R_0 = \frac{\beta N}{\gamma}$ refers to the basic reproductive ratio;

βN is the rate at which a single infective introduced into a susceptible population of size N makes infectious contacts and $\frac{1}{\gamma}$ is the expected length of time such an infective remains infectious. R_0 is the expected number of infectious contacts made by such infective. R_0 is a very important concept in epidemiology.

Also $\varepsilon = \frac{\delta}{\gamma}, (S, E, I, R) = (N, 0, 0, 0)\cdot$

If $R_0 < 1$ the disease dies out in the spatially uniform case while if $R_0 > 1$ an epidemic occurs.

Next, we seek a constant shape travelling wave solution of the form

$$\bar{u}(x,t) = u(s) = u(x+ct),\ \bar{v}(x,t) = v(x+ct)$$
$$\bar{w}(x,t) = w(s) = w(x+ct),\ \bar{z}(x,t) = z(s) = z(x+ct) \tag{2.7}$$

Where $s = x + ct$

$\bar{u}(x,t)$, $\bar{v}(x,t)$, $\bar{w}(x,t)$ *and* $\bar{z}(x,t)$ are travelling waves which moves at a constant speed c in the negative x-direction with positive c.

Substituting (2.7) into (2.6), we have

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$$cu' = -R_0 wu, \quad cv' = R_0 wu - \varepsilon v + v'', \quad cw' = \varepsilon v - w + w'', \quad cz' = w \tag{2.8}$$

where the prime (‘) denotes the differentiation with respect to s. The system (2.8) is to be analysed subject to the boundary conditions

$$\left.\begin{aligned} u(s) \to 1 \ \ as \ \ s \to -\infty, \ \ u(s) \to u_1 \ \ as \ \ s \to \infty \\ v(s), \ w(s) \to 0 \ \ as \ \ s \to \pm\infty \\ z(s) \to 0 \ \ as \ \ s \to -\infty, \ \ z(s) \to z_1 \ \ and \ \ s \to \infty \end{aligned}\right\} \tag{2.9}$$

In order to linearize (2.8), we rearrange the second and third equations of (2.8) and divide through by the first equation in (2.8). This gives

$$cv' + \varepsilon v - v'' = R_0 wu$$

$$\frac{cv' + \varepsilon v - v''}{cu'} = \frac{-R_0 wu}{R_0 wu}$$

so,

$$\frac{dv}{du} + \frac{\varepsilon}{c}\frac{d\left(\frac{v^2}{2}\right)}{du} - \frac{1}{c}\frac{d(v')}{du} = -1$$

Integrating across, we have

$$v + \frac{\varepsilon v^2}{2c} - \frac{1}{c}v' = -u + A$$

Therefore,

$$v' = c\left[v + u + \frac{\varepsilon v^2}{2c} - A\right] \tag{2.10}$$

Also,

$$cw' - \varepsilon v - w'' = -w$$

$$\frac{cw' - \varepsilon v - w''}{cu'} = \frac{w}{R_0 wu}$$

$$\frac{dw}{du} - \frac{\varepsilon}{c}\frac{d\left(\frac{v^2}{2}\right)}{du} - \frac{1}{c}\frac{d(w')}{du} = \frac{1}{R_0 u}$$

Integrating across

$$w - \frac{\varepsilon v^2}{2c} - \frac{1}{c}w' = \frac{1}{R_0}\log_e u + B$$

$$w' = c\left[w - \frac{\varepsilon v^2}{2c} - \frac{1}{R_0}\log_e u - B\right] \tag{2.11}$$

Also, dividing the last equation of (2.8) by the first, we have

$$\frac{z'}{u'} = \frac{-1}{R_0 u} \quad or \quad \frac{dz}{du} = -\frac{1}{R_0 u}$$

Integrating, we have

$$z = -\frac{1}{R_0}\log_e u + D \tag{2.12}$$

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A, B and D in (2.10), (2.11) and (2.12) respectively are constants of integration to be determined by (2.9). Since $u,\ v,\ and\ w$ approach limit as $s \to \pm\infty$, then it implies that $v'\ and\ w'$ do as well. So,

$$v'(s),\ \ w'(s)\ \to 0 \quad as\ \ s \to \pm\infty \tag{2.13}$$

Applying (2.9) and (2.13) on (2.10), (2.11) and (2.12) as $s \to -\infty$, gives
$A = 1,\ B = 0,\ D = 0$. Hence (2.10), (2.11) and (2.12) becomes

$$z = -\frac{1}{R_0}\log_e u = z(u), \quad v' = c\left[v + u + \frac{\varepsilon v^2}{2c} - 1\right]\ and \quad w' = c\left[w - \frac{\varepsilon v^2}{2c} - \frac{1}{R_0}\log_e u\right] \tag{2.14}$$

To determine the wave speed, we analyse (2.14) and

$$u' = -\frac{1}{c}wu \tag{2.15}$$

Applying the second boundary condition as $s \to \infty$, we have that $u_1 + w_1 = 1$ where $u_1 = 1 - z_1$ so that

$$l - z_1 = \exp(-R_0 z_1) \tag{2.16}$$

Since $z = \frac{1}{R_0}\log_e u$ that is $\log_e u = -R_0 z \quad or \quad u = \exp\left(-R_0 z\right)$.

