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Effect of transmission rate in an SIR epidemic model

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Abstract

We extend the transmission rate to nonlinear orders and show that S+I+R is still attractive. When the nonlinearity order is greater than 1, we show that the basic reproduction number decreases as the recruitment increases. We provide the patterns of the stability of the equilibrium states.

Keywords: transmission rate, SIR model, basic reproduction number, stability.

1.0 Introduction

The study of epidemics has a long history with a vast variety of models and explanation for the spread and cause of epidemic outbreaks [2] and [18]. Throughout the history, mankind has suffered many devastating infectious diseases and continues to be fearful of such diseases. Over the past hundred years, mathematics has been used to understand and predict the spread of diseases, relating important public-health questions to basic transmission parameters.
Controlling infectious diseases has been an increasingly complex issue for several countries in recent years. Many scholars have investigated and studied lots of epidemic models of ordinary differential equations [24]. The SIR model proposed by [15] provides an established basis to model the transmission dynamics of infectious diseases. By using the compartmental approach several scholars had extensively analyzed the dynamics of the SIR and SIRS epidemic model where S (t), I (t) and R (t) denote the number of susceptible, infective and recovered at time t, respectively.
One of the fundamental questions of mathematical epidemiology is to find threshold conditions that determine whether an infectious disease will spread in a susceptible population when the disease is introduced into the population (Hyman and Li,2000; [3], [4], [7], [8], [19] and [21]. The threshold conditions are characterized by the so-called reproductive number commonly denoted by R_o in mathematical epidemiology.

The incidence in an epidemiological model is the rate at which susceptible become infectious. Many researchers ([6], [11], [16] and [17]) have proposed transmission laws in which the nonlinearities are more than quadratic. [22] studied an epidemic model with a specific nonlinear incident rate $\frac{\lambda I^P S}{(1+\alpha I^q)}$ were proposed by many researchers and authors ([1], [5], [9], [10], [12], [14], [17] and [23]).
Pathak et al [20] studied an SIR epidemic model with an asymptotically homogenous transmission function. They addressed the stability of the disease-free and the endemic equilibrium.
We extend the work done by Pathak et al, by extending the transmission rate to nonlinear orders and show that SIR model is still attracting. When the nonlinearity order is greater than 1, we show that the basic reproduction number decrease as the recruitment rate increases. We provide the patterns of the stability of the equilibrium states.

2.0 Mathematical Model

We modify the model of [20] by extending the transmission rate to nonlinear orders.

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$$\frac{dS}{dt} = b - dS - \frac{kSI}{1+\alpha S+\beta I} + \gamma R$$

$$\frac{dI}{dt} = \frac{kSI}{1+\alpha S+\beta I} - (d+\mu)I$$

$$\frac{dR}{dt} = \mu I - (d+\gamma)R$$

Modified Model

The modified model is as follows:

$$\frac{dS}{dt} = b - dS - \frac{kSI}{1+\alpha S^{p}+\beta I^{q}} + \gamma R \qquad (2.1)$$

$$\frac{dI}{dt} = \frac{kSI}{1+\alpha S^{p}+\beta I^{q}} - (d+\mu)I \qquad (2.2)$$

$$\frac{dR}{dt} = \mu I - (d+\gamma)R \qquad (2.3)$$

The transmission rate $\phi(S, I) = 1+\alpha S^{p}+\beta I^{q}$

where S (t), I (t) and R (t) denote the number of susceptible, infective and recovered at time t,respectively.b is the recruitment rate of the population is the natural death rate of the population is the proportionality constant, μ is the natural recovery rate of the infective individuals, γ is the rate at which recovered individuals lose immunity and return to the susceptible class, α and β are the parameters which measure the effects of sociological ,psychological or other mechanisms. In literature = q = 1 [20]

3. Stability of the Critical Points

To obtain the critical points, we set equations (2.1) - (2.3) above to zero, i.e $\frac{dS}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0$.

The critical points are (b/d, 0, 0) (3.1)

and $\left(\frac{bd+b\gamma-e^{\sqrt{Q}}(\gamma d-\mu e^{\sqrt{Q}}(d-e^{\sqrt{Q}})d^{2})}{(d+\gamma)d} \quad , \; e^{\sqrt{Q}} \; , \quad \frac{\mu e^{\sqrt{Q}}}{d+\gamma} \right)$.

where

$$Q = Zq - \ln\left(\begin{array}{l} \frac{1}{(d+\gamma)d\beta(d+\mu)}\left(-d^{2}\mu\alpha e^{l^{n}\left(\frac{bd+b\gamma-ez\gamma d-\mu ezd-ezd2}{(d+\gamma)d}\right)p} \right) - d\mu\alpha e^{l^{n}\left(\frac{bd+b\gamma-ez\gamma d-\mu ezd-ezd2}{(d+\gamma)d}\right)p}\gamma + \\ bkd + bk\gamma - d^{2}\mu - d\mu\gamma - ke^{z}\gamma d - d^{3} - d^{2}\gamma - k\mu e^{z}d - ke^{z}d^{2} - d^{3}\alpha e^{l^{n}\left(\frac{bd+b\gamma-ez\gamma d-\mu ezd-ezd2}{(d+\gamma)d}\right)p} \\ - d^{2}\alpha e^{l^{n}\left(\frac{bd+b\gamma-ez\gamma d-\mu ezd-ezd2}{(d+\gamma)d}\right)p}\gamma \end{array} \right)$$

