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On The Stability of Collinear Points In The Photogravitational Elliptic Restricted Three-Body Problem.

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Abstract

This paper investigates the stability of collinear points of a small particle in the photogravitational elliptic restricted three-body problem moving in elliptic orbit about their centre of mass, under the influence of radiation pressures of the primaries, together with the gravitational attraction force. Collinear points in general are unstable, however, the inner collinear point L_2, is seen to be stable in an interval for the mass reduction factor of the bigger primary, under certain conditions depending on the mass ratio, eccentricity and semi-major axis of the orbit. Further, a practical application of the motion of a dust grain in the case of the binary star system in Capella is also discussed.

Keywords: celestial mechanics, elliptic photogravitational RTBP.

1.0 Introduction

The classical circular restricted three-body problem describes the dynamics of a small particle moving in the gravitational field of two finite masses, called primaries, which move in circular orbits around their center of mass on account of their mutual attraction. The equations of motion therefore, most naturally are presented in a non-inertial coordinate system that rotates with the mean motion of the primaries [9]. In the rotating coordinate system, the positions of the primaries are fixed. When the primaries' orbit is elliptic rather than circular a non-uniformly rotating-pulsating coordinate system is commonly used. These new coordinates have the felicitous property that, the positions of the primaries are fixed, however the Hamiltonian is explicitly time-dependent [14]. The infinitesimal mass can be at rest in a rotating coordinate frame, at five libration points (three collinear $L_{1,2,3}$ and two triangular $L_{4,5}$), where the gravitational and centrifugal forces just balance each other. The collinear points are unstable where as the triangular points are linearly stable, when the mass ratio of the primaries is less than the Routhian value [14].

This classical restricted three-body problem is not suited to discuss the case when at least one of the interacting bodies is an intense emitter of radiation. According to [11] and [12] the problem in such a statement is called the photogravitational problem. In certain stellar dynamics problems it is altogether inadequate to consider solely the gravitational interaction force. For example, when a star acts upon a particle in a cloud of gas and dust, the dominant factor is by no means gravity, but the repulsive force of the radiation pressure. Since a large fraction of all stars belong to binary systems, the particle motion in the field of a double star offers special interest. If a satellite flies high enough above the Earth and is large enough in size, but at the same time has sufficiently small mass, then the radiation pressure has a very strong effect on its motion. The distance of the satellite to the Sun practically is unaltered, and so the magnitude of the radiation pressure is practically constant [1]. Following [11] and [12], we express the difference between the gravitational force F_g and the force due to radiation pressures on the infinitesimal body by the means of F_p such that

$$F_g - F_p = F_g\left(1-\frac{F_p}{F_g}\right) = F_g\left(1-\in_1\right) = qF_g$$

where $\in_1 = F_p / F_g$, $q = 1-\in_1$.

It is obvious that

i. If $q=1$, we would have $\in_1 = 0$, implying the radiation pressure has no effect.

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ii. If $0 < q < 1$, we would have $\in_1 > 0$, the gravitational force exceeds the radiational one.

iii. If $q = 0$, then $\in_1 = 1$, in this case the radiation force balances the gravitational one.

iv. If $q < 0$ implies $\in_1 > 1$, here the radiation force overrides the gravitational one.

Here, q is the reduction coefficient, which determines the resultant effect of the forces of gravitation and light pressure on the particle. It turns out [10] that the smaller the absolute dimension of the particles; the stronger the effect of non-gravitational factors associated with the solar radiation on the motion. Since individual particles have their specific coefficients which do not depend on properties of the emitting primary bodies, a gas-dust cloud similar to the Kordylewski clouds [2] is formed in the positions of relative equilibrium. Numerous examples are available in the binary star system where both primaries are radiating. For instance, Alpha centauri A and B, and the binary star system in Capella.

The stability of the collinear points for the circular version of the photogravitational problem was investigated in [4], [5]; [13]. It was demonstrated [4] and [5] that for certain values of the reduction coefficients q_1 and q_2, the inner libration point (L_2) can be stable, while the outer ones (L_1 , L_2) are always unstable. The problem of the influence of the eccentricity e of the orbits of the primary bodies on the existence of the libration points and on the condition of their stability was touched upon to some extent in studies [3] and [8]. In [6] and [7], a simple and physically clear pattern of the influence of the small eccentricity of the orbit of the primary bodies on the position and stability of the triangular and collinear points was obtained.

Our aim in this study is to present a description of the necessary conditions for the stability of the collinear points in the photogravitational elliptic restricted three-body problem, with a numerical example in the case of the binary system in Capella.

