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Stability of Triangular Equilibrium Points in the Photogravitational Elliptic RTBP with Oblateness

Jagadish Singh and Aishetu Umar

Department of mathematics, Faculty of Science,
Ahmadu Bello University, Zaria, Nigeria.

Abstract

This study investigates the motion of an infinitesimal mass around the triangular equilibrium points $L_{4,5}$ in the elliptic restricted three-body problem when the primaries are intense emitters of radiation with further consideration that the bigger is an oblate spheroid. It is found that the motion around the triangular points is stable under certain conditions, which depends on the eccentricity of the orbits, oblateness coefficient and the factors due to radiation of the primaries. We observe that all these parameters have destabilizing tendencies, consequently resulting in a sharp decrease in the region of stability of the triangular points.

Keywords: celestial mechanics.

1.0 Introduction

The restricted three-body problem with or without radiation and oblateness has received attention especially in the two-dimensional case and with respect to its five equilibrium points, i.e. the collinear (or "Eulerian") points L_1, L_2, L_3 and the two isosceles triangular (or "Lagrangian") points L_4, L_5 (e.g. [1] and [11]). The circular restricted three-body problem describes the dynamics of a body having infinitesimal mass and moving in the gravitational field of two massive bodies, called, the primaries, which revolve around their centre of mass on a circular orbit. The equations of motion are, therefore, most naturally presented in a non-inertial coordinate system that rotates with the mean motion of the primaries [8]. In the rotating coordinate system the positions of the primaries are fixed. When the primaries' orbit is elliptic rather than circular a nonuniformly rotating-pulsating coordinate system is commonly used. These new coordinates have the felicitous property that, the positions of the primaries are fixed, however the Hamiltonian is explicitly time-dependent [16].

The participating bodies in the classical restricted three-body problem are strictly spherical in shape, but in actual situations it is found that several celestial bodies, such as Saturn and Jupiter, are oblate. The lack of sphericity, or the oblateness, of the planet or star causes large perturbations from the two-body orbit. The motions of artificial Earth satellites are examples of this. Many researchers have studied the restricted problem by taking into account the shapes (oblateness) of the primaries [4], [12] and [15].

The classical restricted three-body problem is not suited to discuss the motion of the infinitesimal when at least one of the interacting bodies is an intense emitter of radiation. The character of the action of the radiation pressure force pushing the particle away is reduced to a decrease of the mass of the radiating body. In certain stellar dynamics problems it is altogether in adequate to consider solely the gravitational interaction force. For example, when a star acts upon a particle in a cloud of gas and dust, the dominant factor is by no means gravity, but the repulsive force of the radiation pressure. In this connection, it is reasonable to modify the model by superposing a light repulsion field whose source coincides with the source of the gravitational field of the main bodies.
Recent studies of the restricted problem [5], [13] and [14] have included radiation pressure force.

The present study aims to examine the motion of the infinitesimal body in the ER3BP when both primaries are sources of radiation with oblateness of the bigger primary.

Corresponding author: E-mail; umaraishetu33@yahoo.com , Tel. +2348036786146

This paper is organized as follows: section 1, which is introduction; section 2 provides the equations of motion section 3
discuss the locations of the triangular equilibrium solutions; while section 4 focuses on their linear stability. The discussions and conclusions are drawn in sections 5 and 6 respectively.

2.0 Equations of Motion

The equations of motion of the infinitesimal mass in the elliptic restricted three-body problem when both primaries are radiating and the bigger one an oblate spheroid presented below in dimensionless- pulsating-rotating coordinate system have the form [10] as:

$$\xi'' - 2\eta' = \Omega_\xi, \quad \eta'' + 2\xi' = \Omega_\eta, \quad \zeta'' = \Omega_\zeta,$$

where Ω is the force function and expressed as

$$\Omega = \left(1-e^2\right)^{-1/2}\left[\frac{\xi^2+\eta^2}{2}+\frac{1}{n^2}\left(\frac{(1-\mu)q_1}{r_1}+\frac{\mu q_2}{r_2}+\frac{(1-\mu)Aq_1}{2r_1^3}\right)\right] \quad (1)$$

with

$$r_1^2 = (\xi+\mu)^2+\eta^2+\zeta^2, \quad r_2^2 = (\xi+\mu-1)^2+\eta^2+\zeta^2 \quad (2)$$

where q_1 and q_2 are radiation factors of the bigger and smaller primaries, respectively. r_1 and r_2 are distances of the infinitesimal mass from these primaries, A is the oblateness coefficient of the bigger primary, n is the mean motion; e the eccentricity and dashes denotes differentiation with respect to time t.
The mean motion, n , is given by

$$n^2 = \frac{(1+e^2)^{1/2}\left(1+\frac{3}{2}A\right)}{a(1-e^2)} \quad (3)$$

where a is the semi-major axis of the orbits.
Since the distance between the primaries is taken as equal to unity then $\xi_2 - \xi_1 = 1$
so that $\xi_1 = -\mu$ and $\xi_2 = 1-\mu$