(3.2)

The disease-free equilibrium is (b/d, 0, 0) while the second is the endemic equilibrium. We now set

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$$X = S - \frac{b}{d} \quad , \quad I = I \, , \quad R = R \, , \; \textit{so that the equations } (2.1) - (2.3) \; \textit{become}$$

$$(1 + \alpha S^p + \beta I^q) \frac{dX}{dt} = b(1 + \alpha S^p + \beta I^q) - d\left(X + \frac{b}{d}\right)(1 + \alpha S^p + \beta I^q) - kSI + \gamma R(1 + \alpha S^p + \beta I^q)$$

$$(1 + \alpha S^p + \beta I^q) \frac{dI}{dt} = kSI - (d + \mu)(1 + \alpha S^p + \beta I^q) I$$

$$\frac{dR}{dt} = \mu I - (d + \gamma) R$$

Then, we have

$$\frac{dX}{dt} = -dX + \gamma R - \frac{k\frac{b}{d} I}{1 + \alpha\left(\frac{b}{d}\right) p} + \textit{higher order terms}$$

$$\frac{dI}{dt} = \frac{k\frac{b}{d} I}{1 + \alpha\left(\frac{b}{d}\right) p} - (d + \mu) I + \textit{higher order terms}$$

$$\frac{dR}{dt} = \mu I + (d + \gamma) R$$

The Jacobian matrix becomes

$$\begin{pmatrix} \frac{dX}{dt} \\ \frac{dI}{dt} \\ \frac{dR}{dt} \end{pmatrix} = \begin{pmatrix} -d & -\frac{k\frac{b}{d}}{1 + \alpha\left(\frac{b}{d}\right)^p} & \gamma \\ 0 & \frac{k\frac{b}{d}}{1 + \alpha\left(\frac{b}{d}\right)^p} - (d + \mu) & 0 \\ 0 & \mu & -(d + \gamma) \end{pmatrix} \begin{pmatrix} X \\ I \\ R \end{pmatrix} + \textit{higher order terms}$$

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$$P(\lambda) = \begin{vmatrix} -(d+\lambda) & -\dfrac{k\dfrac{b}{d}}{1+\alpha\left(\dfrac{b}{d}\right)^p} & \gamma \\ 0 & \left(\dfrac{k\dfrac{b}{d}}{1+\alpha\left(\dfrac{b}{d}\right)^p} - (d+\mu)\right) - \lambda & 0 \\ 0 & \mu & -(d+\gamma) - \lambda \end{vmatrix} = 0$$

$$-(d+\lambda)\left[\left(\frac{k\frac{b}{d}}{1+\alpha\left(\frac{b}{d}\right)^p} - (d+\mu)\right) - \lambda\left(-(d+\gamma)-\lambda\right)\right] = 0$$

$$\lambda_1 = -d\,, \qquad \lambda_2 = \frac{k\frac{b}{d}}{1+\alpha\left(\frac{b}{d}\right)^p} - (d+\mu), \qquad \lambda_3 = -(d+\gamma)$$

Let $R_0 = \dfrac{k\dfrac{b}{d}}{(d+\mu)\left(1+\alpha\left(\dfrac{b}{d}\right)^p\right)}$

Lemma 1: P(λ) has no positive roots if $R_0 < 1$

Proof: Clearly, $\lambda_1 < 0$, $\lambda_3 < 0$ since $d > 0$ *and* $\gamma > 0$

Now, if $R_o < 1$, *then* $\lambda_2 < 0$

Hence, all the roots are negative.

Lemma 2: The infection free equilibrium is stable if $R_o < 1$.

Proof: The equilibrium is asymptotically stable because $\lambda_1 < 0$, $\lambda_2 < 0$ *and* $\lambda_3 < 0$.

This completes the proof.

Lemma 3: $R_o \to 0$ *as* $b \to \infty$ *for* $\alpha > 1$.

Proof: $R_0 = \dfrac{k\dfrac{b}{d}}{(d+\mu)\left(1+\alpha\left(\dfrac{b}{d}\right)^p\right)}$

$$\frac{dR_0}{db} = \left[\frac{\frac{k}{d}(d+\mu)\left(1+\alpha\left(\frac{b}{d}\right)^p\right) - \alpha\left(\frac{b}{d}\right)^{p-1} k\frac{b}{d} + (d+\mu)}{\left((d+\mu)\left(1+\alpha\left(\frac{b}{d}\right)^p\right)\right)^2}\right]$$

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i.e. $$\frac{dR_0}{db} = \frac{\frac{k}{d}(d+\mu)}{\left((d+\mu)^2\left(1+\alpha\left(\frac{b}{d}\right)^p\right)^2\right)}$$

$$\frac{dR_0}{db} = \frac{k}{d\ (d+\mu)\left(1+\alpha\left(\frac{b}{d}\right)^p\right)^2}$$

Hence, $\frac{dR_0}{db} > 0$

$R_o \to 0 \quad as \quad b \to \infty$

$$\lim_{b\to\infty} R_0 = \frac{\lim \frac{k}{d}}{(d+\mu)\left(P\,\alpha\left(\frac{b}{d}\right)^{p-1}\right)} = 0$$

The Stability of the Endemic Equilibrium:

Let us denote the infected equilibrium (3.2) by

$(S_*, I_*\ R_*)$,

where each component corresponds to an earlier specified value.