2.0 EQUATIONS OF MOTION

The equations of motion of an infinitesimal mass moving in the gravitational field of the luminous primaries revolving in ellipses of eccentricities *e* about their centre of mass in a Keplerian, barycentric, pulsating, rotating coordinate system have the form (Ishwar 2006, with $A=0$ & $q_2 \neq 1$).

$$\xi'' - 2\eta' = \Omega_\xi \ , \ \eta'' + 2\xi' = \Omega_\eta \ , \quad \zeta'' = \Omega_\zeta \tag{1}$$

With the force function

$$\Omega = (1-e^2)^{-\frac{1}{2}}\left[\frac{1}{2}(\xi^2+\eta^2) + \frac{1}{n^2}\left\{\frac{\mu q_1}{r_1} + \frac{(1-\mu)q_2}{r_2}\right\}\right]$$

$$r_i^2 = (\xi - \xi_i)^2 + \eta^2 + \zeta^2, \ (i = 1,2).$$

$$n^2 = \frac{(1+e^2)^{\frac{1}{2}}}{a(1-e^2)}$$

Here, μ, $1\text{-}\mu$ are the dimensionless masses of the primaries; $\xi_1 = -(1-\mu)$, $\xi_2 = \mu$ are their abscissa; r_1 and r_2 are their respective distances from the infinitesimal body; *e*, *a* and *n* are respectively, the eccentricity, semi-major axis and the mean motion of their elliptic orbits; q_1 and q_2 are mass-reduction factors; and the primes denote differentiation with respect to the eccentric anomaly *E* of their orbit.

To locate all the equilibrium points along the 0ξ axis, we denote the expression $(\Omega_\xi)_{\eta=\zeta=0}$ by $f(\xi)$. The coordinates of these points, called collinear points, will be the roots of the equation

$$f(\xi) = \frac{1}{(1-e^2)^{\frac{1}{2}}}\left[\xi - \frac{1}{n^2}\left\{\frac{\mu q_1(\xi-\xi_1)}{|\xi-\xi_1|^3} + \frac{q_2(1-\mu)(\xi-\xi_2)}{|\xi-\xi_2|^3}\right\}\right] = 0 \tag{2}$$

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To find all the real roots of Eq.(2) is a difficult task because of the presence of five parameters: a, e, μ, q_1 and q_2. In order to get rid of this difficulty we find, instead of $\xi(a,e,\mu,q_1, q_2)$ from Eq. (2), say, q_1 as a function of ξ for fixed values of a, e, μ and q_2.

Assuming $\xi \neq \xi_1$, and solving Eq. (2) for q_1, we have,

$$\mu q_1 = \frac{|\xi - \xi_1|^3}{(\xi - \xi_1)}\left[\xi n^2 - \frac{(1-\mu) q_2 (\xi - \xi_2)}{|\xi - \xi_2|^3}\right] \tag{3}$$

The figures below display a simple and physically clear pattern of the influence of the eccentricity of the orbit of the primaries for the fixed value of $\mu = 0.5$ and various selected values of $q_2 \leq 1$, $e < 1$ and $a \leq 1$, as given by Eq.(3). The variations in $q_1(\xi, q_2, \mu, a, e)$ for fixed values of q_2 $(0,1,-2,-4)$ are shown in figures 1,2,3,&4 respectively for increasing values of eccentricity and semi-major axis.

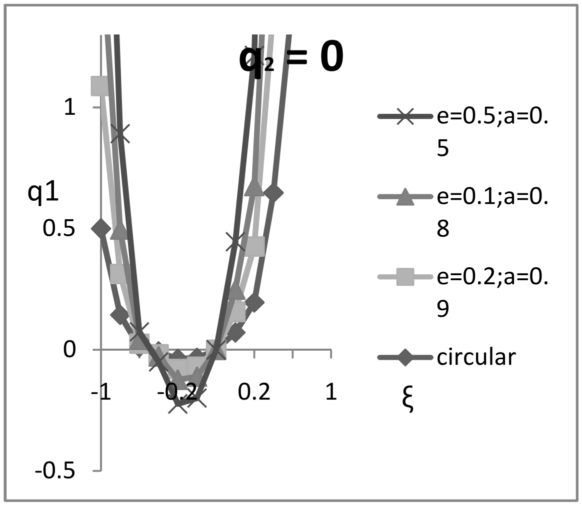

Fig. 1. The family of curves of q_1 for $q_2 = 0$ with increasing eccentricity e and semi-major axis a .