(5.1.16) will always have the solution $z_1 = 0$ representing no epidemic. If $R_0 < 1$, then there is no other positive solution, whereas if $R_0 > 1$ then there is another solution z, satisfying $0 < z_1 < 1 \quad and \quad R_0\left(1 - z_1\right) = R_0 u_1 < 1$.

It follows that in this spatially inhomogeneous case there can be no epidemic if $R_0 < 1$. Let us take $R_0 > 1$, so that $0 < u_1 < \frac{1}{R_0} < 1$. The only critical points of (2.14) and (2.15) are $(u_1,\ 0,\ 0)\ and\ (1,\ 0,\ 0)$. To analyse this, we determine Jacobian matrix of

$$u' = \frac{-1}{c}wu, \quad v' = c\left[v + u + \frac{\varepsilon v^2}{2c} - 1\right], \quad w' = c\left[w - \frac{\varepsilon v^2}{2c} - \frac{1}{R_0}\log_e u\right] \tag{2.17}$$

The Jacobian matrix is

$$\begin{pmatrix} -\frac{1}{c}w & 0 & -\frac{1}{c}u \\ c & c\left(1 + \frac{\varepsilon v}{c}\right) & 0 \\ -\frac{c}{R_0 u} & -\varepsilon v & c \end{pmatrix} \tag{2.18}$$

At the critical (1, 0, 0), we have

$$J^* = \begin{pmatrix} 0 & 0 & -\frac{1}{c} \\ c & c & 0 \\ -\frac{c}{R_0} & 0 & c \end{pmatrix} \tag{2.19}$$

where J^* denotes evaluation of J at the steady state Next, we determine the eigen values λ which are the roots of

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$$\begin{vmatrix} -\lambda & 0 & -\frac{1}{c} \\ c & c-\lambda & 0 \\ -\frac{c}{R_0} & 0 & c-\lambda \end{vmatrix} = 0 \tag{2.20}$$

(2.20) implies

$$-\lambda(c-\lambda)^2 = \frac{1}{c}\left(\frac{c}{R_0}(c-\lambda)\right) = 0$$

$$\lambda^3 - 2c\lambda^2 + \left(c^2 - \frac{1}{R_0}\right)\lambda + \frac{c}{R_0} = 0 \tag{2.21}$$

Now, we simply look for conditions that will make the roots of the polynomial in (2.12) negative, which are the criteria for the critical point (1, 0, 0) to be asymptotically stable. To determine this, we use the popular Routh-Hurwitz Conditions.

The Routh-Hurwitz Conditions for the roots of (2.21) to have negative real parts, requires $R_0\lambda < 0$. This holds if.

$$-2c > 0, \quad \frac{c}{R_0} > 0, \quad -2c\left(c^2 - \frac{1}{R_0}\right) - \frac{c}{R_0} > 0 \tag{2.22}$$

It is obvious that if the second inequality of (2.22) holds, the first and third do not hold except $C < 0$ which again violates the second inequality. Since the Routh-Hurwitz Conditions are not satisfied, we have no travelling wave front solution which approach the steady state (1, 0, 0) as $s \to -\infty$.

For the steady state (1, 0, 0), it is difficult to characterize the travelling wave speed probably due to the fact that we are dealing with more than one infested class. Our analyses were carried out via the method of linearization around the equilibrium point.

However, determine if the wave speed of the second equilibrium point can be characterized, we substitute $(u_1, 0, 0)$ into (2.18).

$$J^* = \begin{pmatrix} 0 & 0 & -\frac{1}{cu_1} \\ c & c & 0 \\ -\frac{c}{R_0 u_1} & 0 & c \end{pmatrix} \tag{2.23}$$

The Eigen values λ of the second equilibrium point $(u_1, 0, 0)$ is the roots of

$$\begin{vmatrix} -\lambda & 0 & -\frac{1}{c}u_1 \\ c & c-\lambda & 0 \\ -\frac{c}{R_0}u_1 & 0 & c-\lambda \end{vmatrix} = 0 \tag{2.24}$$

From (5.1.24), we obtain

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$$\lambda^3 - 2c\lambda + \left(c^2 - \frac{1}{R_0}\right)\lambda + \frac{1}{R_0}c = 0 \qquad (2.25)$$

(2.25) is the same as (2.21), hence the result is the same for both equilibrium points.

Discussion And Conclusion

The Characterization of the wave speed using the method of linearity did not work for the SEIR model since the infectious class has been broken down into two different classes (Latent and infectious). We conjecture that the wave speed for both types of models (SIR and SEIR) will give the same result.

Most times, during the period in which the individuals are is in the latent class, they do not show symptoms of the disease and as a result do not change behaviour thereby not being able to infest and drive the disease. Disease that can fit into this model includes Rabies which has an incubation period of 150 days. HIV/AIDS has a latent period of 3-10 years. We discovered that it is difficult to characterize the wave speed for SEIR epidemic model. The reason for not being able to characterize the wave speed is probably due to the fact that the disease is driven by two infectious classes i.e. the latent and infective classes. Just like the case of Tuberculosis (T.B) which is an SEIR disease, when it becomes an epidemic, it is difficult to track down the disease (i.e. to measure the wave speed). When the latent class and the infective are driving the disease, it is like a pandemonium, it is full blown epidemic – those who are infected and have not yet shown symptoms are spreading the disease just like the infective class.

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