3.0 Positions of triangular Equilibrium points

The positions of the equilibrium points can be found from the equations of motion (1) by putting all velocity and acceleration components equal to zero and solving the resulting system,

$$\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0 \quad , \quad (4)$$

for $\xi,\ \eta, \zeta$.
where

Corresponding author: E-mail; umaraishetu33@yahoo.com , Tel. +2348036786146

$$\Omega_{\xi} = \left(1-e^{2}\right)^{\frac{1}{2}}\left[\xi - \frac{1}{n^{2}}\left(\frac{(1-\mu)q_{1}(\xi-\xi_{1})}{r_{1}^{3}} + \frac{\mu q_{2}(\xi-\xi_{2})}{r_{2}^{3}} + \frac{3Aq_{1}(1-\mu)(\xi-\xi_{1})}{2r_{1}^{5}}\right)\right] = 0$$

$$\Omega_{\eta} = \frac{\left(1-e^{2}\right)^{\frac{1}{2}}\eta}{n^{2}}\left(\frac{(1-\mu)q_{1}}{r_{1}^{3}} + \frac{\mu q_{2}}{r_{2}^{3}} + \frac{3Aq_{1}(1-\mu)}{2r_{1}^{5}}\right) = 0$$

(5) $$\Omega_{\zeta} = \frac{\left(1-e^{2}\right)^{\frac{1}{2}}\eta}{n^{2}}\left(\frac{(1-\mu)q_{1}}{r_{1}^{3}} + \frac{\mu q_{2}}{r_{2}^{3}} + \frac{3Aq_{1}(1-\mu)}{2r_{1}^{5}}\right) = 0$$

The solutions of the first two equations of system (5) with $\eta \neq 0, \;\; \zeta = 0$ provides the positions of the triangular points. From the second equation of system (5), we obtain

$$n^{2} = \frac{q_{1}}{r_{1}^{3}} + \frac{3Aq_{1}}{2r_{1}^{3}} \quad \text{, with} \quad r_{2}^{3} = \frac{q_{2}}{n^{2}}$$

(6)

In the absence of oblateness of the bigger primary

$$r_{1}^{3} = \frac{q_{1}}{n^{2}}$$

(7)

This result differs slightly by α when oblateness of the bigger primary is considered, so that,

$$r_{1} = \alpha + \frac{q_{1}^{1/3}}{n^{2/3}} \tag{8}$$

Considering only terms in A and e^{2} and neglecting their product, equation (3) reduces to

$$n^{2} = \frac{1}{a}\left(1 + \frac{3}{2}A + \frac{3}{2}e^{2}\right) \tag{9}$$

Equations (6) and (9) and gives

$$r_{1} = \left(aq_{1}\right)^{1/3}\left(1 - \frac{e^{2}}{2}\right) \tag{10}$$

Substituting equation (10) in (8), we have

$$r_{1} = \left(aq_{1}\right)^{1/3}\left(1 - \frac{e^{2}}{2}\right) + \alpha \tag{11}$$

where

$$\alpha = \frac{A\left(aq_{1}\right)^{1/3}}{2}\left[-1 + 2e^{2} + \left(aq_{1}\right)^{-2/3} - 2e^{2}\left(aq_{1}\right)^{2/3}\right] \tag{12}$$

Substituting for α in equation (11), we obtain

$$r_{1}^{2} = \left(aq_{1}\right)^{2/3}\left(1 - e^{2} - A + A\left(aq_{1}\right)^{-2/3}\right) \tag{13}$$

Performing same procedure for r_{2}^{2} we get

$$r_{2}^{2} = \left(aq_{2}\right)^{2/3}\left(1 - e^{2} - A\right) \tag{14}$$

From equations of system (2) with $\zeta = 0$ we have

$$r_{1}^{2} - r_{2}^{2} = 2\xi + 2\mu - 1 \tag{15}$$

substituting equations (13) and (14) in (15) we get

Corresponding author: E-mail; umaraishetu33@yahoo.com , Tel. +2348036786146

$$\xi = \frac{1}{2} - \mu + \frac{1}{2}\left[\left(\left(aq_1\right)^{2/3}\right)\left(1 - e^2 - A + A\left(aq_1\right)^{-2/3}\right) - \left(aq_2\right)^{2/3}\left(1 - e^2 - A\right)\right]$$

$$\eta = \pm\left[\left(\left(aq_1\right)^{2/3}\right)\left(1 - e^2 - A + A\left(aq_1\right)^{-2/3}\right) - \frac{1}{4}\left\langle 1 + 2\left(aq_1\right)^{2/3}\left(1 - e^2 - A + A\left(aq_1\right)^{-2/3}\right) - \left(aq_2\right)^{2/3}\left(1 - e^2 - A\right)\right\rangle\right]$$