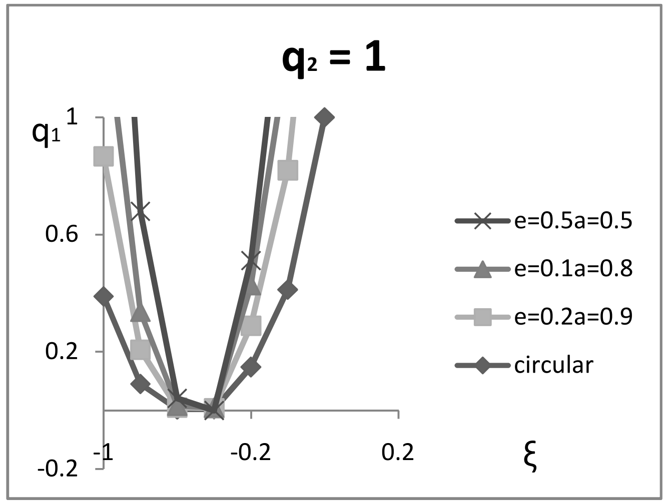

Fig.2. The family of curves of q_1 for $q_2 = 1$ with increasing eccentricity e and semi-major axis a .

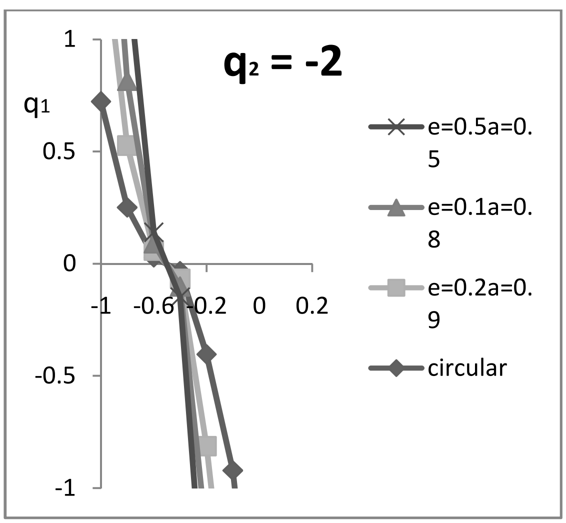

Fig.3. The family of curves of q_1 for $q_2 = -2$ with increasing eccentricity e and semi-major axis a .

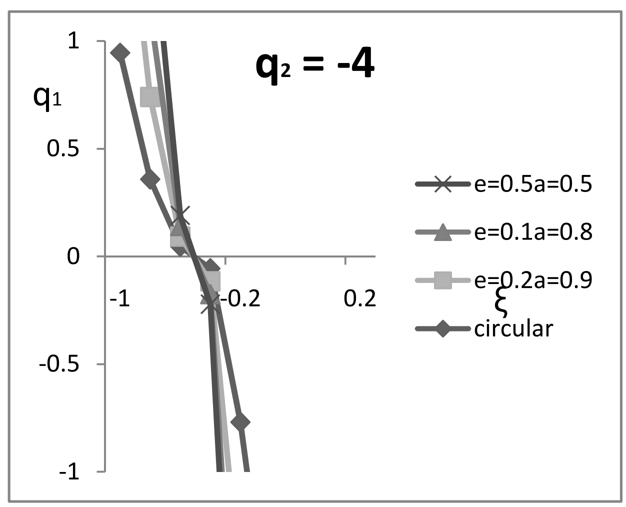

Fig.4. The family of curves of q_1 for $q_2 = -4$ with increasing eccentricity e and semi-major axis a

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3. STABILITY

The question of the real existence of the equilibrium points is tightly connected with the question of their Lyapunov stability. In the classical three-body problem, all the collinear libration points are unstable, but in our problem the positive reply is possible:

If q , q_2 are both positive, the inner libration points may be stable in the range.

$$n^2|\xi_0-\xi_1|^3\left[\frac{8}{9}+\frac{\xi_0}{9\mu}-\frac{5}{9}(\frac{\xi_0}{\mu}-1)e^2\right]\le q_1\le n^2|\xi_0-\xi_1|^3$$

In order to investigate the motion, we assume that the coordinates of the collinear point be $(\xi_o,0,0)$ and the infinitesimal body be displaced to the point $(\xi_o+\delta, \beta, \gamma)$, where (δ, β, γ) are small displacements. Then substituting these quantities in Eq.(1) and expanding in a Taylor's series, we obtain the linear variational equations as:

$$\delta''-2\beta'=\delta\Omega^0_{\xi\xi},\ \beta''+2\delta'=\beta\Omega^0_{\eta\eta},\ \gamma''=\gamma\Omega^0_{\varsigma\varsigma}. \tag{4}$$

Here only linear terms in δ, β, γ have been considered. The second partial derivatives of Ω are denoted by subscripts. The superscript 0 indicates that the derivative is to be evaluated at the collinear point $(\xi_0, 0, 0)$.