Let

$X = S - S_*$, $Y = I - I_*$, $Z = R - R_*$

Then,

$$\frac{dX}{dt} = -dX - \frac{k I_* X}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} - \frac{k S_* Y}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} + \gamma Z + nonlinear\ terms$$

$$\frac{dY}{dt} = \frac{k I_* X}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} + \frac{k S_* Y}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} - (d+\mu)Y + nonlinear\ terms$$

$$\frac{dZ}{dt} = \mu Y - (d+\gamma)Z$$

Then,

$$\begin{pmatrix} X^1 \\ Y^1 \\ Z^1 \end{pmatrix} = A \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

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where

$$A = \begin{pmatrix} -\left(d + \dfrac{k\,I_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}}\right) & -\dfrac{k\,S_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} & \gamma \\ \dfrac{k\,I_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} & \left(\dfrac{k\,S_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} - (d+\mu)\right) & 0 \\ 0 & \mu & -(d+\gamma) \end{pmatrix}$$

The eigenvalues are

$$P(\lambda) = \begin{vmatrix} -\left(d + \dfrac{k\,I_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}}\right) - \lambda & -\dfrac{k\,S_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} & \gamma \\ \dfrac{k\,I_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} & \left(\dfrac{k\,S_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} - (d+\mu)\right) - \lambda & 0 \\ 0 & \mu & -(d+\gamma) - \lambda \end{vmatrix} = 0$$

By solving P(λ),we have

$$\lambda^3 + \left(3d + \gamma + \mu + \frac{k\,I_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} + \frac{k\,S_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}}\right)\lambda^2 +$$

$$\left(3d^2 + 2\gamma d + 2\mu d + \frac{2dk\,I_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} + \frac{\mu k\,I_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} - \frac{dk\,S_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} - \frac{\gamma k\,S_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} + \gamma\mu\right)\lambda$$

$$- \frac{d^2 k\,S_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} + d^3 - \mu d^2 - \frac{\gamma d k\,S_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} + \gamma d^2 + \gamma d\,\mu + \frac{d^2 k\,I_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} + \frac{d\mu k\,I_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}}$$

$$+ \frac{d\gamma k\,I_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} = 0$$

Let

$$\gamma\mu > k\,S_*\,\delta \tag{3.3}$$

$$\text{where } \delta = \frac{d+\gamma}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}}$$

So,

$$d^3 + \gamma d(d+\mu) + dk\delta I_*\left(\frac{\mu}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}}\right) > d\,(k\delta S_* + d\,\mu) \tag{3.4}$$

Put (3.3) into (3.4), we obtain

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$$d^3 + \gamma d^2 + d\,k\,\delta S_* + \frac{\mu dk\delta I_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} > d\,k\,\delta S_* + d^2\,\mu$$

$$d^3 + \gamma d^2 + \frac{\mu dk\delta I_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} > d^2\,\mu$$

$$d^2(d+\gamma) + {}_*\frac{\mu dk\delta I_*}{1+\alpha S_*^{\,p} + \beta I_*^{\,q}} > d^2\,\mu$$

$$d^2(d+\gamma)\left(1+\alpha S_*^{\,p} + \beta I_*^{\,q}\right) + \mu\, d\,k\,\delta I_* > d^2\,\mu\left(1+\alpha S_*^{\,p} + \beta I_*^{\,q}\right)$$

Let $d^2\left(1+\alpha S_*^{\,p} + \beta I_*^{\,q}\right) = A$

we obtain

$$A(d+\gamma) + \mu\, d\,k\,\delta I_* > A\,\mu$$

Theorem: If $A(d+\gamma) + \mu\, d\,k\,\delta I_* > A\,\mu$, the infected equilibrium is asymptotically stable.

Proof: By Routh-Hurwitz criteria (Wang and Lu, 2006).All zeros of P (λ) have negative real parts if and only if $A(d+\gamma) + \mu\, d\,k\,\delta I_* > A\,\mu$.

Therefore, the infected equilibrium is asymptotically stable.

4. Conclusion

In this paper, we have carried out the stability of the equilibrium states for the modified model.In terms of the basic reproduction number R_0 ,the results indicate that when R_0 decreases, the recruitment increases. This implies that the spread of the disease decreases as the transmission rates for the infective increases. These results are in good agreement with those of Pathak et al [17].

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