(16)

These points (16) are denoted by L_4 and L_5 respectively, and form two simple triangles with the line joining the primaries

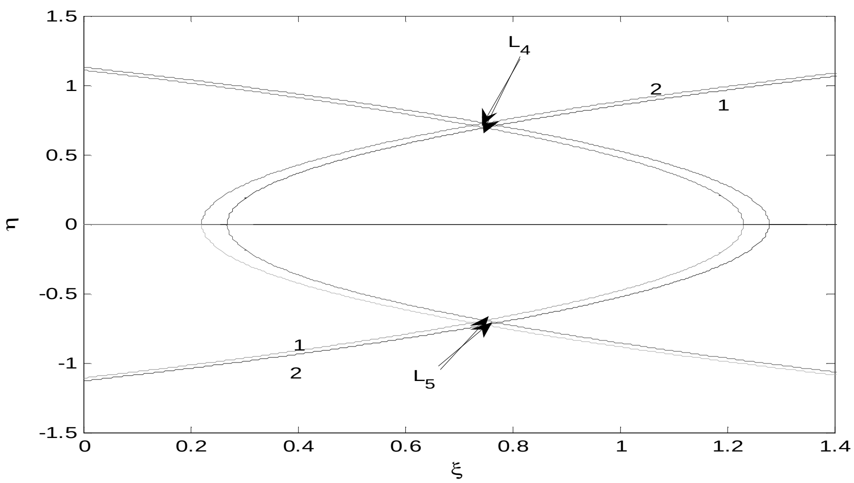

Fig.1.---Positions of the Triangular Equilibruim points L_4 and L_5 for (1) $q_1 = 0.3, q_2=0.25$ (2) $q_1 = 0.6, q_2=0.5$ when $\mu=0.4, A=0.3, a=0.8$ & $e=0.7$

4.0 Linear Stability Of Triangular Points

The stability of linear systems of ordinary differential equations with constant coefficients is determined by the Eigen values. Due to the perturbations induced by the radiation pressure forces of the primaries and oblateness of the bigger primary, the position of the infinitesimal mass would be displaced a little from the equilibrium point. If the resultant motion of the infinitesimal mass is a rapid departure from the vicinity of the point, we can call such a position of equilibrium point an "unstable one", if however the body merely oscillates about the equilibrium point, it is said to be a "stable position" (in the sense of Lyapunov).

We denote the equilibrium points and their positions as $L\left(\xi_0,\eta_0\right)$. Let a small displacement in $\left(\xi_0,\eta_0\right)$ be $\left(u,v\right)$. Then we write

$$\xi = \xi_0 + u \text{ , and } \eta = \eta_0 + v . \qquad (18)$$

Substituting these values in equations (1), we obtain the variational equations,

$$\begin{aligned} u'' - 2v' &= \left(\Omega^0_{\xi\xi}\right)u + \left(\Omega^0_{\xi\eta}\right)v \\ v'' + 2u' &= \left(\Omega^0_{\xi\eta}\right)u + \left(\Omega^0_{\eta\eta}\right)v, \end{aligned} \qquad (19)$$

The characteristic equation corresponding to (19), is

$$\lambda^4 - \left(\Omega^0_{\xi\xi} + \Omega^0_{\eta\eta} - 4\right)\lambda^2 + \Omega^0_{\xi\xi}\Omega^0_{\eta\eta} - \left(\Omega^0_{\xi\eta}\right)^2 = 0 \qquad (20)$$

Where the superscript 0 denote evaluation of the partial derivatives at the equilibrium points, $\left(\xi_0,\eta_0\right)$.