Let $c_{\xi\xi}=-\Omega^0_{\xi\xi}$, $c_{\eta\eta}=-\Omega^0_{\eta\eta}$ *and* $c_{\zeta\zeta}=-\Omega^0_{\zeta\zeta}$. Then

$$c_{\xi\xi}=-\frac{1}{n^2}(1-e^2)^{-\frac{1}{2}}\left[n^2+\frac{2\mu q_1}{|\xi_0-\xi_1|^3}+\frac{2(1-\mu)q_2}{|\xi_0-\xi_2|^3}\right]$$

$$c_{\eta\eta}=-\frac{1}{n^2}(1-e^2)^{-\frac{1}{2}}\left[n^2-\frac{\mu q_1}{|\xi_0-\xi_1|^3}-\frac{(1-\mu)q_2}{|\xi_0-\xi_2|^3}\right]$$

$$c_{\zeta\zeta}=\frac{1}{n^2}(1-e^2)^{-\frac{1}{2}}\left[\frac{\mu q_1}{|\xi_0-\xi_1|^3}+\frac{(1-\mu)q_2}{|\xi_0-\xi_2|^3}\right] \tag{5}$$

It is here noticed that the above are calculated under the values of ξ_0, q_1, q_2, a, e, satisfying Eq.(2).

The positive real parts of the roots of the characteristic equation of the system (4) will be absent if

$$c_{\xi\xi}\le 0,\ c_{\xi\xi}\le 0,\ c_{\zeta\zeta}\ge 0\quad and\quad \sqrt{-c_{\xi\xi}}+\sqrt{-c_{\eta\eta}}\le 2 \tag{6}$$

To analyze these inequalities conveniently, we express q_2 present in coefficient (5) in terms of q_1 by means of Eq. (2). For this, Eq. (2) yields

$$q_2=\frac{|\xi_0-\xi_2|^3}{(1-\mu)(\xi_0-\xi_2)}\left[\xi_0 n^2-\frac{q_1\mu(\xi_0-\xi_1)}{|\xi_0-\xi_1|^3}\right]$$

The substitution for q_2 into Eq. (5) reduces Eq. (6) to

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$$c_{\xi\xi} = -\frac{1}{n^2}(1-e^2)^{-\frac{1}{2}}\left[n^2 + \frac{2\xi_0 n^2}{(\xi_0-\mu)} - \frac{2\mu q_{11}}{(\xi_0-\mu)|\xi_0-\xi_1|^3}\right] \le 0$$

$$c_{\eta\eta} = -\frac{1}{n^2}(1-e^2)^{-\frac{1}{2}}\left[n^2 - \frac{\xi_0 n^2}{(\xi_0-\mu)} - \frac{\mu q_{11}}{(\xi_0-\mu)|\xi_0-\xi_1|^3}\right] \le 0$$

$$c_{\zeta\zeta} = (1-e^2)^{-\frac{1}{2}}\left[\frac{\xi_0}{(\xi_0-\mu)} - \frac{\mu q_{11}}{n^2(\xi_0-\mu)|\xi_0-\xi_1|^3}\right] \ge 0$$

$$q_1^2 - \frac{2n^2|\xi_0-\xi_1|^3}{9\mu}\left[9\xi_0 - 4(\xi_0-\mu)(1-e^{2^{\frac{1}{2}}}\right]q_1 +$$

$$\frac{n^4|\xi_0-\xi_1|^6}{9\mu^2}\left[9\xi_0^2 + (\xi_0-\mu)(1-e^{2^{\frac{1}{2}}}\left\{-24\xi_0 + 16\mu + 16(\xi_0-\mu)(1-e^{2^{\frac{1}{2}}}\right\}\right] \ge 0$$

(7)

Where ξ_0 is the abscissa of the collinear point satisfying Eq.(2), and $\xi_1 = \mu-1$. Eq. (5) shows that the condition $c_{\zeta\zeta} \ge 0$ cannot be satisfied if simultaneously $q_1 < 0$ and $q_2 < 0$. This implies that stability is impossible when none of the stars attracts the particle.