The partial derivatives computed at the triangular points are

Corresponding author: E-mail; umaraishetu33@yahoo.com , Tel. +2348036786146

$$\Omega^{*0}_{\xi\xi} = \left(1-e^2\right)^{-1/2}\left[\frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3(1-\mu)e^2}{4(aq_1)^{2/3}} + \frac{9(1-\mu)A}{4(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)q_2^{2/3}}{2q_1^{2/3}}\right.$$

$$\left. + \frac{3\mu}{4(aq_2)^{2/3}} + \frac{3\mu e^2}{4(aq_2)^{2/3}} - \frac{3\mu A}{4(aq_2)^{2/3}} - \frac{3\mu q_1^{2/3}}{2q_2^{2/3}} + \frac{3\mu}{2}\right] \tag{21}$$

$$\Omega^{*0}_{\eta\eta} = \left(1-e^2\right)^{-1/2}\left[\frac{3\mu}{2} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3(1-\mu)q_2^{2/3}}{2q_1^{2/3}} + \frac{3\mu q_1^{2/3}}{2q_2^{2/3}} - \frac{3\mu}{4(aq_2)^{2/3}}\right.$$

$$\left.\frac{3(1-\mu)A}{4(aq_1)^{2/3}} + \frac{3\mu A}{4(aq_2)^{2/3}} - \frac{3(1-\mu)e^2}{4(aq_1)^{2/3}} - \frac{3\mu e^2}{4(aq_2)^{2/3}}\right]$$

$$\Omega^{*0}_{\xi\eta} = \eta_0\left(1-e^2\right)^{-1/2}\left[\frac{3(1-\mu)}{2(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)q_2^{2/3}}{2q_1^{2/3}} - \frac{3\mu}{2(aq_2)^{2/3}} - \frac{3\mu}{2}\right.$$

$$\left. + \frac{3\mu q_1^{2/3}}{2q_2^{2/3}} + \frac{3(1-\mu)e^2}{2(aq_1)^{2/3}} - \frac{3\mu e^2}{2(aq_2)^{2/3}} + \frac{3(1-\mu)A}{(aq_1)^{2/3}}\right]$$

Substituting the above equation of system (21) in the characteristics equation (20) and restricting ourselves only to the linear terms in $e^2, A, \beta_1, \beta_2, \text{and}\,\beta_3$ for $\alpha = 1-\beta_1,\ q_1 = 1-\beta_2,\ q_2 = 1-\beta_3$, we have

$$4\lambda^4 + 4(4-3\alpha_1)\lambda^2 + 27\mu(1-\mu) + 4\alpha_2 = 0 \tag{22}$$

with

$$\alpha_1 = \left[1 + (1-\mu)A\left(1 + \frac{2}{3}\beta_1 + \frac{2}{3}\beta_2\right)\right]\left(1-e^2\right)^{-1/2}$$

$$= 1 + (1-\mu)A + \frac{e^2}{2}$$

Similarly

$$\alpha_2 = 3\mu(1-\mu)\beta_1 + 3\mu(1-\mu)(\beta_2+\beta_3) + \frac{45}{4}\mu(1-\mu)e^2 + 9\mu(1-\mu)(A)$$

$$= 3\mu(1-\mu)\left[\beta_1 + (\beta_2+\beta_3) + \frac{15}{4}e^2 + 3A\right]$$

Equation (22) is a quadratic equation in λ^2, which yields

$$\lambda^2 = \frac{-(4-3\alpha_1) \pm \left[(4-3\alpha_1)^2 - 27\mu(1-\mu) + 4\alpha_2\right]^{1/2}}{2}$$

For stable motion, we require λ to be pure imaginary i.e., motion must be bounded and periodic, so we choose $\mu,\ \alpha_1,\ \alpha_2$ such that $\lambda^2 < 0$, we get

$$3\alpha_1 - 4 \le 0$$

and the discriminant

$$\Delta = (4-3\alpha_1)^2 - 27\mu(1-\mu) + 4\alpha_2 > 0$$

which yields

$$0 \le e \le \left[1 - \frac{9}{16}\left\langle 1 + (1-\mu)A_1 + \mu A_2\right\rangle^2\right]^{1/2} \tag{23}$$

when $A = 0$, equation (16) becomes

Corresponding author: E-mail; umaraishetu33@yahoo.com , Tel. +2348036786146

$$0 \le e \le \frac{\sqrt{7}}{4} \tag{24}$$

In the case when equation (23) is not satisfied, the characteristic roots will be either real or complex conjugate. In the case of complex roots, the positive real part leads to instability of the investigated equilibrium points.
Now from equation (), we have

$$\begin{aligned}\Delta &= \left[27 + 3\left(4\alpha + 2\beta_1 + 2\beta_2 + 15e^2 + 12A\right)\right]\mu^2 \\ &\quad - \left[27 - 6A + 3\left(4\alpha + 2\beta_1 + 2\beta_2 + 15e^2 + 12A\right)\right]\mu \\ &\quad + \left(1 - 6A + 3e^2\right) > 0\end{aligned} \tag{25}$$

The necessary conditions for the stability of the triangular points (23, 25) have thus been derived. The solution of the quadratic equation $\Delta = 0$; i.e., when the discriminant vanishes for μ gives the critical value μ_C of the mass parameter given as

$$\mu_C = \frac{1}{2}\left(1 - \sqrt{\frac{23}{27}}\right) - \frac{1}{9}\left(1 + \frac{13}{\sqrt{69}}\right)A - \frac{2}{27\sqrt{69}}\left(\beta_1 + \beta_2\right) - \frac{4}{27\sqrt{69}}\alpha - \frac{14}{9\sqrt{69}}e^2 \tag{26}$$

The equation (26) represents the effect of radiation pressures of the primaries and oblateness of the bigger primary and eccentricity on the critical mass value μ_C.
It is seen from equation (26) that the radiation pressure(s) and oblateness always have a destabilizing tendency which confirms the results of [1], [2] and [13] when eccentricity is taken as zero. Hence, the overall effect of radiation pressures, oblateness and eccentricity always result in a decrease in the region of stability of the triangular points.

5.0 Discussion

The system (1) of equations of motion is different from those obtained by [17] due to the introduction of oblateness of the primaries and absence of eccentric and true anomaly. We observe that the assumption that the primaries are oblate in shape still permits the existence of the triangular points $L_{4,5}$; though these points are affected by the oblateness coefficients A. If we put $A = 0$ and $e = 0$, the triangular points (16) will fully coincide with those of [6]. If the smaller primary is taken as a non radiating one and both primaries are spherical; the points (16) will be analogous to that of [7] in the absence of eccentric and true anomaly.

In the case when eccentricity ($e = 0$) i.e. when orbits are circular, the equation (12) fully coincide with that of [6] and [13]when $A = 0$; [2] when $A = 0$ and $\beta_2 = 0$; the classical case of [16] when the primaries are spherical (i.e. $A = 0$) and non emitters of radiation, i.e. $\beta_1 = \beta_2 = 0$.

The characteristic equation of the triangular points obtained by [7] is different from our characteristic equation (14), due to oblateness of the radiating primaries and absence of eccentric and true anomaly. On ignoring eccentricity ($e = 0$) i.e. if orbits are circular; the equation (14) differ from that of [1] due to the perturbations in the Coriolis and centrifugal forces; analogous to that of [13]; differs from the characteristic equation of [2] due to the inclusion of oblateness of the primaries and radiation tendency of the smaller primary; and reduces to the classical case of [16], when the bigger primary is spherical and both non emitters of radiation.

Equation (29) gives the critical values μ_C for different values of oblateness, radiation pressures and eccentricity. The critical mass ratios are a tool in determining the region of stability and also serve as the technicalities in analyzing the behaviors or effects of these parameters on the motion of the infinitesimal mass around1the triangular equilibrium points. When eccentricity is absent (i.e. $e = 0$), μ_C differ from that of [1] due to the perturbations in the Coriolis and centrifugal forces; same as worked out by [14]; different from the critical mass of [2] due to the presence of oblateness of the primaries and radiation pressure of the smaller primary; and reduces to the classical case of [16], when the primaries are spherical and non emitters of radiation.

Corresponding author: E-mail; umaraishetu33@yahoo.com , Tel. +2348036786146

It is observed from the critical mass parameter μ_C, that the radiation pressure(s) of the primaries have destabilizing tendency which validates the findings of [2], [3], [6], [7], [9], [13] and [17] that the radiation pressures of the primaries always have a destabilizing effect. We also observe form the critical mass ratio that oblateness of the bigger primary also possesses a destabilizing tendency and this verifies the results of [1] and [13]. The eccentricity is like seen to possess a destabilizing behavior on the stability of motion around the triangular equilibrium points and confirms the assertions of [7] and [17]. The overall effect is that the region of stability of the triangular points in the photogravitational elliptic restricted three-body problem decreases fast.

6.0 Conclusion

The stability of the triangular libration points is investigated in the photogravitational elliptic restricted three-body problem, in which the two primary bodies radiating and the bigger an oblate spheroid. The conditions of stability of the triangular libration points are obtained and the stability regions are determined in the space of the parameters of mass, eccentricity, radiation pressures and oblateness. It is found that radiation pressures, oblateness of the primaries and the eccentricity exert a significant quantitative influence on the stability regions. Consequently the overall effect of the destabilizing behaviors of the radiation pressures, oblateness of the bigger primary and the eccentricity on the region of stability of the triangular points in the photogravitational elliptic restricted three-body problem decreases the region of stability fast.

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Corresponding author: E-mail; umaraishetu33@yahoo.com , Tel. +2348036786146