Now, we show that none of the external collinear points is stable. For this, it will be sufficient to prove that at least one of the inequalities (7) does not hold. From the middle two inequalities (7) together with $\xi_0 < \xi_1 < 0$, we can get an interval for q_1:

$$\frac{\xi_0 n^2}{\mu}|\xi_0-\xi_1|^3 \le q_1 \le n^2|\xi_0-\xi_1|^3$$

(8)

Also, from Eq. (2) and $q_2 \le 1$ we have, $-\xi_0 n^2(\xi_0-\xi_1)^2 - (1-\mu)\left(\frac{\xi_0-\xi_1}{\xi_0-\xi_2}\right)^2 \le \mu q_1$

(9)

This determines the domain of existence of the collinear points in question. We see here that inequality (8) fails in the domain (9). Therefore, all collinear points are unstable for $\xi_0 < \xi_1$. Further, from the second of inequalities ((7) and $\xi_0 > \xi_2$ we can derive

$q_1 \ge n^2|\xi_0-\xi_1|^3$. But if $\xi_0 > \xi_2$, the above inequality will be true only if $q_1 > 1$, which has no any physical sense. Hence, the external collinear points are unstable.

Now, we examine the stability of the internal libration points lying in the interval $\xi_1 < \xi_0 < \xi_2$. In this case, inequalities (8) are valid. So we consider the last of inequalities (7), which is satisfied by the value of q_1 decided by one of the inequalities

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$$q_1 \le \frac{n^2|\xi_0 - \xi_1|^3}{\mu}\left[\xi_0 + (\xi_0 - \mu)e^2\right]$$

$$q_{11} \ge n^2|\xi_0 - \xi_1|^3\left[\frac{8}{9} + \frac{\xi_0}{9\mu} - \frac{5}{9}(\frac{\xi_0}{\mu} - 1)e^2\right]$$

Considering these inequalities together with (8), we find the following interval for q_1:

$$n^2|\xi_0 - \xi_1|^3\left[\frac{8}{9} + \frac{\xi_0}{9\mu} - \frac{5}{9}(\frac{\xi_0}{\mu} - 1)e^2\right] \le q_1 \le n^2|\xi_0 - \xi_1|^3 \qquad (10)$$

This is a solution of the system (7) and thus guarantees the stability of the internal collinear points to a first approximation. Hence, inequalities (10) are the only necessary conditions for stability.

4 A Practical Application

Considering the binary star system in Capella, in the constellation Auriga, with masses $2.6M_{sun}$ and $2.7M_{sun}$ respectively; about 43 yrs from the Earth. Capella has an Absolute magnitude 0.4 and an Apparent magnitude 0.08. So, the Absolute Bolometric magnitude is -6.52 and $\mu = \frac{M_2}{M_1 + M_2} = 0.49 \;\; so \;\; that \quad 1 - \mu = 0.51$.

The mass reduction factor for the bigger primary for $\kappa=1$ say, is $q_1 = 1 - 0.0956594 \; \frac{1}{a\rho}$ [calculated on the basis of Stefan-Boltzmann's law [15], $q = 1 - \frac{A\kappa P}{a\rho M}$, where M is the mass and P the luminosity of the star; a and ρ are the radius and density of a moving body; κ is the radiation pressure efficiency factor of a star; $A = \frac{3}{16\pi CG}$ is a constant]. Let the radius and density of some dust grain in Capella be respectively, $a=2x10^{-2}cm$ and $\rho=1.4g/cm$, then $q_1=-2.414$. The luminosity of the bigger primary is thus $0.45x10^4 \, P_{sun}$ where P_{sun} is the luminosity of the Sun. $\xi_1=\mu-1=-0.51$ and from $\xi_1<\xi_0<\xi_2$ we assume $\xi_0=-0.24$, $e=0.25$ and $a=0.6$ in, the last of Eq.(1) $n^2=1.83249$. Then from Eq.(10) we obtain the necessary interval for the stability of the internal collinear point: $0.031964 \le q_1 \le 0.036069$.

5. Conclusions

Considering both primaries as sources of radiation in the elliptic restricted three-body problem, we found that the outer collinear points remain unstable. Only the inner collinear point for certain values of the reduction coefficient q_1 given by Eq.(10) is stable. This agrees with [5] with $e=0$ in our problem. It can be seen (figures 1,2,3 & 4) that a growth of the eccentricity leads to a reduction of the size of the domain of necessary stability, agreeing with [16].

Further, a numerical investigation for the motion of a dust grain in the binary star system in Capella shows the reduction coefficient to be negative on account of the small size of the particle. It also provides a necessary interval for the stability of the inner collinear point: $0.031964 \le q \le 0.036069$.

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